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A divisibility property for a subgroup of Riordan matrices

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Abstract

We identify a subgroup of Riordan matrices whose entries share the well-known divisibility property displayed by the entries of the Pascal matrix. We also establish a one-to-one correspondence between the matrices of the subgroup and sets of weighted lattice walks. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is a well-known fact that given the Pascal triangular array, $(a_{nk})_{n\geqslant k\geqslant 0}=(\binom{n}{k})_{n\geqslant k\geqslant 0}$, and a prime p, that p divides a_{pk} for $k=1,2,\ldots,p-1$. In this paper, we generalize this result to a large set of important combinatorial triangular arrays. These triangular arrays form a subgroup of a group called the Riordan group. We describe the Riordan group sufficiently to keep this paper self-contained, but see [2–5] for many more examples and applications. The Riordan group is a set of infinite lower triangle matrices defined so that each matrix has columns generated as follows: The generating function for the elements of the first column (zeroth column) has the form $g(x)=1+g_1x+g_2x^2+\cdots$, and the generating function for the ith column has the form $g(x)[f(x)]^i, i\geqslant 1$, where $f(x)=x+f_2x^2+f_3x^3+\cdots$. The coefficients f_i and g_i are integers for all i. We often denote a Riordan matrix by $(g(x), f(x))$. The set of all Riordan matrices forms a group under matrix multiplication. See Section 3 for a brief description of the group properties.

In this paper we are concerned with a subgroup H of the Riordan group with the elements of H having the form $(xf'(x)/f(x),\ f(x))$. It is easy to verify that the Pascal triangle $(1/(1-x), x/(1-x))$ belongs to the subgroup H. Like the Pascal matrix, the

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entries of each matrix in the subgroup H exhibit a divisibility property defined as follows.

Definition. A subset of Riordan matrices is said to have the “divisibility property” if each matrix $(m_{nk})_{n,k\geqslant 0}$ of the subset satisfies the property that n divides $k \cdot m_{nk}$ whenever $0 < k < n$.

Example 1.1. As an illustration of this divisibility property, consider the element of the subgroup with $g(x) = 1/\sqrt{(1-2x-3x^2)}$, and $f(x) = (1-x-\sqrt{1-2x-3x^2})/2x$. The entries in the first eight rows and first eight columns are given by

$$\begin{bmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 3 & \mathbf{2} & 1 & & & & & \\ 7 & \mathbf{6} & \mathbf{3} & 1 & & & & \\ 19 & 16 & 10 & 4 & 1 & & & \\ 51 & \mathbf{45} & \mathbf{30} & \mathbf{15} & \mathbf{5} & 1 & & \\ 141 & 126 & 90 & 50 & 21 & 6 & 1 & \\ 393 & \mathbf{357} & \mathbf{266} & \mathbf{161} & \mathbf{77} & \mathbf{28} & \mathbf{7} & 1 \end{bmatrix}.$$

The rows and columns are numbered starting with 0, so that the first row is the zeroth row, the second is row 1, and so on. Now, observe that the bold-faced entries in the pth row are divisible by p, where p is a prime. In Section 2, we give more details and prove the divisibility property of the subgroup H. In Section 3, we prove that H is a subgroup of the Riordan group.

There is a very interesting connection between the elements of the subgroup H and certain weighted lattice walks. In Section 4, we describe this connection.

2. Divisibility property of the subgroup *H*

The following theorem establishes the divisibility property of the subgroup H.

Theorem 2.1. *Let* $M = (m_{nk})_{n,k\geqslant 0} = (xf'(x)/f(x), f(x))$. *Then* n *divides* $k \cdot m_{nk}$ *for all* $0 < k < n$.

Proof. The generating function for the kth column of M is given by

$$c_k(x) = \sum_{n\geqslant 0} m_{nk}x^n = xf'(x)[f(x)]^{k-1} = x\left(\frac{[f(x)]^k}{k}\right)'.$$

Therefore, $km_{nk}=$ coefficient of x^n in $x((f(x))^k)' =$ coefficient of x^n in $x(\sum_{n\geqslant 0} d_{nk}x^n)'$, where d_{nk} is the coefficient of x^n in $(f(x))^k$. Therefore, $km_{nk} = nd_{nk}$. □

Corollary 2.1. *If* p *is a prime, then* p *divides* m_{pk} *for* $0 < k < p$.

There is an interesting generalization of Theorem 2.1. We can establish a divisibility property for a larger subset of the Riordan group.

Theorem 2.2. *Let* $M = (xh(f(x))f'(x)/f(x), f(x)) = (m_{nk})_{n,k\geqslant 0}$, *where* $h(x)$ *is a polynomial of degree* l *with integer coefficients and constant term* 1. *Then* n *divides* $k(k+1)\dots(k+l)\cdot m_{nk}$ *for* $0<k<n-l$.

Proof. Let $h(x) = 1 + h_1 x + \cdots + h_l x^l$. Then

$$\begin{aligned} c_k(x) &= \sum_{n\geqslant 0} m_{nk}x^n = xh(f(x))f'(x)[f(x)]^{k-1} \\ &= x(f'(x)[f(x)]^{k-1} + h_1 f'(x)[f(x)]^k + \cdots + h_l f'(x)[f(x)]^{l+k-1}) \\ &= x\left(\frac{[f(x)]^k}{k} + \frac{h_1[f(x)]^{k+1}}{k+1} + \cdots + \frac{h_l[f(x)]^{k+l}}{k+l}\right)'. \end{aligned}$$

Therefore,

$$k(k+1)\dots(k+l)\sum_{n\geqslant 0} m_{nk}x^n = x\left(\sum_{n\geqslant 0} b_n(k,h_1,\dots,h_l)\cdot x^n\right)',$$

where $b_n(k,h_1,\dots,h_l)$ is an integer depending on $k,h_1,\dots,h_l$. Equating coefficients, we get

$$k(k+1)\dots(k+l)\cdot m_{nk} = n\cdot b_n(k,h_1,\dots,h_l). \quad \square$$

Corollary 2.2. *If* p *is a prime, then* p *divides* m_{pk} *for* $0<k<p-l$. *Example* 5.3 *in Section* 5 *is an illustration of Corollary* 2.2.

3. Riordan group and subgroup properties

Here, we give a brief description of the Riordan group. A more detailed description together with examples can be found in [3]. The set of all Riordan matrices defined in Section 1 forms a group under ordinary matrix multiplication $*$. The product is given by

$$(g(x), f(x)) * (h(x), l(x)) = (g(x)h(f(x)), l(f(x))).$$

The identity is $(1,x)$. The inverse of $(g(x), f(x))$ is $(1/g(\bar{f}), \bar{f})$, where $\bar{f}$ is the compositional inverse of f.

To see that the members of the group with the form $(xf'(x)/f(x), f(x))$ belong to a subgroup denoted by H, note the following:

(i) The identity

$$(1,x) = \left(\frac{x(x)'}{x}, x\right) \in H.$$

(ii) The product

$$\left(\frac{xf'(x)}{f(x)}, f(x)\right) * \left(\frac{xh'(x)}{h(x)}, h(x)\right) = \left(\frac{xf'(x)}{f(x)} \cdot \frac{f(x)h'(f(x))}{h(f(x))}, h(f(x))\right)$$

$$= \left(\frac{x(h(f(x)))'}{h(f(x))}, h(f(x))\right) \in H.$$

(iii) The inverse of $(xf'(x)/f(x), f(x))$ is

$$\left(\frac{1}{\bar{f}(x) \cdot \frac{f'(\bar{f}(x))}{f(\bar{f}(x))}}, \bar{f}(x)\right) = \left(x \cdot \frac{(\bar{f})'(x)}{\bar{f}(x)}, \bar{f}(x)\right) \in H.$$

4. Lattice walks and the divisibility property

In this section, we will show that certain weighted lattice walks lead to a Riordan matrix in the subgroup H. Recall that the matrices in H have the divisibility property defined in Section 1. Conversely, we show that a Riordan matrix in the subgroup H corresponds to a given set of lattice walks.

In general, consider a lattice walk that starts at the origin $(0,0)$ and ends at (n,k) and has the form

$$(0,0) \to (1,k_1) \to (2,k_2) \to \cdots \to (n-1,k_{n-1}) \to (n,k).$$

The step $(i,k_i) \to (i+1,k_{i+1})$ is assigned the weight $w_{k_{i+1}-k_i}$. The weight of a walk is the product of the weights of its steps. For example, the walk $(0,0) \to (1,1) \to (2,1) \to (3,2)$ has weight $w_1 w_0 w_1$, while the walk $(0,0) \to (1,-1) \to (2,-1) \to (3,-1) \to (4,-5)$ has weight $w_{-1} w_0 w_0 w_{-4}$.

In this work, the step weights that we consider are integers and satisfy $w_1 = 1$, and $w_{-i} = 0$ for $i \leqslant 2$. Let a_{nk} be the sum of the weights of all walks starting at $(0,0)$ and ending at (n,k). Also let b_{nk} be the sum of the weights of all walks from $(0,0)$ to (n,k) with each lattice point having positive second coordinate except the origin.

Example 4.1. Suppose $w_k = 1$ for all $k \leqslant 1$ and $w_k = 0$ for $k \geqslant 2$, then some values of a_{nk} are given by

$n \setminus k$	-5	-4	-3	-2	-1	0	1	2	3	4	5
0	0	0	0	0	0	1	0	0	0	0	0
1	1	1	1	1	1	1	1	0	0	0	0.
2	8	7	6	5	4	3	2	1	0	0	0
3	45	36	28	21	15	10	6	3	1	0	0
4	220	165	120	84	56	35	20	10	4	1	0

The right-half of this array $(a_{nk})_{n \geqslant k \geqslant 0}$ is the Riordan matrix with the divisibility property which corresponds to the set of lattice walks. In the representation $(g(x), f(x))$

of this Riordan matrix, we have from Theorem 4.1 that $f(x)$ is given by $f(x) = xa(f(x)) = x\cdot\sum_{k\geqslant -1} w_{-k}(f(x))^{k+1} = x/(1-f(x))$. So, we get $f(x)=(1-\sqrt{1-4x})/2$ and $g(x)=xf'(x)/f(x)=2x/(4x-1+\sqrt{1-4x})$.

The matrix $(b_{nk})_{n\geqslant k\geqslant 0}$ comes out as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & . \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & . \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & . \\ 0 & 5 & 5 & 3 & 1 & 0 & 0 & . \\ 0 & 14 & 14 & 9 & 4 & 1 & 0 & . \\ 0 & 42 & 42 & 28 & 14 & 5 & 1 & . \\ . & . & . & . & . & . & . & . \end{bmatrix}.$$

The entries of this matrix are the ballot numbers (see for example [6]), and $f(x) = \sum_{n\geqslant 0} b_{n1}x^n = xC(x)$, where $C(x) = 1+x+2x^2+5x^3+\cdots$ is the generating function for the Catalan numbers. Therefore, in the general case, $f(x)$ can be regarded as a generalized Catalan generating function and $(b_{nk})_{n\geqslant k\geqslant 0}$ contains generalized ballot numbers.

We now proceed to examine the general case with step weights w_{-i}, where $w_1 = 1, w_{-i} = 0$ for $i\leqslant -2$. We will use the Lagrange inversion formula as stated in [7].

Theorem 4.1. Let $a(x)=\sum_{k\geqslant -1} w_{-k}x^{k+1}$ *be the generating function for the weights,* $g(x)=\sum_{n\geqslant 0} a_{n0}x^n$ *the generating function for the zeroth column of* $(a_{nk})_{n\geqslant k\geqslant 0}$, *and* $f(x)=\sum_{n\geqslant 0} b_{n1}x^n$ *the generating function for column* 1 *of* $(b_{nk})_{n\geqslant k\geqslant 0}$. *Then*

(i) $\sum_{k\leqslant n} a_{nk}x^{n-k} = (a(x))^n$ *for* $n\geqslant 1$,
(ii) $\sum_{n\geqslant 0} a_{nk}x^n = \left(\sum_{n\geqslant 0} a_{n,k-1}x^n\right) f(x)$ *for* $k\geqslant 1$,
(iii) $\sum_{n\geqslant 0} b_{nk}x^n = (f(x))^k$ *for* $k\geqslant 1$,
(iv) $f(x) = xa(f(x))$,
(v) $b_{nk} = \frac{k}{n}a_{nk}$ *for* $n\geqslant k > 0$,
(vi) $g(x) = \frac{xf'(x)}{f(x)}$.

So, given the step weights, we obtain the Riordan matrix $(g(x), f(x))$ in the subgroup H from $f(x)=xa(f(x))$ and $g(x)=xf'(x)/f(x)$. Conversely, if we start with the Riordan matrix $(g(x), f(x))$, where $g(x) = xf'(x)/f(x)$, we produce the lattice walk step weights w_{-k} from $w_{-k} = [x^{k+1}]\{a(x)\}$ where $a(x)$ is given by $f(x) = xa(f(x))$.

Proof. (i): Note that

$$a_{1k} = w_k \quad \text{and} \quad a_{nk} = \sum_{l=-1}^{n-k-1} w_{-l}a_{n-1,l+k} = \sum_{l} w_{1-l}a_{n-1,l-1+k}.$$

Therefore, by induction on n, we get $\sum_k a_{n,n-k}x^k = (a(x))^n$.

So, $\sum_k a_{nk}x^{n-k} = (a(x))^n$.

(ii):

$$a_{nk} = \sum_{m=0}^{n} a_{m,\,k-1} \cdot [\text{sum of weights of walks from } (m,k-1) \textit{ to } (n,k)$$

$$\text{with each lattice point having second coordinate greater than } k-1 \text{ except } (m,k-1)]$$

$$= \sum_{m=0}^{n} a_{m,\,k-1} b_{n-m,1} = [x^n]\left\{\left(\sum_{l\geqslant 0} a_{l,k-1}x^l\right)\left(\sum_{l\geqslant 0} b_{l1}x^l\right)\right\}.$$

(iii): For $k\geqslant 2$, we have $b_{nk} = \sum_{m=0}^{n} b_{m,\,k-1}b_{n-m,1}$. Therefore, $\sum_{n\geqslant 0} b_{nk}x^n = f(x)\cdot \sum_{n\geqslant 0} b_{n,\,k-1}x^n$. Induction on k then gives (iii).

(iv): For $n\geqslant 1$, we have

$$[x^n]\{a(f(x))\} = w_0[x^n]\{f(x)\} + w_{-1}[x^n]\{(f(x))^2\} + \cdots + w_{1-n}[x^n]\{(f(x))^n\}$$
$$= w_0 b_{n1} + w_{-1}b_{n2} + \cdots + w_{1-n}b_{nn} = b_{n+1,1} = [x^{n+1}]\{f(x)\}.$$

(v): Applying the Lagrange inversion formula to (iv), and using (i) and (ii), we obtain

$$[x^n]\{(f(x))^k\} = \frac{1}{n}[x^{n-1}]\{kx^{k-1}\cdot (a(x))^n\} = \frac{k}{n}a_{nk}.$$

Therefore, $b_{nk} = (k/n)a_{nk}$.

(vi): From (v), $nb_{n1} = a_{n1}$. Then, using (ii), we get $\sum_{n\geqslant 1} nb_{n1}x^n = \sum_{n\geqslant 1} a_{n1}x^n = f(x)\cdot \sum_{n\geqslant 0} a_{n0}x^n$.

That is, $xf'(x) = f(x)g(x)$. □

Example 4.2. As a second example in this section, consider the lattice walks with step weights $w_{-1} = 1$, $w_0 = 1$, $w_1 = 1$, $w_{-i} = 0$ for all $i\leqslant 2$.

We have $f = x(1+f+f^2) = (1-x-\sqrt{1-2x-3x^2})/2x$, and $g = \frac{xf'}{f} = x(\ln f)' = 1/(\sqrt{1-2x-3x^2})$. The entries in the first eight columns and the first eight rows of the Riordan matrix are given in Example 1.1. Observe the divisibility property that for $0<k<n$, n divides $k\cdot m_{nk}$. In the first column $k=0$, and m_{nk} represents the sum of the weights of all walks from $(0,0)$ to $(n,0)$. The entry $m_{7,2} = 266$ is the sum of the weights of all paths of length 7 from $(0,0)$ to $(7,2)$. The first column contains the central trinomial coefficients. The weight of each walk is 1. Therefore, the central trinomial coefficients count the number of walks of length n from $(0,0)$ to $(n,0)$. The central trinomial coefficients also count the number of king walks down a chess board.

5. Further examples

In each of the following examples, the bold-faced entries in the pth row of the matrix are divisible by p.

Consider the lattice walks with step weights $w_1 = 1$, $w_0 = a$, $w_{-1} = b$, $w_{-i} = 0$ for all $i \geqslant 2$, where a and b are positive integers. For the Riordan matrix

$$f(x) = \frac{1 - ax - \sqrt{(a^2 - 4b)x^2 - 2ax + 1}}{2bx}$$

and

$$g(x) = \frac{xf'(x)}{f(x)} = \frac{1}{\sqrt{(a^2 - 4b)x^2 - 2ax + 1}}.$$

Example 5.1. With $a = 2$, $b = 1$, we obtain $(g(x), f(x)) = (1/(\sqrt{1-4x}), (1 - 2x - \sqrt{1-4x})/2x)$. The first eight rows of this matrix are given by

$$\begin{bmatrix} 1 \\ 2 & 1 \\ 6 & \mathbf{4} & 1 \\ 20 & \mathbf{15} & \mathbf{6} & 1 \\ 70 & 56 & 28 & 8 & 1 \\ 252 & \mathbf{210} & \mathbf{120} & \mathbf{45} & \mathbf{10} & 1 \\ 924 & 792 & 495 & 220 & 66 & 12 & 1 \\ 3432 & \mathbf{3003} & \mathbf{2002} & \mathbf{1001} & \mathbf{364} & \mathbf{91} & \mathbf{14} & 1 \end{bmatrix}.$$

Here $g(x)$ is the generating function for the central binomial coefficients. Therefore the total weight of all walks of length n from $(0,0)$ to $(n,0)$ is $\binom{2n}{n}$.

Example 5.2. With $a = 3$, $b = 2$, we get the lattice walks with step weights $w_{-1} = 2$, $w_0 = 3$, $w_1 = 1$, $w_i = 0$, otherwise. Then

$$(g(x), f(x)) = \left(\frac{1}{\sqrt{x^2 - 6x + 1}}, \frac{1 - 3x - \sqrt{x^2 - 6x + 1}}{4x} \right).$$

The entries in the first eight rows and eight columns of this matrix are given by

$$\begin{bmatrix} 1 \\ 3 & 1 \\ 13 & \mathbf{6} & 1 \\ 63 & \mathbf{33} & \mathbf{9} & 1 \\ 321 & 180 & 62 & 12 & 1 \\ 1683 & \mathbf{985} & \mathbf{390} & \mathbf{100} & \mathbf{15} & 1 \\ 8989 & 5418 & 2355 & 720 & 147 & 18 & 1 \\ 48639 & \mathbf{29953} & \mathbf{13923} & \mathbf{4809} & \mathbf{1197} & \mathbf{203} & \mathbf{21} & 1 \end{bmatrix}.$$

The numbers in the first column give the number of walks from $(0,0)$ to (n,n) using steps $\{(0,1),(1,0),(1,1)\}$. Details can be found in [1, p. 81].

Example 5.3. This example is an illustration of Theorem 2.2 and Corollary 2.2. Let $h(x)=1+2x+x^2$. Take $f(x)=(1-x-\sqrt{1-2x-3x^2})/2x$ as in Example 1.1. Then the entries in the first eight rows and first eight columns of $M=(xh(f(x))f'(x)/f(x),\ f(x))$ are given by

$$\begin{bmatrix} 1 &&&&&&& \\ 3 & 1 &&&&&& \\ 8 & 4 & 1 &&&&& \\ 22 & 13 & 5 & 1 &&&& \\ 61 & 40 & 19 & 6 & 1 &&& \\ 171 & \mathbf{120} & \mathbf{65} & 26 & 7 & 1 && \\ 483 & 356 & 211 & 98 & 34 & 8 & 1 & \\ 1373 & \mathbf{1050} & \mathbf{665} & \mathbf{343} & \mathbf{140} & 43 & 9 & 1 \end{bmatrix}.$$

The bold-faced entries in the pth row are divisible by p.

In the general case, we have an interpretation of $M=(xh(f(x))f'(x)/f(x),\ f(x))$ in terms of two-stage weighted lattice walks. For the first step of the walk, we use the weights $w_{-m}=h_{m+1}$ for $m=-1,0,\ldots,l-1$; $w_{-m}=0$, otherwise, where $h(x)=1+h_1x+\cdots+h_lx^l$. For all other steps we use the weights $w_{-k}=[x^{k+1}]\{a(x)\}$, for $k\geqslant -1$, where $a(x)$ is given by $f(x)=xa(f(x))$. Now if $\tilde{m}_{nk}=$ sum of the weights of all such two-stage lattice walks starting at $(0,0)$ and ending at (n,k), $n\geqslant k\geqslant 0$, then $(\tilde{m}_{ij})_{i,j\geqslant 0}=(G(x),f(x))$, where $G(x)f(x)/x=xh(f(x))f'(x)/f(x)$. In other words, the Riordan matrix $(xh(f(x))f'(x)/f(x),\ f(x))=(m_{nk})_{n,k\geqslant 0}$, can be obtained from $(\tilde{m}_{nk})_{n,k\geqslant 0}$ by deleting the zeroth column. This means that $m_{nk}=\tilde{m}_{n+1,k+1}=$ sum of the weights of the two stage lattice walks from $(0,0)$ to $(n+1,k+1)$.

In the example, for the first step of the walks we use step weights $w_{-1}=1$, $w_0=2$, $w_1=1$, $w_{-i}=0$, otherwise. For all other steps $w_{-1}=1$, $w_0=1$, $w_1=1$, $w_{-i}=0$, otherwise. The entry $m_{53}=26=$ sum of the weights of all walks from $(0,0)$ to $(6,4)$.

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