

RODRIGUES' FORMULAS FOR JACOBSTHAL-TYPE POLYNOMIALS

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1. INTRODUCTION

Motivation

Recently [2], some second-order differential properties of generalized Fibonacci polynomials and generalized Lucas polynomials were exhibited.

Here, we intend to

- (i) obtain similar differential equations from a slightly different viewpoint in the more general context of the polynomials $W_n(x)$ and ${}^{\circ}W_n(x)$ [3], and
- (ii) discover analogous equations for Jacobsthal polynomials $J_n(x)$ and Jacobsthal-Lucas polynomials $j_n(x)$ [4], i.e., non-Fibonacci and non-Lucas polynomials.

Central to the process is the question:

Can we determine Rodrigues' formulas for $J_n(x)$ and $j_n(x)$ corresponding to those (in a somewhat different notation) for $U_n(x)$ and $V_n(x)$ in [2]?

Background

Essentially, the following basic material [3] is needed:

$$W_{n+2}(x) = p(x)W_{n+1}(x) + q(x)W_n(x), \quad W_0(x) = 0, W_1(x) = 1, \quad (1.1)$$

$${}^{\circ}W_{n+2}(x) = p(x){}^{\circ}W_{n+1}(x) + q(x){}^{\circ}W_n(x), \quad {}^{\circ}W_0(x) = 2, {}^{\circ}W_1(x) = p(x), \quad (1.2)$$

leading to (if we drop the functional notation)

$$W_n = \frac{\alpha^n - \beta^n}{\Delta}, \quad (1.3)$$

$${}^{\circ}W_n = \alpha^n + \beta^n, \quad (1.4)$$

where

$$\left. \begin{aligned} \alpha &= \frac{1}{2}\{p + \Delta\}, \\ \beta &= \frac{1}{2}\{p - \Delta\}, \\ \Delta &= \sqrt{p^2 + 4q} = \alpha - \beta. \end{aligned} \right\} \quad (1.5)$$

Differentiating once w.r.t. x gives

$$\Delta' = \frac{pp' + 2q'}{\Delta}. \quad (1.6)$$

Specialized cases of (1.1) and (1.2) are generalized the Fibonacci and Lucas polynomials $F_n = W_n$ and $L_n = {}^{\circ}W_n$, for which $p = x, q = 1$, and the Jacobsthal and Jacobsthal-Lucas polynomials J_n and j_n , for which $p = 1, q = 2x$. (See [3] for other examples of "Fibonacci-type" polynomials, e.g., Pell, Chebyshev, and Fermat.)

Two dichotomous situations thus arise:

- A. $q' = 0$ for "Fibonacci-type" polynomials like F_n and L_n ;
- B. $p' = 0$ for J_n and j_n .

Immediately from (1.6) we have

$$\Delta' = \begin{cases} \frac{pp'}{\Delta}, & (1.6A) \\ \frac{2q'}{\Delta}. & (1.6B) \end{cases}$$

Crucial to the theory are the derivatives [3]

$${}^{\circ}W'_n = \begin{cases} np'W_n & (q' = 0), \\ nq'W_{n-1} & (p' = 0), \end{cases} \quad (1.7)$$

so, in particular,

$$j'_n = 2nJ_{n-1}. \quad (1.8)$$

Finally, we record for later use the notation [2]

$$c_{n,0} = 2 \frac{n!}{(2n)!} \quad (n \geq 0), \quad (1.9)$$

and

$$c_{n,r} = 2 \frac{n!n(n+r)!}{(2n)!(n+r)(n-r)!} \quad (n \geq r \geq 1), \quad (1.10)$$

whence

$$c_{n,r+1} = (n^2 - r^2)c_{n,r} \quad (n \geq r+1 \geq 1). \quad (1.11)$$

Notation for Theorems: Letters F and $J(j)$ will be appended as subscripts to the Theorem number of theorems relating to Fibonacci-type polynomials and Jacobsthal-type polynomials, respectively. In this symbolism, we will have Theorem $1_F, \dots, \text{Theorem } 3_J$.

2. SOME BASIC DIFFERENTIAL EQUATIONS FOR RECURRENCES

A. Fibonacci-type Polynomials ($q' = 0$)

From (1.3)-(1.7), double differentiation of ${}^{\circ}W_n$ leads to

$$\Delta^2 {}^{\circ}W''_n = n^2 (p')^2 {}^{\circ}W_n - np(p')^2 W_n$$

whence, with ${}^{\circ}W_n = y$,

$$\Delta^2 y'' + pp'y' - (np')^2 y = 0. \quad (2.1)$$

Alternatively, if we follow the procedure in [2], while using our notation, then we arrive at (2.1) also, a process left to the reader.

Differentiating (2.1) r times in conjunction with Leibniz' rule, we deduce that $z = y^{(r)} = {}^{\circ}W_n^{(r)}$ satisfies the differential equation

$$\Delta^2 z'' + (2r + 1)pp'z' + (p')^2(r^2 - n^2)z = 0, \tag{2.2}$$

of which (2.1) is the special case when $r = 0$.

Illustrations of (2.1) are:

- (i) the associated Morgan-Voyce polynomial $C_n = y$, for which $p = 2 + x, q = -1$, leading to [2]

$$x(x + 4)y'' + (x + 2)y' - n^2y = 0;$$

- (ii) the Chebyshev polynomial $T_n = y$, in which $p = 2x, q = -1$ ($x = \cos\theta$), yielding

$$(1 - x^2)y'' - xy' + n^2y = 0,$$

in conformity with [6, p. 260].

Starting now with the double differentiation of W_n in (1.3), we eventually arrive at the differential equation

$$\Delta^2 W_n'' + 3pp'W_n' - (p')^2(n^2 - 1)W_n = 0. \tag{2.3}$$

Compare this with (2.1). A quick check confirms that $r = 1$ in (2.2) does indeed give us (2.3), where we invoke (1.7) for $q' = 0$. Particular instances of (2.3) are

- (a) the Morgan-Voyce polynomial B_n , for which $p = 2 + x, q = -1$, giving

$$x(x + 4)B_n'' - 3(x + 2)B_n' - (n^2 - 1)B_n = 0,$$

in conformity with [2, p. 455] on making the transformation $n \rightarrow n - 1$ for our B_n ;

- (b) The Chebyshev polynomial S_n (in the notation of [2, p. 453]), where $p = 2x, q = -1$ ($x = \cos\theta$), for which

$$(1 - x^2)S_n'' - 3xS_n' + (n^2 - 1)S_n = 0$$

as in [6, p. 260], n being replaced by $n - 1$ for our S_n .

Now (1.7), where $q' = 0$, immediately shows that ${}^oW_n^{(r)} = np'W_n^{(r-1)}$ ($r \geq 1$), i.e.,

$$W_n^{(r-1)} = \frac{1}{np'} {}^oW_n^{(r)}. \tag{2.4}$$

Hence, $W_n^{(r-1)}$ satisfies (2.2). Combining this with (2.2), we deduce that

Theorem 1_F: $W_n^{(r-1)}$ and ${}^oW_n^{(r)}$ both satisfy (2.2).

Example ($r = 2, n = 4; p = 2x, q = 1$, Pell-type polynomials [3]): $P_4^{(1)} = (8x^3 + 4x)'$ and $Q_4^{(2)} = (16x^4 + 16x^2 + 2)''$ both satisfy

$$(x^2 + 1)z'' + 5xz' - 12z = 0.$$

Observe that (2.2) can be cast in the more general form (cf. [2]):

$$[\Delta^{2r+1}z']' = (p')^2(n^2 - r^2)\Delta^{2r-1}z. \tag{2.5}$$

Following the technique in [2] and using (2.5), we may establish the results corresponding to equations (2.9)-(2.11) in [2], namely (with $D^{(r)} \equiv \frac{d^r}{dx^r}$):

$$D[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}] = (p')^2(n^2 - r^2)\Delta^{2r-1}D^{(n+r-1)}\Delta^{2n-1}, \tag{2.6}$$

$$D[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}] = (p')^2(n^2 - (r+1)^2)\Delta^{-2r-3}D^{(n-r-2)}\Delta^{2n-1}, \tag{2.7}$$

$$D[\Delta D^{(n+1)}\Delta^{2n+1}] = (p')^2(n+1)^2\Delta^{-1}D^{(n)}\Delta^{2n+1}. \tag{2.8}$$

B. Jacobsthal (\equiv non-Fibonacci)-type Polynomials ($p' = 0$)

Trying to apply the method used in [2], or variations of it, to J_n and j_n is likely to lead to frustration.

Therefore, we abandon this approach and start afresh.

Differentiate twice in the pivotal relation (1.7) for $p' = 0$. Then

$$\Delta^2 W_n'' + (q')^2 W_n' - n(n-1)(q')^2 W_{n-2} = 0, \tag{2.9}$$

wherein the diminished subscript in the undifferentiated polynomial is particularly to be noted. [Check (2.9) when, for example, $j_4 = 8x^2 + 8x + 1$, $j_6 = 16x^3 + 36x^2 + 12x + 1$, for which $p = 1$, $q = 2x$, $\Delta^2 = 1 + 8x$.]

Continued differentiation with recourse to Leibniz' rule, as in [2], reveals the generalized form of (2.9) to be ($z_n = W_n^{(r)}$)

$$\Delta^2 z_n'' + (4r + q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.10}$$

Putting $r = 0$ in (2.10) obviously leads us back to (2.9).

Repeated differentiation in (1.3) next yields, with little difficulty,

$$\Delta^2 W_n'' + 3(q')^2 W_n' - n(n-1)(q')^2 W_{n-2} = 0. \tag{2.11}$$

Contrast this with (2.3). One may readily verify (2.11) for, say, $J_5 = 4x^2 + 6x + 1$, $J_7 = 8x^3 + 24x^2 + 10x + 1$.

Proceeding for the sake of interest to differentiate (2.11) many times, we eventually arrive at the generalization ($z_n = W_n^{(r-1)}$)

$$\Delta^2 z_n'' + (4r + q')q'z_n' - n(n-1)(q')^2 z_{n-2} = 0. \tag{2.12}$$

Substituting $r = 1$ clearly reproduces (2.11), since $q' = 2$.

Bearing in mind (1.7) with $p' = 0$ and (2.12), we conclude that

Theorem 1_J: $J_n^{(r-1)}$ and $j_n^{(r)}$ both satisfy (2.10).

Analogously to (2.5), we see that (2.10) may be reformulated as

$$[\Delta^{2r+1}z_n']' = (q')^2 n(n-1)\Delta^{2r-1}z_{n-2}.$$

Corresponding to (2.6)-(2.8), we derive

$$D[\Delta^{2r+1}D^{(n+r)}\Delta^{2n-1}] = (q')^2 n(n-1)\Delta^{2r-1}D^{(n+r-3)}\Delta^{2n-1}, \tag{2.13}$$

$$D[\Delta^{-2r-1}D^{(n-r-1)}\Delta^{2n-1}] = (q')^2 n(n-1)\Delta^{-2r-3}D^{(n-r-4)}\Delta^{2n-1}, \tag{2.14}$$

$$D[\Delta D^{(n+1)}\Delta^{2n+1}] = (q')^2 n(n+1)\Delta^{-1}D^{(n-2)}\Delta^{2n+1}. \tag{2.15}$$

3. RODRIGUES' FORMULAS

Rodrigues' formulas for $W_n, {}^{\circ}W_n$ (when $q' = 0$) and for J_n, j_n (when $p' = 0$) are now determined.

A. Case $q' = 0$.

Procedures followed in [2] using (1.9) will largely be applied here.

Theorem 2_F:

$$(i) \quad W_n = \frac{nc_{n,0}}{(p')^{n-1}} \Delta^{-1} D^{(n-1)} \Delta^{2n-1},$$

$$(ii) \quad {}^{\circ}W_n = \frac{c_{n,0}}{(p')^n} \Delta D^{(n)} \Delta^{2n-1}.$$

Proof: Definitions (1.3) and (1.4) disclose that

$${}^{\circ}W_{n+1} = \frac{1}{2} [p {}^{\circ}W_n + \Delta^2 W_n]. \tag{3.1}$$

Assuming (i), (ii) in Theorem 2_F, we then have, on simplifying,

$${}^{\circ}W_{n+1} = \frac{n! \Delta}{(2n)!(p')^n} [p D^{(n)} \Delta^{2n-1} + np' D^{(n-1)} \Delta^{2n-1}]. \tag{3.2}$$

But, by Leibniz' rule,

$$\begin{aligned} D^{(n+1)} \Delta^{2n+1} &= D^{(n)} \{ (2n+1) p p' \Delta^{2n-1} \} \\ &= (2n+1) p' \{ p D^{(n)} \Delta^{2n-1} + n p' D^{(n-1)} \Delta^{2n-1} \}, \end{aligned} \tag{3.3}$$

since $p'' = 0$. Accordingly, (3.2), (3.3) yield

$${}^{\circ}W_{n+1} = \frac{2(n+1)! \Delta}{(2n+2)!(p')^{n+1}} D^{(n+1)} \Delta^{2n+1}$$

in conformity with Theorem 2_F(ii) and (1.9).

Furthermore, from (1.7),

$$\begin{aligned} W_{n+1} &= \frac{1}{(n+1)p'} {}^{\circ}W_{n+1} \\ &= \frac{1}{(n+1)p'} \frac{c_{n+1,0}}{(p')^{n+1}} D(\Delta D^{(n+1)} \Delta^{2n+1}) \quad \text{by Theorem 2_F(ii)} \\ &= \frac{2(n+1)}{(p')^n} c_{n+1,0} \Delta^{-1} D^{(n)} \Delta^{2n+1} \quad \text{by (2.8)} \end{aligned}$$

in agreement with Theorem 2_F(i). Consequently, Theorem 2_F is completely proved.

Example (Chebyshev polynomials [3], $p = 2x, q = -1$):

$$W_5 = 16x^4 - 12x^2 + 1 \quad (= U_4 \text{ [5, p. 256]}),$$

$${}^{\circ}W_5 = 2(16x^5 - 20x^3 + 5x) \quad (= 2T_5 \text{ [5, p. 256]}).$$

See also [7, p. 755]. Be it noted that in [6] the Rodrigues formulas for Chebyshev polynomials are given in terms of Gamma functions.

More generally,

Theorem 3_F:

$$(i) \quad W_n^{(r)} = \frac{c_{n,r+1}}{n(p')^{n-2r-1}} \Delta^{-2r-1} D^{(n-r-1)} \Delta^{2n-1};$$

$$(ii) \quad {}^qW_n^{(r)} = \frac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1}.$$

Proof:

(i) Induction on r is employed. The Theorem is true for $r = 0$ [Theorem 2_F(i)] and may be verified for $r = 1, 2$. Assume it is true for $r = k$. Then

$$\begin{aligned} W_n^{(k+1)} &= \frac{c_{n,k+1}}{n(p')^{n-2k-1}} D[\Delta^{-2k-1} D^{(n-k-1)} \Delta^{2n-1}] \text{ by Theorem 3}_{F}(i) \\ &= \frac{c_{n,k+2}}{n(p')^{n-2(k+1)-1}} [\Delta^{-2(k+1)-1} D^{(n-(k+1)-1)} \Delta^{2n-1}] \text{ by (2.7)} \end{aligned}$$

as expected. Thus, the Theorem is true for $r = k + 1$. Hence, it is true for all r .

$$\begin{aligned} (ii) \quad {}^qW_n^{(r)} &= np' W_n^{(r-1)} \text{ by (1.7)} \\ &= np' \frac{c_{n,r}}{n(p')^{n-2r+1}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1} \text{ by Theorem 3}_{F}(i) \\ &= \frac{c_{n,r}}{(p')^{n-2r}} \Delta^{-2r+1} D^{(n-r)} \Delta^{2n-1} \end{aligned}$$

as desired. Thus, Theorem 3_F is completely established.

Examples:

Chebyshev: $W_5^{(1)} = 8x(8x^2 - 3);$

Fermat: ${}^qW_4^{(2)} = 36(27x^2 - 4).$ (Here, $p = 3x, q = -2$.)

B. Case $p' = 0$.

Efforts to exploit the techniques of the theory when $q' = 0$ to the related situation when $p' = 0$ are doomed to disappointment, due mainly to the differing natures of Δ' in (1.6A) and (1.6B). A fresh approach is therefore necessary.

Computations rapidly show that, since $\Delta' = 2q' / \Delta$ (1.6B),

$$\begin{aligned} D^{(1)} \Delta^{2n-1} &= (2n-1)(2q') \Delta^{2n-3}, \\ D^{(2)} \Delta^{2n-1} &= (2n-1)(2n-3)(2q')^2 \Delta^{2n-5}, \\ &\dots \\ D^{(n-1)} \Delta^{2n-1} &= (2n-1)(2n-3)(2n-5) \dots 3(2q')^{n-1} \Delta, \end{aligned} \tag{3.4}$$

whence

$$\begin{aligned}
 \binom{n}{1} \frac{c_{n,0}}{(q')^{n-1}} \Delta^{-1} D^{(n-1)} \Delta^{2n-1} &= \binom{n}{1}, \\
 \binom{n}{3} \frac{c_{n-2,0}}{(q')^{n-3}} \Delta^{-1} D^{(n-2)} \Delta^{2n-1} &= \binom{n}{3} \Delta^2, \\
 &\dots \\
 \left\{ \begin{aligned} \binom{n}{n-1} \frac{c_{2,0}}{(q')^1} \Delta^{-1} D^{(1)} \Delta^{2n-1} &= \binom{n}{n-1} \Delta^{n-2}, \quad n \text{ even,} \\ \binom{n}{n} \frac{c_{1,0}}{(q')^1} \Delta^{-1} D^{(1)} \Delta^{2n-1} &= \binom{n}{n} \Delta^{n-1}, \quad n \text{ odd.} \end{aligned} \right.
 \end{aligned} \tag{3.5}$$

Differentiating once more in (3.4) gives rise to

$$D^{(n)} \Delta^{2n-1} = (2n-1)(2n-3)(2n-5) \cdots 3 \cdot 1 (2q')^n \Delta^{-1}. \tag{3.6}$$

Initially

$$D^{(0)} \Delta^{2n-1} = \Delta^{2n-1}. \tag{3.7}$$

Reassembling the ideas in (3.4), (3.5), and (3.6), we arrive at

$$\begin{aligned}
 \binom{n}{0} \frac{c_{n,0}}{(q')^n} \Delta D^{(n)} \Delta^{2n-1} &= \binom{n}{0}, \\
 \binom{n}{2} \frac{c_{n-2,0}}{(q')^{n-2}} \Delta D^{(n-1)} \Delta^{2n-1} &= \binom{n}{2} \Delta^2, \\
 &\dots \\
 \left\{ \begin{aligned} \binom{n}{n} \frac{c_{2,0}}{(q')^0} \Delta D^{(1)} \Delta^{2n-1} &= \binom{n}{n} \Delta^n, \quad n \text{ even,} \\ \binom{n}{n-1} \frac{c_{1,0}}{(q')^0} \Delta D^{(1)} \Delta^{2n-1} &= \binom{n}{n-1} \Delta^{n-1}, \quad n \text{ odd.} \end{aligned} \right.
 \end{aligned} \tag{3.8}$$

Because the left-hand forms in (3.5) and (3.8) resemble the Rodrigues formulas in Theorem 2_F, we feel justified to appropriate to them the name of *Rodrigues-type* expressions.

Now, $p = 1$ and $q = 2x$ in (1.1), (1.3), and (1.5) indicate that

$$\begin{aligned}
 J_n &= \frac{(1+\Delta)^n - (1-\Delta)^n}{\Delta} \quad (\Delta^2 = 1+8x) \\
 &= \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} \Delta^{2k} = \frac{1}{2^{n-1}} \left[\binom{n}{1} + \binom{n}{3} \Delta^2 + \binom{n}{5} \Delta^4 + \cdots + \begin{cases} \binom{n}{n-1} \Delta^{n-2} \\ \binom{n}{n} \Delta^{n-1} \end{cases} \right] \begin{cases} n \text{ even,} \\ n \text{ odd,} \end{cases} \\
 &= \text{a sum of expressions of Rodrigues-type (3.5)}.
 \end{aligned} \tag{3.9}$$

Similarly, use of (1.2), (1.4), and (1.5) gives rise to

$$\begin{aligned}
 j_n &= (1+\Delta)^n + (1-\Delta)^n = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} \Delta^{2k} \\
 &= \frac{1}{2^{n-1}} \left[\binom{n}{0} + \binom{n}{2} \Delta^2 + \binom{n}{4} \Delta^4 + \cdots + \begin{cases} \binom{n}{n} \Delta^n \\ \binom{n}{n-1} \Delta^{n-1} \end{cases} \right] \begin{cases} n \text{ even,} \\ n \text{ odd,} \end{cases} \\
 &= \text{a sum of expressions of Rodrigues-type (3.8)}.
 \end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we then conclude that

Theorem 2_J: The Rodrigues formula analogs for J_n and j_n are given by (3.9) and (3.10).

Examples:

$$J_6 = \frac{1}{32} \left[\binom{6}{1} + \binom{6}{3}(1+8x) + \binom{6}{5}(1+8x)^2 \right] = 1 + 8x + 12x^2;$$

$$j_6 = \frac{1}{32} \left[\binom{6}{0} + \binom{6}{2}(1+8x) + \binom{6}{4}(1+8x)^2 + \binom{6}{6}(1+8x)^3 \right] = 1 + 12x + 36x^2 + 16x^3.$$

Our last major program is to generalize Theorem 2_J. Recall, first, that $j_n^{(1)} = 2nJ_{n-1}$ (1.8). Elementary calculations involving (1.3) and (1.4) for J_n and j_n quickly tell us that

$$J_n^{(1)} = \frac{4}{\Delta^2} \left(\frac{n}{2} j_{n-1} - J_n \right). \tag{3.11}$$

Subsequent differentiation reveals that

$$J_n^{(2)} = \frac{4}{\Delta^2} [n(n-1)J_{n-2} - 3J_n^{(1)}],$$

$$J_n^{(3)} = \frac{4}{\Delta^2} [n(n-1)J_{n-2}^{(1)} - 5J_n^{(2)}],$$

$$J_n^{(4)} = \frac{4}{\Delta^2} [n(n-1)J_{n-2}^{(2)} - 7J_n^{(3)}],$$

and so on, suggesting the proposition that

Theorem 3_J: $J_n^{(r)} = \frac{4}{\Delta^2} \{n(n-1)J_{n-2}^{(r-2)} - (2r-1)J_n^{(r-1)}\}$, $r \geq 2$.

Proof: Induction on r demonstrates the validity of this assertion.

Successive differentiations in (1.8) then establish that

Theorem 3_j: $j_n^{(r)} = 2nJ_{n-1}^{(r-1)} = \frac{8n}{\Delta^2} [(n-1)(n-2)J_{n-3}^{(r-3)} - (2r-3)J_{n-1}^{(r-2)}]$, $r \geq 3$.

Example of Theorems 3_J, 3_j ($r = 2, n = 9$):

$$J_9^{(2)} = \frac{12}{8x+1} [24J_7 - J_9^{(1)}] = 24(8x^2 + 20x + 5) = \frac{1}{20} j_{10}^{(3)}.$$

Observations

- (i) Summation procedures beginning with the definitions (1.3) and (1.4), and ending with (3.9) and (3.10), cannot be applied to the Fibonacci-type polynomials. This is because (3.9) and (3.10) are tied irrevocably to (3.5) and (3.8), both of which depend on $p' = 0$.
- (ii) Corresponding to (3.1) for Fibonacci-type polynomials, for J_n and j_n we may derive

$${}^qW_{n+1} = \Delta^2 W_n - q {}^qW_{n-1}. \tag{3.12}$$

Use of the Leibniz rule nexus in Theorem 2_F is impossible in the case of Jacobsthal-type polynomials J_n and j_n because of the diminished subscript for qW on the right-hand side.

(iii) In (3.11), where $r = 1$, the appearance of $\frac{n}{2} j_{n-1}$, which seems to break the pattern of the theorem, requires explanation. From (1.8),

$$\frac{n}{2} j_{n-1} = \frac{n}{2} \int_0^x \frac{d}{dx} j_{n-1} dx = \frac{n}{2} \cdot 2(n-1) \int_0^x j_{n-2} dx = n(n-1) J_{n-2}^{(-1)},$$

where integration is represented by the negative unit superscript. With this symbolism, the pattern in Theorem 3_J is valid for $r \geq 1$, and hence that in Theorem 3_J for $r \geq 2$.

4. ILLUSTRATION OF THEORY WHEN $n = 5$ (i.e., $2n - 1 = 9$)

Now

$$\begin{aligned} D^{(1)}\Delta^9 &= 9(pp' + 2q')\Delta^7, \text{ where } \Delta^2 = p^2 + 4q \text{ (1.5),} \\ D^{(2)}\Delta^9 &= 9\{7(pp' + 2q')^2\Delta^5 + (p')^2\Delta^7\}, \\ D^{(3)}\Delta^9 &= 9\{7[5(pp' + 2q')^3\Delta^3 + 3(p')^2(pp' + 2q')\Delta^5]\}, \\ D^{(4)}\Delta^9 &= 9 \cdot 7 \cdot 3[5(pp' + 2q')^4\Delta + 10(p')^2(pp' + 2q')^2\Delta^3 + (p')^4\Delta^5]. \end{aligned} \tag{I}$$

Therefore,

$$\frac{5\Delta^{-1}D^{(4)}\Delta^9}{9 \cdot 7 \cdot 5 \cdot 3 \cdot (p')^4} = \frac{1}{(p')^4} \left\{ \binom{5}{4} (pp' + 2q')^4 + \binom{5}{2} (p')^2 (pp' + 2q')^2 \Delta^2 + \binom{5}{0} (p')^4 \Delta^4 \right\}. \tag{I}$$

So, for $q' = 0$, on simplifying,

$$\begin{aligned} \frac{1}{2^4} (\text{R.H.S.}) &= p^4 + 3p^2q + q^2 = W_5, \\ &= \begin{cases} 16x^4 + 12x^2 + 1 & \text{for the Pell polynomial } P_5 [3]: p = 2x, q = 1, \\ (1 + 6x + 4x^2) & \text{for the Jacobsthal polynomial } J_5: p = 1, q = 2x. \end{cases} \end{aligned}$$

Differentiate (I) again to get

$$D^{(5)}\Delta^9 = 9 \cdot 7 \cdot 5 \cdot 3 \cdot \left[\frac{(pp' + 2q')^5}{\Delta} + 10(p')^2(pp' + 2q')^3\Delta^2 + 5(p')^4(pp' + 2q')\Delta^3 \right]. \tag{II}$$

Then

$$\frac{\Delta D^{(5)}\Delta^9}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 \cdot (p')^5} = \frac{1}{(p')^5} \left\{ \binom{5}{5} (pp' + 2q')^5 + \binom{5}{3} (p')^2 (pp' + 2q')^3 \Delta^2 + \binom{5}{1} (p')^4 (pp' + 2q') \Delta^4 \right\}, \tag{II}$$

whence, for $q' = 0$,

$$\begin{aligned} \frac{1}{2^4} (\text{R.H.S.}) &= p^5 + 5p^3q + 5pq^2 = {}^oW_5, \\ &= \begin{cases} 32x^5 + 40x^3 + 10x & \text{for Pell - Lucas polynomials [3],} \\ (1 + 10x + 20x^2) & \text{for Jacobsthal - Lucas polynomials.} \end{cases} \end{aligned}$$

On the other hand, when $p' = 0$, we obtain the results (3.4)-(3.8) and hence (3.9) and (3.10).

Notice, particularly, that the general expressions for W_5 and oW_5 above are valid for both Fibonacci-type and Jacobsthal-type polynomials, even though $q' = 0$.

This is because the binomial coefficients associated with the powers of Δ in (I') and (II') are the same as those in (3.9) and (3.10), since $\binom{n}{m} = \binom{n}{n-m}$.

Expressions for W_n and oW_n may be sighted in [5] in a notation slightly varied from that used here.

5. CONCLUSION

While the author of [2] evidently did not consider this theory as applying to Jacobsthal-type polynomials, one observes that if his numerical parameter q in [2, eqns. (1.1), (1.2)] is allowed to be functional $q(x) = -2x$ with accompanying change in his x and p , then J_n and j_n can be incorporated into his system. For example, his U_5 ([2, eqn. (1.12)] reduces to $1 + 6x + 4x^2 = J_5$.

So we come to our rest, having achieved the objectives (i) and (ii) in Section 1 which motivated our undertaking. Many facets of the work were revealed with others to be investigated. The unexpected complications in the patterns of behavior of J_n and j_n (and W_n and oW_n) have added zest to the hunt.

Questions: Does there exist a general formula for the coefficients of the Jacobsthal-type polynomials in terms of the Gamma function in the sense of [1, Table 22.3]? If so, is it attainable using the techniques of this paper? Can, further, the theory be extended to the situation when both $p(x)$ and $q(x)$ are linear polynomials?

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