**International Mathematical Forum, Vol. 7, 2012, no. 47, 2333 - 2351**

# On the Rodrigues' Formula Approach to Operator Factorization

**W. Robin**

Engineering Mathematics Group
Edinburgh Napier University
10 Colinton Road, Edinburgh, EH10 5DT, UK
b.robin@napier.ac.uk

**Abstract**

In this paper, we derive general formulae that reproduce well-known instances of recurrence relations for the classical orthogonal polynomials as special cases. These recurrence relations are derived, using only elementary mathematics, *directly* from the general Rodrigues' formula for the classical orthogonal polynomials – a 'first-principles' derivation – and represent a unified presentation of various approaches to the exact solution of an important class of second-order linear ordinary differential equations. When re-expressed in ladder-operator form, the recurrence relations are seen to represent to a basic development of the work of Jafarizadeh and Fakhri [5] and allow a 'Schrödinger operator factorization' of the defining equation of the classical orthogonal polynomials, as well as an operational formula for the solution of this defining equation. The identity between the Rodrigues' formula and the operational formula is determined and standard examples involving the application of the ladder-operator approach presented. The relationship with previous work is discussed.

**Mathematics Subject Classification:** 33C45; 42C05

**Keywords:** classical orthogonal polynomials; ladder-operators; operational formula

## 1. Rodrigues' Formula Solutions to Second-Order Differential Equations

In this paper we consider the second-order linear ordinary differential equation

$$p(z)y_n''(z) + q(z)y_n'(z) + \lambda_n y_n(z) = 0 \qquad (1)$$

where $n \geq 0$ is a *non-negative integer*, the dashes denote differentiation with respect to the function argument z, $\lambda_n$ is *independent* of z, and

$$p(z) = \frac{p''}{2}z^2 + p'(0)z + p(0) \text{ and } q(z) = q'z + q(0) \qquad (2)$$

that is, p(z) is a quadratic function, q(z) a linear function of z. Equation (1) has known (classical) orthogonal polynomial solutions, with normalising factors $K_n$

$$y_n(z) = \frac{1}{K_n w(z)}\left(\frac{d}{dz}\right)^n [w(z)p^n(z)] \qquad (3)$$

with respect to the weight function $w(z)$ in the interval (a, b) – which interval need not be finite – provided that [3]

$$n(1-n)p'' - 2nq' = 2\lambda_n \qquad (4)$$

Given an ordinary differential equation (1.1), the weighting function $w(z)$ is determined by a first-order ordinary differential equation, a Pearson equation [2]

$$\frac{d}{dz}[w(z)p^{k+1}(z)] = [q(x) + kp'(z)][w(z)p^k(x)] \qquad (5)$$

with $k \geq 0$ a non-negative integer. As p(z) and q(z) are given, (5) may be solved for w(z) and we have then (up to a normalisation factor) a solution (3) of (1), provided that p(z), q(z) and $\lambda_n$ satisfy the 'integration condition' (4) (which we assume throughout).

Now, another well-known approach to the solution of (1), is to (effectively) 'factorize' the second-order linear differential operator on the left-hand side of (1) – called here the 'Schrödinger factorization' approach to the solution of (1). Since the solution of (1) is unique, it is apparent that the two solution processes for (1) – the Rodrigues' formula solution and the factorization approach – must yield the same answers under the same set of circumstances, and the question arises as to the exact connection between the Rodrigues' formula solution to (1) and the Schrödinger factorization approach to (1). It is this question that we address, below, in the body of the paper.

There is a considerable volume of literature on the Rodrigues' formula and factorization approaches to equation (1) and its solution. For the purpose of comparison, however, we restrict the discussion of this literature to a few of the most pertinent references. Of particular interest here, by way of comparison with the results presented below, is the work of Erdelyi et al [3], Jafarizadeh and Fakhri [5], Lorente [8], Kaufman [6], Nikiforov and Uvarov [11], Van Iseghem [13] and Yanez, Dehesa and Nikiforov [14], which we discuss in detail in section 5,

(Of course reference [3] has been a 'standard' for many years.)

The approach to the Schrödinger factorization that we develop below is based on the demonstration, in section 2, of recurrence relations for the Rodrigues' formula (3). The demonstrations in section 2 are based on techniques going well-back [1], [4] that depend on the mathematical structure of the Rodrigues' formula (3). The ladder-operator formalism that emerges from the analysis of the mathematical structure of the Rodrigues' formula (3), presented in section 3, is 'first cousin' to that developed by Jafarizadeh and Fakhri [5], who, however, obtain their ladder-operators by *factorizing* (the equivalent of) (1) *directly*. As a development, an extra element is here extracted and an assumption eliminated from the Jafarizadeh and Fakhri formalism and an operational *identity* established between the

Rodrigues' formula solution and the ladder-operator solution to (1). Examples of the ladder-operators method are presented in section 4 and we round-off our presentation with a brief discussion and acknowledgement of the work of previous authors, and some further closing remarks, in section 5.

## 2. The Recurrence Relations for the Rodrigues' Formula

The basic relations that link the factorization methodology with the Rodrigues' formula solution to (1) [(3)] are recurrence relations for (3). In this section the derivations of differential recurrence relations and a three term recurrence relation for the Rodrigues' formula solution, (3), to equation (1) are outlined.
In the next section we show how the differential recurrence relations for (3) can be used to set-up the Schrödinger factorization approach to solving (1).
The technique that we adopt in the derivations below is to derive and manipulate, through elimination, equations involving derivatives of $w(z)p^k(z)$. The philosophy motivating this approach is obvious on examining the structure of (3). We require five equations in total, of which the first is the Pearson equation (5). The remaining four equations are obtained by differentiating (5) and by the application of Leibnitz' rule for differentiating a product.
So, from (5) we find that $(k = n)$

$$\left(\frac{d}{dz}\right)^{n+1}[wp^{n+1}] = (q + np')\left(\frac{d}{dz}\right)^{n}[wp^n] + n(q' + np'')\left(\frac{d}{dz}\right)^{n-1}[wp^n] \tag{6}$$

while, from the direct application of Leibnitz' rule to $p(z)[w(z)p^n(z)]$, we get

$$\left(\frac{d}{dz}\right)^{n+1}[wp^{n+1}] = p\left(\frac{d}{dz}\right)^{n+1}[wp^{n}] + (n+1)p'\left(\frac{d}{dz}\right)^{n}[wp^{n}] + \frac{n(n+1)}{2}p''\left(\frac{d}{dz}\right)^{n-1}[wp^{n}]$$

---------- (7)

The fourth and fifth equations are obtained in a manner similar to (6) and (7).

From (5) we find that $(k = n-1)$

$$\left(\frac{d}{dz}\right)^{n-1}[wp^{n}] = (q+(n-1)p')\left(\frac{d}{dz}\right)^{n-2}[wp^{n-1}] + (n-2)(q'+(n-1)p'')\left(\frac{d}{dz}\right)^{n-3}[wp^{n-1}]$$

---------- (8)

and, from the direct application of Leibnitz' rule to $p(z)[w(z)p^{n-1}(z)]$, we get

$$\left(\frac{d}{dz}\right)^{n-1}[wp^{n}] = p\left(\frac{d}{dz}\right)^{n-1}[wp^{n-1}] + (n-1)p'\left(\frac{d}{dz}\right)^{n-2}[wp^{n-1}] + \frac{(n-1)(n-2)}{2}p''\left(\frac{d}{dz}\right)^{n-3}[wp^{n-1}]$$

---------- (9)

We write-out, first, the expression for the 'raising' differential recurrence relation, then we present the three term recurrence relation and, finally, we write-out the expression for the 'lowering' differential recurrence relation.

So, eliminating $\left(\frac{d}{dz}\right)^{n+1}[wp^{n}]$ and $\left(\frac{d}{dz}\right)^{n-1}[wp^{n}]$ from (5), (8) and (7), with (3) in mind, we get the 'raising' differential recurrence relation

$$p\frac{dy_n(z)}{dz} + \frac{(2q'+(n-1)p'')(q+np')}{2(q'+np'')}y_n(z) = \frac{(2q'+(n-1)p'')}{2(q'+np'')}\frac{K_{n+1}}{K_n}y_{n+1}(z)$$

---------- (10)

(Note that, from (3) and (5) with $k = 0$, we find that [11]

$$p\left(\frac{d}{dz}\right)^{n+1}[wp^n] = K_n w[py'_n + (q-p')y_n])$$

Next, from (7), (8) and (9) we may eliminate $\left(\frac{d}{dz}\right)^{n-1}[wp^n]$,

$\left(\frac{d}{dz}\right)^{n-2}[wp^{n-1}]$ and $\left(\frac{d}{dz}\right)^{n-3}[wp^{n-1}]$ to get, with (3) in mind, the three term recurrence relation in the form

$$\begin{aligned}&(2q'+(n-1)p'')(q'+(n-1)p'')K_{n+1}y_{n+1}(z) = \\ &\quad (2q'+(2n-1)p'')\{n(2q'+(n-1)p'')p'+(q'-p'')Q\}K_n y_n(z) \\ &+ n(q'+np'')\{2(q'+(n-1)p'')^2 p-(q+(n-1)p')[(2q''+(n-1)p'')p'-p''q]\}K_{n-1}y_{n-1}(z)\end{aligned}$$

---------- (11)

Finally, eliminating $y_{n+1}(z)$ from (10) and (11) we find the 'lowering' differential recurrence relation

$$\begin{aligned}&p\frac{dy_n(z)}{dz} - \frac{n[(2q'+(n-1)p'')p'-p''q]}{2(q'+(n-1)p'')}y_n(z) \\ &\quad = \frac{n\{(2q'+(n-1)p'')^2 p-(q+(n-1)p')[(2q'+(n-1)p'')p'-p''q]\}}{2(q'+(n-1)p'')}\frac{K_{n-1}}{K_n}y_{n-1}(z)\end{aligned}$$

---------- (12)

## 3. The Schrödinger Factorization Method

One of the best-known ways of providing solutions to equation (1) is to introduce 'ladder operators' [5], [6], [8]. Ladder-operators are first-order linear differential operators that relate different solutions of (1) to one another via differentiation and elementary algebra. The big advantage of employing ladder-operators is that given any one of the ladder-operator solutions, any other solution can be obtained simply by applying the appropriate ladder-operator a sufficient number of times. In addition, it is possible to re-arrange the basic equation, (1), so that it may be represented in a particular factorized form that we have called here the Schrödinger factorized form or Schrödinger factorization. Specifically, the ladder operators come in 'matched pairs': a 'raising' ladder operator, $L_n^+$

$$L_n^+ = p(z)\left(\frac{d}{dz}\right) + (\alpha_n^+ z + \beta_n^+) \tag{13}$$

with $\alpha_n^+$ and $\beta_n^+$ independent of z, and a 'lowering' ladder operator, $L_n^-$

$$L_n^- = p(z)\left(\frac{d}{dz}\right) + (\alpha_n^- z + \beta_n^-) \tag{14}$$

with $\alpha_n^-$ and $\beta_n^-$ independent of z also. The basic property of the 'raising' ladder operator, $L_n^+$, is that it relates successive solutions of (1) through

$$L_n^+ y_n(z) = \gamma_n^+ y_{n+1}(z) \tag{15}$$

while the corresponding relation for the 'lowering' ladder operator, $L_n^-$, is

$$L_n^- y_n(z) = \gamma_n^- y_{n-1}(z) \tag{16}$$

with $\gamma_n^+$ and $\gamma_n^-$ independent of z. Using the 'ladder operator representation', (13) and (14), we may reproduce the second-order linear ODE (1) (or its *equivalent*) by combining (15) with (16) as either, $L_{n-1}^+ L_n^-$ or $L_n^- L_{n-1}^+$. Indeed if we define

$$E_n = \gamma_{n-1}^+ \gamma_n^- \tag{17}$$

see immediately that

$$L_{n-1}^{+}L_{n}^{-}y_{n}(z) = E_{n}y_{n}(z) \tag{18}$$

and

$$L_{n}^{-}L_{n-1}^{+}y_{n-1}(z) = E_{n}y_{n-1}(z) \tag{19}$$

The relationship between the Rodrigues' formula solution and the ladder operator approach to (1) is, in the light of section 2, immediate. If we rewrite (15) and (16) in full, using (13) and (14) respectively, then we get

$$p(z)\frac{dy_{n}(z)}{dz} + (\alpha_{n}^{+}z + \beta_{n}^{+})y_{n}(z) = \gamma_{n}^{+}y_{n+1}(z) \tag{20}$$

and

$$p(z)\frac{dy_{n}(z)}{dz} + (\alpha_{n}^{-}z + \beta_{n}^{-})y_{n}(z) = \gamma_{n}^{-}y_{n-1}(z) \tag{21}$$

Comparing (20) with (10), we identify the coefficients of (13) as

$$\alpha_{n}^{+} = q' + \frac{(n-1)}{2}p'' \tag{22a}$$

$$\beta_{n}^{+} = \frac{(2q' + (n-1)p'')[q(0) + np'(0)]}{2(q' + np'')} \tag{22b}$$

and

$$\gamma_{n}^{+} = \frac{(2q' + (n-1)p'')}{2(q' + np'')}\frac{K_{n+1}}{K_{n}} \tag{22c}$$

Also, by comparing (21) with (12), we find that

$$\alpha_{n}^{-} = -\frac{np''}{2} \tag{23a}$$

$$\beta_{n}^{-} = -\frac{n[(2q' + (n-1)p'')p'(0) - p''q(0)]}{2(q' + (n-1)p'')} \tag{23b}$$

and

$$\gamma_{n}^{-} = \frac{n\{(2q' + (n-1)p'')^{2}p - (q + (n-1)p')[(2q' + (n-1)p'')p' - p''q]\}}{2(q' + (n-1)p'')}\frac{K_{n-1}}{K_{n}}$$

---------- (23c)

Further to this, combining the results from equations (22c) and (23c), we get an expression for $E_n = \gamma^+_{n-1}\gamma^-_n$ as

$$E_n = \frac{n(2q' + (n-2)p'')\{2(q' + (n-1)p'')^2 p - (q + (n-1)p')[(2q' + (n-1)p'')p' - p''q]\}}{4(q' + (n-1)p'')^2}$$

---------- (24)

Finally, we still have a three-term recurrence relation between the different levels of solution to (1), that is

$$y_{n+1}(z) + (a_n z + b_n)y_n(z) + c_n y_{n-1}(z) \tag{25}$$

where the coefficients $a_n$, $b_n$ and $c_n$, with $a_n, b_n$ and $c_n$ independent of z, can be evaluated by a basic comparison between (25) and (11). Interestingly, the normalization factors for the $y_n(z)$, the classical orthogonal polynomials, may be obtained directly from (25) [7].

The ladder operator representation gives us, then, an alternative solution method for the ODE (1), as mentioned above. In fact given $y_0(z)$, we may obtain all other solutions $y_n(z)$, $n \geq 1$, from equation (15), since $\gamma^+_n \neq 0, n \geq 0.$ To obtain $y_0(z)$, we make use of the fact, from (21) and (23), that $y_0(z)$ is obtained as a solution to

$$\frac{dy_0(z)}{dz} = 0 \tag{26}$$

In other words, $y_0(z)$ is a *constant*, which is well-known [3, 11] from the theory of classical orthogonal polynomials. This is, of course, only the case if we identify the equivalence between (18), say, and the original equation (1). Indeed, if we write-out (18) in full and compare the result with (1) in the form

$$p(z)[p(z)y''_n(z) + q(z)y'_n(z) + \lambda_n y_n(z)] = 0 \tag{27}$$

then we see that we require the consistency conditions

$$p'(z) + (\alpha_{n-1}^{+} + \alpha_{n}^{-})z + (\beta_{n-1}^{+} + \beta_{n}^{-}) \equiv q(z) \tag{28}$$

and

$$\alpha_{n}^{-}p(z) + (\alpha_{n-1}^{+}z + \beta_{n-1}^{+})(\alpha_{n}^{-}z + \beta_{n}^{-}) - E_{n} \equiv p(z)\lambda_{n} \tag{29}$$

Similarly, if we write-out (19) in full and compare the result with the *adjusted* equation (27) ($n \to n-1$) then we see that we require the further consistency condition

$$\alpha_{n-1}^{+}p(z) + (\alpha_{n-1}^{+}z + \beta_{n-1}^{+})(\alpha_{n}^{-}z + \beta_{n}^{-}) - E_{n} \equiv p(z)\lambda_{n-1} \tag{30}$$

A straightforward calculation, a matter of substitution and simplification, shows that the identities (28) to (30) *are* indeed satisfied. In Table 1, we present the factorized form of the Legendre, the Laguerre and the Hermite equations, along with their respective Rodrigues' formulae.

By construction, the ladder operator representation gives identical answers to the Rodrigues formula. Indeed, by repeated application of the raising operator on $y_0(z)$, we get a so-called *operational formula* for $y_n(z)$, that is

$$y_n(z) = \left(\prod_{k=0}^{n-1}(\gamma_k^{+})^{-1}L_k^{+}\right)\cdot y_0(z) \tag{31}$$

By choosing (see [3] and [11]) $y_0(z) \equiv 1$, we must therefore expect to find

$$y_n(z) = \left(\prod_{k=0}^{n-1}(\gamma_k^{+})^{-1}L_k^{+}\right)\cdot 1 = \frac{1}{K_n w(z)}\left(\frac{d}{dz}\right)^{n}[w(z)p^{n}(z)] \tag{32}$$

To confirm this, we may proceed by mathematical induction on n. First, for $n = 0$, we have $y_0(z) \equiv 1$ in both cases, as $K_0 \equiv 1$ by *convention* (see [3] and [11] again). Next, we assume the result (31) is true for arbitrary $n > 0$. Finally, we have

$$y_{n+1}(z) = \left( \prod_{k=0}^{n} (\gamma_k^+)^{-1} L_k^+ \right) \cdot 1 = [(\gamma_k^+)^{-1} L_k^+] \left( \prod_{k=0}^{n-1} (\gamma_k^+)^{-1} L_k^+ \right) \cdot 1 \tag{33}$$

or, by (32) and the induction assumption

$$y_{n+1}(z) = \left( \prod_{k=0}^{n} (\gamma_k^+)^{-1} L_k^+ \right) \cdot 1 = [(\gamma_k^+)^{-1} L_k^+] \cdot \frac{1}{K_n w(z)} \left( \frac{d}{dz} \right)^n [w(z) p^n(z)] \tag{34}$$

However, from (13), (22) and (10), we see that

$$[(\gamma_k^+)^{-1} L_k^+] \cdot \frac{1}{K_n w(z)} \left( \frac{d}{dz} \right)^n [w(z) p^n(z)] = \frac{1}{K_{n+1} w(z)} \left( \frac{d}{dz} \right)^n [w(z) p^{n+1}(z)] \tag{35}$$

and, from (33) and (34), the assertion holds true.

## 4. Examples

It is not a difficult task to apply the general formulae we have presented above to any specific class of classical orthogonal polynomials. We consider in detail the generation of the first few terms of the Jacobi polynomials, while presenting similar results for the Legendre, Laguerre and Hermite polynomials in tabular form in Table 2. The Jacobi polynomials arise as solutions [3] of the Jacobi equation

$$(1 - z^2) P_n^{(\alpha,\beta)''} + [\beta - \alpha - (\alpha + \beta + 2)z] P_n^{(\alpha,\beta)'} + n(n + \alpha + \beta + 1) P_n^{(\alpha,\beta)} = 0 \tag{36}$$

With the normalization encapsulated in

$$K_n = (-1)^n 2^n n! \tag{37}$$

we obtain the recurrence relation generating the Jacobi polynomials, via (36), (20), (22), and (31), as

$$P_n^{(\alpha,\beta)}(z) = \left(\prod_{k=0}^{n-1} \frac{(\alpha+\beta+2k+2)(z^2-1)}{2(k+1)(\alpha+\beta+k+1)} \frac{d}{dz} - \frac{(\beta-\alpha-(\alpha+\beta+2k+2)z)}{2(k+1)}\right) \cdot P_0^{(\alpha,\beta)}(z)$$

---------- (38)

Choosing [3]

$$P_0^{(\alpha,\beta)}(z) = 1 \qquad (39a)$$

we see from (38) that

$$P_1^{(\alpha,\beta)}(z) = \frac{\alpha-\beta+(\alpha+\beta+2)z}{2} \qquad (39b)$$

and

$$P_2^{(\alpha,\beta)}(z) = \frac{(\alpha+\beta+3)(\alpha+\beta+4)z^2 + 2(\alpha-\beta)(\alpha+\beta+3)z + \alpha^2 + \beta^2 - \alpha(2\beta+1) - \beta - 4}{2^3}$$

---------- (39c)

and so on and so forth. We recognise $P_0^{(\alpha,\beta)}(z)$, $P_0^{(\alpha,\beta)}(z)$ and $P_2^{(\alpha,\beta)}(z)$ as the first three terms of the sequence of Jacobi polynomials generated by (38) and (39a). Naturally, the Rodrigues' formula (3), with the appropriate choice of $K_n$, $w(z)$ and $p(z)$ develops the identical sequence of Jacobi polynomials also, as may be checked from the actual formula, that is, from [3]

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\left(\frac{d}{dz}\right)^n [(1-z)^{\alpha+n}(1+z)^{\beta+n}] \qquad (40)$$

In Table 2 we summarize the above procedure for the Legendre, Laguerre and Hermite polynomials. As 'basic data' for the construction of Table 2, we have the following. For the Legendre polynomials we have the equation and normalization

$$(1-z^2)P_n'' - 2zP_n' + n(n+1)P_n = 0, \quad K_n = (-1)^n 2^n n! \qquad (41)$$

while for the Laguerre polynomials and Hermite polynomials we have, respectively

$$zL_n'' + (1-z)L_n' + nL_n = 0, \qquad K_n = n! \tag{42}$$

and

$$zH_n'' - 2zH_n' + 2nH_n = 0, \qquad K_n = (-1)^n \tag{43}$$

Finally, in all three cases, Legendre, Laguerre and Hermite polynomials, we assume that $y_0(z) = 1$. Again, the Rodrigues' formula (3), with the appropriate choice of $K_n$, $w(z)$ and $p(z)$ develops the identical sequence of Legendre, Laguerre and Hermite polynomials also. This may be checked using the explicit Rodrigues' formulae from Table 1.

Of course, the analysis presented here may be applied to the other examples of classical orthogonal polynomials also (see references [3] and [11] for their detailed description).The basic process, summarized by (20) and (22) (essentially) defines a generic infinite sequence for the general classical orthogonal polynomial solutions of equation (1). As the process is *iterative*, it may be implemented on a computer and as many or as few terms in the sequence of polynomials 'turned-out' as is required. The same is equally true of the recurrence relation (25), given $y_0(z)$ and $y_1(z)$. In this case we require the coefficients $a_n, b_n$ and $c_n$, but this is a straightforward determination as mentioned already. Also, it is not difficult to show that

$$\alpha_n^- = \alpha_n^+ - \gamma_n^+ a_n \ , \ \beta_n^- = \beta_n^+ - \gamma_n^+ \ b_n \ , \ \gamma_n^- = \gamma_n^+ \ c_n \tag{44}$$

from which the coefficients $a_n, b_n$ and $c_n$ may be found by basic arithmetic.

## 5. Summary and Discussion

We have derived 'ladder-operator' differential recursive formulae – and an associated operational formula – for the solutions of (1), by deducing the recurrence relations (10), (11) and (12) directly from the Rodrigues formula (2), using the Pearson equation (4) and elementary mathematics only. The ladder-operators (13) and (14), following from the recurrence relations (10) and (12), lead then to a Schrödinger factorization for (1). In Tables 1 and 2, a short collection of some of the results of this analysis is presented.

The derivations in section 2, inspired (mostly) by the methodology of references [1] and [4], constitute a 'first-principles' deduction of the recurrence relations (10), (11) and (12) for the classical orthogonal polynomials (3) and so for the Schrödinger ladder-operators (13) and (14) for equation (1) also. Note, however, that the presentations of references [1] and [4] are based on a *different starting formula* than (3) so that their derivations differ *in detail* from those of section 2, as do the specific form of their final results.

The recurrence formulae of section 2 are well and widely known. However, their derivation through the manipulation of the general Rodrigues' formula (3) is not of such a wide currency and previous essays in this direction differ from the approach of section 2 either in starting point, as in references [1] and [4], or else in scope of application. So, on the one hand, the recurrence relations of section 2 have been obtained from the Rodrigues formula solution of (1) by Nikiforov and Uvarov [11] and also, in a development of the work of Nikiforov and Uvarov [11], by Yanez, Dehesa and Nikiforov [14]. However, Nikiforov and Uvarov [11] and Yanez et al [14] do not derive the recurrence relations from the Rodrigues formula (1.3) *directly*, but through an analysis, via contour integration, of a contour integral representation of (3). Similarly, Erdelyi et al [3] quote their equivalent to (10), (11) and (12) (with 'their' (10) *implicit* in the discussion). However, their derivation differs in starting point (and in details) also. On the other hand, parallel to the method developed in section 2 is that presented by Van Iseghem [13]. Van Iseghem *does* manipulate (3) directly, but derives equations (6) and (7) *only*, and so produces *solely* the 'raising' differential recurrence relation, that is, equation (10). (Van Iseghem [13] does derive her equivalent of (11), but again from the viewpoint of Nikiforov and Uvarov [11].)

In addition to these differences between the work in section 2, and that of references [3], [11], [13] and [14], we note that, the coefficients in the recurrence formulae of section 2 are given in terms of the coefficients of the original second-order linear differential equation, (1), *directly*, thus making applying these recurrence formulae, to any particular instance of a differential equation, simpler. None of the above quoted authors relate their recurrence relations to the factorization of equation (1). So, we move on and consider previous work associated with the development of ladder-operators along the lines of section 3. Again, such formulae as (20), (21) or (25) are well known, but, once more, essays in developing formulae such as (20), (21) or (25) differ from the approach of section 3 either in starting point or in scope of application. For example, Lorente [8] derives a 'Schrödinger operator factorization' for, and solutions to (orthogonal polynomials) equation (1), through the application of recurrence relations derived from its Rodrigues' formula by Nikiforov and Uvarov [11] and so (implicitly) Lorente has a different starting point from that of section

3. Actually, Lorente [8] produces a factorized form of an equation related to (1). As a second example, we consider Kaufman [6], who also started with 'ladder operator relations' and proceeded to construct second-order ordinary differential equations in the manner of Lorente. Kaufman, however, considered only special cases of ladder operators [6], which he considered as *given*. In the Rodrigues' formula procedure, ladder operators are derived for the *general* Rodrigues' formula, (3) (or see Lorente [8] again).

Another ladder-operator methodology, and one which is closest to that presented in section 3, is that of Jafarizadeh and Fakhri [5]. In fact, the present work ends-up, as reported in section 3, as a *development* of the factorization methodology of Jafarizadeh and Fakhri [5]. However, Jafarizadeh and Fakhri [5] start, not with the Rodrigues' formula (3), but with the assumed factorizations (18) and (19) of equation (1). In addition, Jafarizadeh and Fakhri [5] do *not* assume the specific forms (20) and (21) for the 'raising' and 'lowering' operations but utilize, instead, the 'composite' format of

$$p(z)\frac{d\hat{y}_n(z)}{dz} + (\alpha_n^+ z + \beta_n^+)\hat{y}_n(z) = \hat{y}_{n+1}(z) \tag{45}$$

and

$$p(z)\frac{d\hat{y}_n(z)}{dz} + (\alpha_n^- z + \beta_n^-)\hat{y}_n(z) = E_n\hat{y}_{n-1}(z) \tag{46}$$

and, to determine the unknown coefficients $\alpha_n^{\pm}, \beta_n^{\pm}$ and $E_n$, require the further assumption that $E_0 = 0$ [5]. The use, in section 3, of the direct approach of section 2 produces the factorized form $E_n = \gamma_{n-1}^+\gamma_n^-$, from which, as $\gamma_0^- = 0$, we get $E_0 = 0$ *identically*, as a result. Further, the operational formula (31) *depends* on the different approach of section 3. Apparently, by comparison, the relationship between the approach of Jafarizadeh and Fakhri [5] and the present approach, is determined through

$$y_n(z) = \left( \prod_{k=1}^{n} (\gamma_k^{+}) \right)^{-1} \hat{y}_n(z), \quad n \geq 1 \tag{47}$$

provided we agree that $y_0(z) = \hat{y}_0(z) = 1$.

We round-off our discussion with a few general points of interest. First, we note that the Schrödinger factorizations in addition to being well-known in the theory of Sturm-Liouville eigenvalue problems and intimately related to the concepts of supersymmetry and 'shape invariance' [5], are basic to the theory of special functions [10]. Indeed the factorization approach to (1) lends itself to further developments, especially where the task of linking generating functions and recurrence relations is concerned. In particular, by making use of recurrence relations, we can utilise existing methodology (see [9] and [12]) to produce a multitude of generating functions, which are known to be of great utility in applications. Finally, we note that we have restricted ourselves to the basic classical orthogonal polynomials. The methodology for producing 'raising' and 'lowering' operators and factorized forms can be extended to a consideration of the *associated* polynomials and their defining differential equation, although the formulae are somewhat more involved.

## References

[1] Beale F.S., Annals of Math. Stats. **12** (1941) 97.

[2] Chihara T. S., *An Introduction to Orthogonal Polynomials* (Gordon and Breach) 1978.

[3] Erdelyi A., Magnus W., Oberhettinger F. and Tricomi F. G., *Higher Transcendental Functions* Volume II (McGraw-Hill) 1953.

[4] Hildebrandt E.H., Annals of Math. Stats. **2** (1931) 379.

[5] Jafarizadeh M. A. and Fakhri H., Physics Letters A **230** (1997) 164; Ann. Phys. (N.Y.) **262** (1998) 260.

[6] Kaufman B., J. Math. Phys. **7** (1966) 447.

[7] Law A. G. and Sledd M. B., Proc. Am. math. Soc. **48** (1975) 505.

[8] Lorente M., J. Phys. A: Math. Gen. **34** (2001) 569.

[9] McBride E. B., *Obtaining Generating Functions* (Springer-Verlag) 1971.

[10] Miller W., *Lie Theory and Special Functions* (Academic Press) 1968.

[11] Nikiforov A. F. and Uvarov V. B., *Special Functions of Mathematical Physics* (Birkhauser) 1988.

[12] Truesdell C., *An Essay Toward a Unified Theory of Special Functions* (Princeton University Press) 1948.

[13] Van Iseghem J., ScientificCommons (1995)

[14] Yanez R. J., Dehesa J. S. and Nikiforov A. F., J. Math Anal. Appl, **188** (1994) 855.

| Equation | Equation Form | Factorized Form | Rodrigues' Formula |
|---|---|---|---|
| Legendre | $(1-z^2)P''_n - 2zP'_n + n(n+1)P_n = 0$ | $[(1-z^2)\frac{d}{dz} - nz][(1-z^2)\frac{d}{dz} + nz]P_n = -n^2 P_n$ | $P_n = \frac{(-1)^n}{2^n n!}\left(\frac{d}{dz}\right)^n [(1-z^2)^n]$ |
| Hermite | $H''_n - 2zH'_n + 2nH_n = 0$ | $(\frac{d}{dz} - z)(\frac{d}{dz})H_n = -2nH_n$ | $H_n = (-1)^n e^{z^2}\left(\frac{d}{dz}\right)^n [e^{-z^2}]$ |
| Laguerre | $zL''_n + (1-z)L'_n + nL_n = 0$ | $(z\frac{d}{dz} + n - z)(z\frac{d}{dz} - n)L_n = -n^2 L_n$ | $L_n = \frac{1}{n!} e^x \left(\frac{d}{dz}\right)^n [z^n e^{-z}]$ |

**Table 1. Important Equations, Factorized with their Rodrigues' Formula [3]**

| Term | Legendre Recurrence | Laguerre Recurrence | Hermite Recurrence |
|---|---|---|---|
| n | $P_n(z) = \left( \prod_{k=0}^{n-1} -\frac{(1-z^2)\frac{d}{dz} - (k+1)z}{k+1} \right) \cdot 1$ | $L_n(z) = \left( \prod_{k=0}^{n-1} \frac{zL'_k + (k+1-z)L_k}{k+1} \right) \cdot 1$ | $H_n(z) = \left( \prod_{k=0}^{n-1} -(H'_k(z) - 2zH_k) \right) \cdot 1$ |
| 0 | 1 | 1 | 1 |
| 1 | $z$ | $1 - z$ | $2z$ |
| 2 | $\frac{3z^2 - 1}{2}$ | $\frac{z^2 - 4z + 2}{2}$ | $4z^2 - 2$ |
| 3 | $\frac{5z^3 - 3z}{2}$ | $\frac{-z^3 + 9z^2 - 18z + 6}{6}$ | $8z^3 - 12z$ |
| 4 | $\frac{35z^4 - 30z^2 + 3}{8}$ | $\frac{z^4 - 16z^3 + 72z^2 - 96z + 24}{24}$ | $16z^4 - 48z^2 + 12$ |

**Table 2. Recurrence Relations and the First Few Terms of their Sequences**

**Received: April, 2012**