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Some algebraic structure of the Riordan group

Candice Jean-Louis, Asamoah Nkwanta *

Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, MD 21251, United States

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ABSTRACT

In this paper we highlight some of the algebraic properties of the Riordan group. We present some new properties and provide proofs for some known properties. For instance, proofs of properties of the power-Bell and derivative subgroups are presented. We also give a new property for generating stochastic Riordan matrices which are elements of the stochastic Riordan subgroup. Centralizers, stabilizers and certain Riordan group isomorphisms are also given and proved. We conclude by proving some properties connecting similar Riordan matrices and pseudo-involutions in the Riordan group.

1. Introduction

The focus of this paper is to highlight some of the algebraic properties of the Riordan group. There are many interesting papers on Riordan arrays that discuss algebraic properties. For instance, Merlini et al. [17] present the theory of implicit Riordan arrays and use the arrays to give explicit results for solving sums involving combinatorial inversions. He and Sprugnoli [12] show that the multiplication of Riordan arrays can be performed in terms of the formulation rules determining the entries of an array. The rules are called the A and Z sequences of a Riordan array. He [11] formulates a relationship between a pair of Laurent series and Riordan arrays. Cheon et al. [4] show that all pseudo-involutions

* Corresponding author. Tel.: +1 443 885 4652; fax: +1 443 885 8216.
E-mail addresses: candice.jean-louis@morgan.edu (C. Jean-Louis), asamoah.nkwanta@morgan.edu (A. Nkwanta).

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of the Bell subgroup of the Riordan group have corresponding Δ-sequences. The Δ-sequences are single sequences of real numbers that characterize the Bell subgroup. Luzon and Moron [15] show how Banach's fixed point theorem is used to iterate a contractive first degree polynomial induced by a certain complete metric in the ring of formal power series whose fixed point is $1/Q$ where Q is any quadratic polynomial $Q(x) = a + bx + cx^2$ where $a \neq 0$. Also, Luzon and Moron [16] show how self-inverse sequences of Sheffer polynomials are used to describe all involutions of the Riordan group.

Although much research has been done on the combinatorics of the Riordan group [3,18,19,21,28,31,32], few papers focus exclusively on the algebraic structure of the group. Some structural properties of the group are known and appear in the Riordan array literature. However, the proofs of some properties are either missing or not well documented. One purpose of this paper is to provide more details of proofs of some known properties. Another purpose is to present some new algebraic properties. By focusing more on the algebraic properties of the group, we hope to help fill some of the gaps on this topic and spark more interest on the structure of the group.

This paper is arranged as follows. The definition of a Riordan matrix and the Fundamental Theorem of the Riordan group are presented in Section 2. A few known properties of the Riordan group are covered in Section 3. In addition, we prove that the centralizer of a certain Riordan element is the checkerboard subgroup. We also present a new similarity property for certain generalized Riordan matrices. Some new and known facts are also given and proved in Section 4. Moreover, the elements of the stabilizer subgroup are described. A relationship between the pair of generating functions $g(z)$ and $f(z)$ of a Riordan array which leads to the derivation of stochastic Riordan matrices is established. The relationship is then used to generate elements of the stochastic subgroup of the Riordan group. We prove that the set of power-Bell matrices is a subgroup of the Riordan group. We also prove that the derivative subgroup is isomorphic to the associated and Bell subgroups. Properties of similar Riordan matrices and pseudo-involutions of the Riordan group are also described, and a new result for constructing certain pseudo-involutions is presented. Some problems for possible future research projects are presented in Section 5. Some of the new results in this paper are given in [14].

2. The Riordan group

Let $\mathbb{N}^*$ denote the natural numbers (including 0) and $\mathbb{C}$ the complex numbers. The definition of a Riordan matrix and two important theorems for proving properties on the algebraic structure of the Riordan group are presented.

Definition 1. An infinite matrix $L = (l_{n,k})_{n,k\in\mathbb{N}^*}$ with entries in $\mathbb{C}$ is called a Riordan matrix if the kth column satisfies

$$\sum_{n\geqslant 0} l_{n,k}\, z^n = g(z)(f(z))^k$$

where $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + \cdots$ belong to the ring of formal power series $\mathbb{C}[[z]]$ and $f_1 \neq 0$ and $g_0 \neq 0$.

A formal power series of the form

$$b(z) = b_0 + b_1 z + b_2 z^2 + \cdots = \sum_{n\geqslant 0} b_n z^n$$

where z is an indeterminate is called the ordinary generating function of the sequence $\{b_n\}$. A Riordan matrix L can also be defined by exponential generating functions. The matrices given in this paper are defined by ordinary generating functions and are the proper Riordan arrays as given by Sprugnoli [31]. We are also only concerned with Riordan matrices with real valued entries. The matrices are typically written in pair form as $(g(z), f(z))$ or (g, f). Pascal's triangle written in lower triangular form is denoted by

$$P = (1/(1-z), z/(1-z))\,.$$

The following theorem is called the Fundamental Theorem of the Riordan Group. It leads to the next theorem by applying the fundamental theorem to an arbitrary Riordan matrix N, one column of N at a time.

Theorem 1 [29]. *If $L = (l_{n,k})_{n,k\in\mathbb{N}^*} = (g(z), f(z))$ is a Riordan matrix and $h(z)$ is the generating function of the sequence associated with the entries of the column vector $h = \langle h_k\rangle^T_{k\in\mathbb{N}^*}$, then the product of L and $h(z)$, defined by*

$$L \otimes h(z) = g(z)h(f(z)),$$

is the generating function of the sequence associated with the entries of the column vector $\left\langle \sum_{k=0}^{n} l_{n,k}h_k\right\rangle^T_{n\in\mathbb{N}^*}$.

Let us denote by $L * N$, or by simple juxtaposition LN, the row-by column product of two Riordan matrices.

Theorem 2 [29]. *If*

$$L = (l_{n,k})_{n,k\in\mathbb{N}^*} = (g(z), f(z)) \quad \text{and} \quad N = (\nu_{n,k})_{n,k\in\mathbb{N}^*} = (h(z), d(z))$$

are Riordan matrices, then

$$LN = \left(\sum_{j=0}^{n} l_{n,j}\nu_{j,k}\right)_{n,k\in\mathbb{N}^*} = (g(z), f(z)) * (h(z), d(z)) = (g(z)h(f(z)), d(f(z))),$$

and the set $\mathcal{R}$ of all Riordan matrices is a group under the operation of matrix multiplication.

The notation $(\mathcal{R}, *)$ denotes the Riordan group. See [28,29,31] for more information on the Riordan group. The Riordan group is also known as the Sheffer or 1-umbral group [13,24].

3. Some structural properties

Some properties of the Riordan group are given in this section. We start with the basic group properties. Then, we present similar Riordan matrices, various known subgroups with some related properties, and properties involving pseudo-involutions. We also give a new similarity property and prove that the Appell subgroup is a normal subgroup of $(\mathcal{R}, *)$.

The Riordan group is noncommutative and closed under usual matrix multiplication. It is easy to show associativity. The identity element of $(\mathcal{R}, *)$ is denoted by $I = (1, z)$, where I is the usual unit diagonal matrix. The inverse elements of the group are Riordan matrices of the form

$$L^{-1} = (g(z), f(z))^{-1} = \left(1/g(\bar{f}(z)), \bar{f}(z)\right)$$

where $f(z)$ and $\bar{f}(z)$ are compositional inverses.

Example 1. The matrix $F = (1, z(1+z))$ is called a Fibonacci matrix since its row sums are the Fibonacci numbers (sequence A800045 [30]) with the first term of the sequence equal to 1. Then,

$$F^{-1} = (1, z(1+z))^{-1} = (1, zc(-z)) = \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & -1 & 1 & & & \cdots \\ 0 & 2 & -2 & 1 & & \\ 0 & -5 & 5 & -3 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}$$

where $c\,(z)$ denotes the Catalan generating function

$$c(z) = \left(1 - \sqrt{1-4z}\right)/2z. \tag{1}$$

See Examples 10 and 13, respectively, for the first few Fibonacci numbers and entries of F.

Definition 2. Two Riordan matrices A and B are said to be similar if there exists a Riordan matrix L such that $A = L^{-1}BL$.

Example 2. Recall $P = (1/(1-z), z/(1-z))$ is the Pascal matrix. Let $s(z)$ denote the RNA generating function

$$s(z) = \left(1 - z + z^2 - \sqrt{1 - 2z - z^2 - 2z^3 + z^4}\right)\Big/ 2z^2$$

from discrete mathematical biology. The matrix $R = (s(z), zs(z))$ is the RNA matrix since the RNA numbers [30] are the entries of the leftmost column of the matrix. The entries of R count certain RNA secondary structures [19,20,25].

R in the above example is similar to P since

$$R = C_0^{-1}PC_0$$

where

$$C_0 = \left(c(z^2), zc(z^2)\right), \quad C_0^{-1} = \left(1\Big/\left(1+z^2\right), z\Big/\left(1+z^2\right)\right),$$

and $c\,(z)$ is given by Eq. (1).

We now give a property for certain similar generalized Riordan matrices, and we also generalize Example 2.

Theorem 3. *If A and B are similar Riordan matrices, then A^k and B^k are similar Riordan matrices for integers $k \geqslant 0$.*

Proof. The proof is by induction on k, Riordan matrix multiplication and applying associativity. $\square$

Example 3. Let P^k and R^k denote, respectively, the generalized Pascal and RNA matrices where

$$P^k = (1/(1-kz), z/(1-kz))$$

and

$$R^k = (s(z), zs(z))^k = \left(tc(t^2), ztc(t^2)\right)$$

where $t = \left(1/(1-kz+z^2)\right)$ and $c\,(z)$ is given by Eq. (1). Then, the generalized RNA and Pascal matrices are similar since P and R are similar. The matrices are also similar since

$$R^k = C_0^{-1}P^kC_0.$$

More examples of similar Riordan matrices are given in [20,21].

3.1. Some Riordan subgroups

There are several known subgroups of $(\mathcal{R}, *)$. We highlight in this subsection some of the subgroups and briefly describe some of their properties.

The associated subgroup [29]: Elements of the associated subgroup are of the form $(1, f(z))$. The Fibonacci matrices given in Example 1 are elements of the associated subgroup. The associated subgroup is isomorphic to the group of formal power series under composition [11].

The Appell subgroup [29]: Elements of the Appell subgroup are of the form $(g(z), z)$. The Appell subgroup is isomorphic to the group of invertible formal power series under multiplication [11] . If

$$g(z) = \sum_{n \geqslant 0} g_n z^n = g_0 + g_1 z + g_2 z^2 + \cdots ,$$

then the coefficients of $g(z)$ determine the entries of the elements of the Appell subgroup. The elements are Riordan matrices of the form

$$(g(z), z) = \begin{pmatrix} g_0 & & & & \\ g_1 & g_0 & & & \cdots \\ g_2 & g_1 & g_0 & & \\ g_3 & g_2 & g_1 & g_0 & \\ & \vdots & & & \ddots \end{pmatrix}.$$

We call the elements Appell matrices. An Appell matrix $(g(z), z)$ is the transpose of a semi-circulant matrix [21]. Semi-circulant matrices are of the form $(g(z), z)^T$ where $(g(z), z)^T$ denotes the transpose of $(g(z), z)$.

We now provide a proof that the Appell subgroup is normal.

Theorem 4 [29]. *The Appell subgroup is normal.*

Proof. Let $a = (h(z), d(z))$ be an arbitrary element of $(\mathcal{R}, *)$ and $N = (g(z), z)$ be an element of the Appell subgroup. Then

$$\begin{aligned} a^{-1}Na &= (h(z), d(z))^{-1} * (g(z), z) * (h(z), d(z)) \\ &= \left(1/h(\bar{d}(z)), \bar{d}(z)\right) * (g(z), z) * (h(z), d(z)) \\ &= \left(1/h(\bar{d}(z)), \bar{d}(z)\right) * (g(z) \cdot h(z), d(z)) \\ &= \left(g(\bar{d}(z)) \cdot h(\bar{d}(z))/h(\bar{d}(z)), d(\bar{d}(z))\right) \\ &= \left(g(\bar{d}(z)), z\right). \end{aligned}$$

Thus, this shows that $a^{-1}Na$ is an element of the Appell subgroup. □

Given normality in $(\mathcal{R}, *)$, we now present a semi-direct product in $(\mathcal{R}, *)$. That is, the Riordan group is a semi-direct product of the Appell and associated subgroups [29]. The product is given by

$$(g(z), z) * (1, f(z)) = (g(z), f(z)) .$$

The Bell subgroup [29]: Elements of the Bell subgroup are of the form $(g(z), zg(z))$ or $(f(z)/z, f(z))$. The RNA, Pascal, C_0, and C_0^{-1} matrices in Examples 2 and 3 are elements of the Bell subgroup. The Riordan group is also a semi-direct product of the Bell and Appell subgroups [1]. The product here is given by

$$(zg(z)/f(z), z) * (f(z)/z, f(z)) = (g(z), f(z)) .$$

The checkerboard subgroup [1,28]: Elements of the checkerboard subgroup are of the form $(g_e(z), f_o(z))$ where $g_e(z)$ denotes an even function and $f_o(z)$ denotes an odd function. The following matrix is an interesting example of an element of the checkerboard subgroup

$$J = (g_e(z), f_o(z)) = \begin{pmatrix} 1 & & & & & & & \\ 0 & 1 & & & & & & \\ 1 & 0 & 1 & & & & & \cdots \\ 0 & 4 & 0 & 1 & & & & \\ 6 & 0 & 7 & 0 & 1 & & & \\ 0 & 31 & 0 & 10 & 0 & 1 & & \\ 53 & 0 & 65 & 0 & 13 & 0 & 1 & \\ & & \vdots & & & & & \ddots \end{pmatrix}$$

where

$$g_e(z) = \frac{\left(1-\sqrt{1+4z}\right)\left(1-\sqrt{1-4z}\right)}{z\left(\sqrt{1-4z}-\sqrt{1+4z}\right)}$$

and

$$f_o(z) = \frac{2-\left(\sqrt{1-4z}+\sqrt{1+4z}\right)}{\sqrt{1+4z}-\sqrt{1-4z}}.$$

The row sums of J are $c(z)$ where $c(z)$ is given by Eq. (1). The matrices C_0 and C_0^{-1} in Examples 2 and 3 are also elements of the checkerboard subgroup.

We now prove that the centralizer of the Riordan element $(1, -z)$ is the checkerboard subgroup.

Theorem 5 [28]. *The centralizer of the element $M = (1, -z)$ is the checkerboard subgroup.*

Proof. In order to prove that the centralizer of $M = (1, -z)$ is the checkerboard subgroup, we need to show that an arbitrary element $L = (g(z), f(z))$ of $(\mathcal{R}, *)$ is in the checkerboard subgroup if and only if

$$(g(z), f(z)) * (1, -z) = (1, -z) * (g(z), f(z))$$

$\Longleftrightarrow$

$$(g(z), -f(z)) = (g(-z), f(-z))$$

$\Longleftrightarrow$ $g(z) = g(-z)$ and $-f(z) = f(-z)$ $\Longleftrightarrow$ $g(z)$ is even and $f(z)$ is odd. Thus, we obtain the checkerboard subgroup. $\square$

Cheon subgroup [6]: Let $\mathcal{H}_0$ denote the set of all complex valued functions such that $h_0 \neq 0$. Then, the set of all Riordan matrices of the form $(g(z), zf(z^m))$ where $g(z), f(z) \in \mathcal{H}_0$ and m is any integer is a subset of $(\mathcal{R}, *)$. We note that the checkerboard subgroup is a subgroup of the Cheon subgroup since they are both subgroups of the Riordan group and the elements of the checkerboard subgroup form a subset of the elements of the Cheon subgroup.

The stochastic Riordan subgroup [28]: The elements of this subgroup are the Riordan matrices whose row sums equal one. Equivalently, this is the stabilizer of the column vector $\langle 1, 1, 1, \ldots\rangle^T$ where $\langle 1, 1, 1, \ldots\rangle^T$ denotes the transpose of the row vector $\langle 1, 1, 1, \ldots\rangle$. We note that the definition of stochastic matrix here differs from the usual definition since the matrices are infinite lower-triangular and some entries of some Riordan matrices can be negative.

The derivative subgroup [26]: Elements of this subgroup are of the form $(f'(z), f(z))$ where $f'(z)$ denotes the first derivative of $f(z)$. The following matrix is an element of the derivative subgroup

$$(1+2z, z(1+z)) = \begin{pmatrix} 1 & & & & & & \\ 2 & 1 & & & & & \\ 0 & 3 & 1 & & & \cdots & \\ 0 & 2 & 4 & 1 & & & \\ 0 & 0 & 5 & 5 & 1 & & \\ 0 & 0 & 2 & 9 & 6 & 1 & \\ & & \vdots & & & & \ddots \end{pmatrix}. \quad (2)$$

The hitting time subgroup [22]: The elements of this group are of the form $(zf'(z)/f(z), f(z))$. The Pascal matrix is also an element of this subgroup.

We now present a definition of divisibility for the hitting time subgroup. The divisibility property of the Pascal triangle is well-known. An interesting feature of Theorem 6 below is that it generalizes the divisibility property of Pascal's triangle to all elements of the hitting time subgroup.

Definition 3 [22]. A subset of Riordan matrices is said to have the "divisibility property" if each matrix $(m_{n,k})_{n,k\geqslant 0}$ of the subset satisfies the property that n divides $k \cdot m_{n,k}$ whenever $0 < k < n$.

Theorem 6 [22]. *Let* $M = (m_{n,k})_{n,k\geqslant 0} = (zf'(z)/f(z), f(z))$*. Then n divides* $k \cdot m_{n,k}$ *for all* $0 < k < n$.

Corollary 1 [22]. *If p is a prime, then p divides* $m_{p,k}$ *for* $0 < k < p$.

3.2. Riordan group involutions

We now present Riordan group elements of finite order. Recall, in this paper we are only concerned with Riordan matrices with real valued entries. The only elements of finite order in the group of non-zero real numbers under multiplication are -1 and 1. Thus, it follows that any Riordan matrix with real valued entries and of finite order must have order 2 [7]. Elements of order 2 are called involutions.

Example 4. The element

$$M = (1, -z) = \begin{pmatrix} 1 & & & & \\ 0 & -1 & & & \\ 0 & 0 & 1 & & \cdots \\ 0 & 0 & 0 & -1 & \\ & & \vdots & & \ddots \end{pmatrix}. \quad (3)$$

is an involution in $(\mathcal{R}, *)$.

Proposition 1 [2]. *If* $T = (1, f(z))$ *is an element of the associated subgroup, then T is an involution if and only if* $f(z) = \bar{f}(z)$.

Example 5. Let $T = (1, -z/(1+z))$. Then T is an involution in $(\mathcal{R}, *)$ since $f(z) = \bar{f}(z)$. The result here can also be obtained by Riordan matrix multiplication.

There are uncountably many formal power series $f(z)$ of compositional order 2. Thus, there are uncountably many elements of the associative subgroup $(1, f(z))$ of order 2. So, there are uncountably many Riordan group elements of order 2 [8].

We now present the definition of a pseudo-involution of $(\mathcal{R}, *)$ and give some examples. The results for the examples can be obtained by Riordan matrix multiplication.

Definition 4 [2]**.** An element L of the Riordan group is called a pseudo-involution or is said to have pseudo-order 2 if LM or equivalently ML has order 2 where M is given by Eq. (3).

Example 6. The identity element I is a pseudo-involution that is not an involution. The element $T = (1, -z/\,(1+z))$ in Example 5 is an involution but not a pseudo-involution.

Example 7. The Pascal matrix P is not an involution but is a pseudo-involution. More generally, for $k \geqslant 0$ the generalized Pascal matrix P^k is not an involution but is a pseudo-involution since $(P^kM)^2 = I$. The set of matrices P^k is a subset of a larger subset of Riordan matrices of the form

$$(1/(1-rz), z/(1-rz))$$

where r is any real or complex number. These matrices are elements of the Bell subgroup and are also pseudo-involutions.

Example 8. The RNA matrix R is not an involution but is a pseudo-involution since $(RM)^2 = I$.

Proposition 2. *If $L = (g(z), f(z))$ is a pseudo-involution, then*

$$L^{-1} = (g(-z), -f(-z)).$$

Example 9. Consider the RNA matrix R. Then,

$$R^{-1} = (s(-z), zs(-z)).$$

Note that a computer algebra system like Maple or Mathematica can be used to confirm that $RR^{-1} = I$.

An interesting feature of Proposition 2 is that it can be used to find the inverse of a pseudo-involution with complicated computations involving the compositional inverse.

Proposition 3. *The generalized RNA matrix*

$$R^k = C_0^{-1} P^k C_0$$

is a pseudo-involution.

Proof. We must show $(MR^k)^2 = I$. By associativity, we have

$$\begin{aligned}(MR^k)^2 &= \left(M\left(C_0^{-1}P^kC_0\right)\right) * \left(M\left(C_0^{-1}P^kC_0\right)\right)\\ &= \left(\left(MC_0^{-1}\right)P^kC_0\right) * \left(\left(MC_0^{-1}\right)P^kC_0\right).\end{aligned}$$

Then, by $MC_0^{-1} = C_0^{-1}M$, associativity and P^k being a pseudo-involution we have

$$\begin{aligned}(MR^k)^2 &= \left(\left(C_0^{-1}M\right)P^kC_0\right) * \left(\left(C_0^{-1}M\right)P^kC_0\right)\\ &= \left(C_0^{-1}MP^k\right) * (C_0C_0^{-1}) * (MP^kC_0)\\ &= C_0^{-1}\left(\left(MP^k\right)\left(MP^k\right)\right)C_0\\ &= C_0^{-1}C_0\\ &= I. \quad \square\end{aligned}$$

Recall that the matrices P^k and R^k are similar. A link between pseudo-involutions and similar Riordan matrices is given in the next section. More information on pseudo-involutions in the Riordan group can be found in [1,3–5,16,20].

4. More structural properties

In this section, we present more structural properties. We start with the stabilizer subgroup.

4.1. Stabilizer subgroup

Recall that the stochastic Riordan subgroup is the stabilizer of the column vector $\langle 1, 1, 1, \ldots \rangle^T$. In order for a Riordan group element to stabilize a column vector whose entries are given by the generating function $h(z)$, the following condition must be satisfied

$$(g(z), f(z)) \otimes h(z) = h(z). \tag{4}$$

This leads to the following propositions.

Proposition 4. *Let $h(z)$ be the generating function associated with the coefficients whose entries form the column vector $\langle h_0, h_1, h_2, \ldots \rangle^T$. The stabilizer of the column vector is the set of all Riordan matrices of the form*

$$(h(z)/h(f(z)), f(z)) .$$

Proof. Applying the fundamental theorem to Eq. (4) gives

$$g(z) \cdot h(f(z)) = h(z). \tag{5}$$

Solving Eq. (5) for $g\,(z)$ gives

$$g(z) = h(z)/h(f(z)). \quad \square$$

Remark 1. It is well-known that the stabilizer of an element of a set on which a group G acts forms a subgroup [10]. Nonetheless, we present a short proof below using Riordan matrix multiplication.

Proposition 5. *The set of elements of the form $(h(z)/h(f(z)), f(z))$ is a subgroup of the Riordan group.*

Proof. Let

$$a = (h(z)/h(f_1(z)), f_1(z)) \quad \text{and} \quad b = (h(z)/h(f_2(z)), f_2(z))$$

be two elements of the set. Then,

$$ab = (h(z)/h(f_2(f_1(z))), f_2(f_1(z))) .$$

This shows closure. The inverse element is

$$a^{-1} = \left(h(z)/h(\bar{f}(z)), \bar{f}(z)\right) .$$

Hence, the set forms a subgroup. $\square$

If $h(z) = 1$, then the associated subgroup is obtained as mentioned by Shapiro [28]. By applying the fundamental theorem we obtain

$$(g(z), f(z)) \otimes h(z) = 1 \iff g(z) \cdot 1 = 1 \iff (g(z), f(z)) = (1, f(z))$$

where $h(z)$ is the generating function associated with the column vector $\langle 1, 0, 0, \ldots \rangle^T$.

4.2. Generating stochastic Riordan arrays

We now give a definition and property for a Riordan array to be stochastic. The definition differs from the usual definition of a stochastic matrix as given in [23].

Definition 5. A stochastic Riordan array is a Riordan array written in Riordan pair form $(g(z), f(z))$ where the kth column is of the form $g(z) \cdot (f(z))^k$ and the row sums equal one.

Lemma 1. *A Riordan array $L = (g(z), f(z))$ is stochastic if and only if*

$$f(z) = -g(z) + zg(z) + 1.$$

Proof. The stabilizer of the column vector whose coefficients are associated with the generating function $1/(1-z)$ is the stochastic Riordan subgroup. That is, $(g(z), f(z))$ is an element of the stochastic Riordan subgroup if and only if

$$(g(z), f(z)) \otimes 1/(1-z) = 1/(1-z).$$

By the fundamental theorem, this is true if and only if

$$g(z) \cdot 1/(1 - f(z)) = 1/(1-z)$$

$$\Longleftrightarrow$$

$$g(z) \cdot (1-z) = 1 - f(z)$$

$$\Longleftrightarrow$$

$$f(z) = -g(z) + zg(z) + 1. \quad \square$$

As a result of Lemma 1, we can generate stochastic Riordan arrays for certain sequences of counting numbers. We now present an example of a stochastic Riordan array generated for the Fibonacci numbers with the first term equal to 1. The array is not a proper Riordan array but is of interest since the Fibonacci numbers appear in the leftmost column.

Example 10. The generating function for the Fibonacci numbers with the first term equal to 1 is

$$F(z) = 1/(1 - z - z^2).$$

By Lemma 1, if $g(z) = F(z)$, then $f(z) = -z^2F(z)$. Then, in matrix form

$$F_1 = (F(z), -z^2F(z)) = \begin{pmatrix} 1 & & & & & & \\ 1 & 0 & & & & & \\ 2 & -1 & 0 & & & & \cdots \\ 3 & -2 & 0 & 0 & & & \\ 5 & -5 & 1 & 0 & 0 & & \\ 8 & -10 & 3 & 0 & 0 & 0 & \\ & & \vdots & & & & \ddots \end{pmatrix}$$

Note that the row sums of the above first few entries are 1. This can be confirmed by multiplying the pair form by $1/(1-z)$. This also gives the following interesting relationship involving the Fibonacci generating function

$$F(z) \Big/ \left(1 + z^2F(z)\right) = 1/(1-z).$$

Remark 2. The Riordan array given in Example 10 is a stochastic Riordan array that is not invertible. Thus, F_1 is not a Riordan matrix. The array; however, is an example of a vertically stretched Riordan array. We note that the vertically stretched arrays have left inverses. For more information on left inverses of Riordan arrays see [9].

The following theorem gives conditions for a stochastic Riordan array to be a stochastic Riordan matrix.

Theorem 7. *A stochastic Riordan array is a stochastic Riordan matrix provided $g_0 = 1$ and $g_1 \neq 1$ where g_0 and g_1 are the leading coefficients of the generating function*

$$g(z) = \sum_{n \geqslant 0} g_n z^n = g_0 + g_1 z + g_2 z^2 + \cdots .$$

Proof. Recall by Lemma 1, a Riordan array $(g(z), f(z))$ is stochastic if

$$f(z) = 1 - g(z) + zg(z).$$

From this relationship, we obtain

$$\begin{aligned} f(z) &= (1 - g_0 - g_1 z - g_2 z^2 - \cdots) + (g_0 z + g_1 z^2 + g_2 z^3 + \cdots) \\ &= 1 - g_0 + (g_0 - g_1)z + (g_1 - g_2)z^2 + (g_2 - g_3)z^3 + \cdots \end{aligned}$$

From the definition of a Riordan matrix, the constant term of $f(z)$ is 0. So $1 - g_0 = 0$ implies $g_0 = 1$. In addition, f_1 cannot be 0, hence $g_0 - g_1 \neq 0$ implies $1 - g_1 \neq 0$ or $g_1 \neq 1$. $\square$

We now give an example of a stochastic Riordan matrix. An interesting feature of the example is that the Hex numbers (sequence A002212 [30]), excluding the first term of the sequence, appear in the leftmost column of the matrix. The Hex numbers count the number of restricted hexagonal polyominoes with n cells [30].

Example 11. If

$$g(z) = \left(1 - 3z - \sqrt{1 - 6z + 5z^2}\right) \Big/ 2z^2,$$

then

$$f(z) = \left(1 - 2z - z^2 - (z+1)\sqrt{1 - 6z + 5z^2}\right) \Big/ 2z^2.$$

Now, let $H = (g(z), f(z))$. Thus, H is a stochastic Riordan matrix and an element of the stochastic Riordan subgroup. In matrix form

$$H = \begin{pmatrix} 1 & & & & & \\ 3 & -2 & & & & \cdots \\ 10 & -13 & 4 & & & \\ 36 & -67 & 40 & -8 & & \\ 137 & -321 & 277 & -108 & 16 & \\ & \vdots & & & & \ddots \end{pmatrix}.$$

As expected, the row sums here are also 1.

4.3. The power-Bell subgroup

The following problem was mentioned by Shapiro in 2008. The problem is whether Riordan matrices of the form $(g(z), zg^n(z))$, $n \geqslant 0$ (fixed integer), is a subgroup of $(\mathcal{R}, *)$. When $n = 0, 1$, we respectively, get the Appell and Bell subgroups. The question is what matrices do we get when $n \neq 0, 1$? We generalize the problem to Riordan matrices of the form $(g(z), zg^r(z))$ where r is a fixed real number. We call these matrices power-Bell matrices and provide a solution to the problem in the following proposition.

Proposition 6. *The set of Riordan matrices $(g(z), zg^r(z))$, where r is a fixed real number, is a subgroup of the Riordan group.*

Proof. When $r = 0$, we get the Appell subgroup. Now, let's consider when $r \neq 0$. Note that the pair form of the matrices can be rewritten as

$$\left((f(z)/z)^{\frac{1}{r}}, f(z)\right).$$

This simplifies the proof of the proposition since expressing the compositional inverse of $f(z)$ is much simpler than that of $zg^r(z)$. In order to show that these matrices form a subgroup, we need to show closure under multiplication and taking inverses.

Consider a and b where

$$a = \left((f_1(z)/z)^{\frac{1}{r}}, f_1(z)\right) \quad \text{and} \quad b = \left((f_2(z)/z)^{\frac{1}{r}}, f_2(z)\right)$$

are elements of the set. Then, multiplying a and b gives the following

$$\begin{aligned} ab &= \left((f_1(z)/z)^{\frac{1}{r}}, f_1(z)\right) * \left((f_2(z)/z)^{\frac{1}{r}}, f_2(z)\right) \\ &= \left(\left((f_1(z)/z)^{\frac{1}{r}}\right) \cdot \left((f_2(f_1(z))/f_1(z))^{\frac{1}{r}}\right), f_2(f_1(z))\right) \\ &= \left((f_2(f_1(z))/z)^{\frac{1}{r}}, f_2(f_1(z))\right). \end{aligned}$$

This shows closure.

Next, we show closure under inverses. Thus, for an arbitrary element of the set, we obtain

$$\begin{aligned} \left((f(z)/z)^{\frac{1}{r}}, f(z)\right)^{-1} &= \left(\frac{1}{\left(f(\bar{f}(z))/\bar{f}(z)\right)^{\frac{1}{r}}}, \bar{f}(z)\right) \\ &= \left(\left(f(\bar{f}(z))/\bar{f}(z)\right)^{-\frac{1}{r}}, \bar{f}(z)\right) \\ &= \left(\left(\bar{f}(z)/z\right)^{\frac{1}{r}}, \bar{f}(z)\right). \end{aligned}$$

Thus, the set is a subgroup of $(\mathcal{R}, *)$. $\square$

We call this subgroup the power-Bell subgroup of $(\mathcal{R}, *)$.

Example 12. For $r = 2$, we give an example where $F(z)$ is the Fibonacci generating function. Let $g(z) = (F(z))^{\frac{1}{2}}$ and $f(z) = zF(z)$. Then,

$$((F(z))^{\frac{1}{2}}, zF(z)) = \begin{pmatrix} 1 & & & & & & \\ 1/2 & 1 & & & & & \\ 7/8 & 3/2 & 1 & & & & \\ 17/16 & 27/8 & 5/2 & 1 & & & \cdots \\ 203/128 & 95/16 & 55/8 & 7/2 & 1 & & \\ 583/256 & 1395/128 & 245/16 & 91/8 & 9/2 & 1 & \\ & & \vdots & & & & \ddots \end{pmatrix}.$$

Note that if we take $g(z) = 1/\sqrt{1-4z}$, then we get an element of the power-Bell subgroup with integral entries. Another element of the power-Bell subgroup with integral entries can also be obtained by using the Fine numbers (sequence A000957 [30]). The Fine numbers count Dyck paths with no hills [1,30].

4.4. Isomorphism of subgroups

We establish that the derivative subgroup is isomorphic to the associated and Bell subgroups. Starting with the associative subgroup, if we remove the leftmost column and topmost row of an element, then we get an element of the Bell subgroup [28].

Example 13. Removing the leftmost column and topmost row of the Fibonacci matrix $F = (1, z(1+z))$ gives

$$\begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 0 & 2 & 1 & & & \cdots \\ 0 & 0 & 1 & 3 & 1 & & \\ 0 & 0 & 0 & 3 & 4 & 1 & \\ & & & \vdots & & & \ddots \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 0 & 2 & 1 & & & \cdots \\ 0 & 1 & 3 & 1 & & \\ 0 & 0 & 3 & 4 & 1 & \\ & & \vdots & & & \ddots \end{pmatrix}.$$

This mapping is given by $(1, z(1+z)) \to (1+z, z(1+z))$.

The above example illustrates that the associated subgroup is isomorphic to the Bell subgroup via the mapping that takes $(1, f(z))$ to $(f(z)/z, f(z))$. Thus, we have the following proposition.

Proposition 7. *The mapping $\phi : (1, f(z)) \to (f(z)/z, f(z))$ is an isomorphism.*

Proof. Consider the mapping ϕ which takes $(1, f(z))$ to $(f(z)/z, f(z))$. Clearly ϕ is one-to-one and onto. We now show ϕ is a homomorphism. By Riordan matrix multiplication and the definition of the mapping we obtain

$$\begin{aligned} \phi\left((1, f(z)) * (1, h(z))\right) &= \phi\left(1, h(f(z))\right) \\ &= (h(f(z))/z, h(f(z))) \\ &= (f(z)/z, f(z)) * (h(z)/z, h(z)) \\ &= \phi\left(1, f(z)\right) * \phi\left(1, h(z)\right). \end{aligned}$$

This proves the result. □

Remark 3. Similar to Proposition 7, the mapping

$$\Phi : (1, f(z)) \rightarrow \left((f(z)/z)^{1/r}, f(z)\right)$$

is an isomorphism between the associative and power-Bell subgroups.

We now establish an isomorphism between the derivative and associated subgroups.

Proposition 8. *The mapping* $\alpha : (f'(z), f(z)) \rightarrow (1, f(z))$ *is an isomorphism.*

Proof. Consider the mapping α that takes (f', f) to $(1, f)$. Clearly, α is one to one and onto. We now show α is a homomorphism. By Riordan matrix multiplication and the definition of the mapping we obtain

$$\begin{aligned}\alpha[(f_1', f_1)*(f_2', f_2)] &= \alpha[(f_1'f_2'(f_1), f_2(f_1))] \\ &= (1, f_2(f_1)) \\ &= (1, f_1)*(1, f_2) \\ &= \alpha(f_1', f_1)*\alpha(f_2', f_2).\end{aligned}$$

This proves the result. □

The fact that the derivative and associated subgroups are isomorphic and the associated and Bell subgroups are isomorphic, we know that the derivative and Bell subgroups are isomorphic since isomorphism is an equivalence relation [10]. By using the Fundamental Theorem of Calculus, we give a proof of the isomorphism between the derivative and Bell subgroups. The proof gives a nice application of the Fundamental Theorem of Calculus.

Proposition 9. *The mapping* $\beta : (f'(z), f(z)) \rightarrow (f(z)/z, f(z))$ *is an isomorphism.*

Proof. Consider the mapping β defined by

$$\beta(f', f) = \left(1/z \int_0^z f'(t)dt, f(z)\right).$$

Clearly β is one-to-one and onto. We now show β is a homomorphism. By Riordan matrix multiplication, the definition of the mapping, and the Fundamental Theorem of Calculus we get the following.

$$\begin{aligned}\beta[(f', f) * (h', h)] &= \beta\left(f' \cdot h'(f), h(f)\right) \\ &= \left((1/z) \int_0^z \left((h(f))'(t)dt\right), h(f)\right) \\ &= ((1/z) \cdot h(f), h(f)) \\ &= \left((1/z) \int_0^{f(z)} h'(t)dt, h(f)\right) \\ &= \left((1/z)f(z)(1/f(z)) \int_0^{f(z)} h'(t)dt, h(f)\right) \\ &= \left((1/z) \int_0^z f'(t)dt(1/f(z)) \int_0^{f(z)} h'(t)dt, h(f)\right) \\ &= \left((1/z) \int_0^z f'(t)dt, f(z)\right) * \left((1/z) \int_0^z h'(t)dt, h(z)\right) \\ &= \beta(f', f)*\beta(h', h).\end{aligned}$$

This proves the result. □

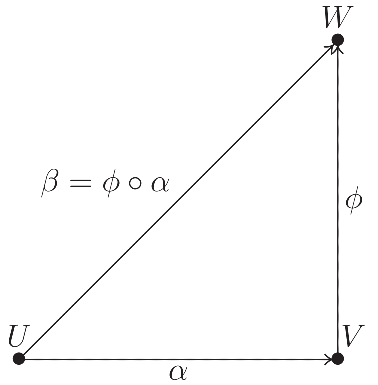

Fig. 1. Commutative diagram.

By combining Propositions 7, 8 and 9, we give in Fig. 1 an illustration of a commutative diagram where $U = (f', f)$, $V = (1, f)$, and $W = (f/z, f)$.

4.5. Similar Riordan matrices and pseudo-involutions

We now give some properties in this subsection that link similar Riordan matrices and pseudo-involutions. We start by giving a property that can be used to construct certain generalized pseudo-involutions of $(\mathcal{R}, *)$. Then, we give a lemma and theorem connecting similar Riordan matrices and pseudo-involutions where similarity involves elements of the checkerboard subgroup.

Proposition 10. *Let L be an arbitrary Riordan matrix, $M = (1, -z)$ and D a pseudo-involution. Then, a Riordan matrix A of the form*

$$A = (L^{-k}M)D(L^kM)$$

is a pseudo-involution in the Riordan group.

Proof. (Sketch) By Definition 4, the fact that $M^2 = I$ and D is a pseudo-involution gives

$$(AM)^2 = \left(\left((L^{-k}M)D(L^kM)\right)M\right)^2 = I. \quad \square$$

Example 14. If $D = R$ is the RNA matrix, then by Proposition 10

$$A = \left(L^{-k}M\right) R \left(L^kM\right)$$

is a pseudo-involution. This result can also be obtained by Riordan matrix multiplication, associativity and the fact that M and R are pseudo-involutions.

We showed in Examples 2 and 3 that the Pascal and RNA matrices are similar. We also showed in Example 7 that the Pascal and generalized Pascal matrices are pseudo-involutions and in Example 8 that the RNA matrix is a pseudo-involution. In addition, we proved in Proposition 3 that the generalized RNA matrix is a pseudo-involution. Therefore, we have similar Riordan matrices that are also pseudo-involutions. However, there are cases when Riordan matrices are similar and neither matrix is a pseudo-involution. In addition, there are cases when two Riordan matrices are similar and one is a pseudo-involution and the other is not. We now give an example to illustrate this last point.

Recall that P denotes the Pascal matrix which is a pseudo-involution. For $k > 0$ (integer), let Q_k denote the following generalized Riordan matrix

$$Q_k = \left(\left((1-z)^{2k-1} \,/\, (1-2z)^k\right), z/(1-z)\right).$$

Thus, Q_k is similar to P since

$$Q_k = E^{-k} P E^k$$

where

$$E^k = \left(1/(1-z)^k, z\right) \quad \text{and} \quad E^{-k} = \left((1-z)^k, z\right).$$

We now show that Q_k is not a pseudo-involution. Consider

$$Q_k M = \left(\left((1-z)^{2k-1} \,/\, (1-2z)^k\right), z/(1-z)\right) * (1, -z).$$

Then, we obtain

$$\begin{aligned}(Q_k M)^2 &= \left(\left((1+z)^{2k-1} \,/\, (1+2z)^k\right), -z/(1+z)\right)^2 \\ &= \left(\left((1+z) \Big/ \left(1+z-2z^2\right)\right)^k, z\right).\end{aligned}$$

Hence, $(Q_k M)^2 \neq I$ since

$$\left((1+z) \Big/ \left(1+z-2z^2\right)\right)^k \neq 1.$$

Thus, Q_k is similar to P where P is a pseudo-involution and Q_k is not. We now give properties that show when similar Riordan matrices are also pseudo-involutions.

Lemma 2. *If A and B are similar Riordan matrices of the form $B = S^{-1}AS$ where S is an element of the checkerboard subgroup, then A is a pseudo-involution if and only if B is a pseudo-involution.*

Proof. The proof uses associativity and the fact that the centralizer of $M = (1, -z)$ is the checkerboard subgroup. In order to prove that B is a pseudo-involution, we show that $(BM)^2 = I$. Now, we assume that A is a pseudo-involution. Then,

$$\begin{aligned}(BM)^2 &= \left(\left(S^{-1}AS\right) M\right) \left(\left(S^{-1}AS\right) M\right) \\ &= S^{-1}A(SMS^{-1})ASM \\ &= S^{-1}AMASM \\ &= S^{-1}(AMA)SM \\ &= S^{-1}(AMA)MS \\ &= S^{-1}(AMAM)S \\ &= S^{-1}(AM)^2S \\ &= S^{-1}S \\ &= I.\end{aligned}$$

The reverse direction of the proof follows since similarity is an equivalence relation. □

Theorem 8. *If*

$$B = S^{-1}AS$$

where S is an element of the checkerboard subgroup, then A^k is a pseudo-involution if and only if B^k is a pseudo-involution.

Proof. The proof follows from Theorem 3 and Lemma 2. □

4.6. Conclusion

This paper presents some properties on the algebraic structure of the Riordan group. The elements of the Riordan group have been mainly used as combinatorial objects with little emphasis placed on the algebraic structure. There is still much about the group that is unknown. For instance, some work has been done on finding the commutator subgroup [26,27]. This is of interest since the commutator is a proper normal subgroup which seems to be relatively rare. Also, finding other commutative subgroups and cyclic subgroups of $(R, *)$ is of interest. We showed that the associated subgroup stabilizes the column vector associated with the coefficients of the generating function $h(z) = 1$. The stochastic Riordan subgroup stabilizes the column vector associated with the coefficients of the generating function $h(z) = 1/(1 - z)$. Thus, finding other column vectors that are stabilized by subgroups of the Riordan group is of interest. We presented a property linking similarity of Riordan matrices, elements of the checkerboard subgroup, and pseudo-involutions. Determining if there are other Riordan matrices not in the checkerboard subgroup for which the matrices are similar and pseudo-involutions is of interest. Given that the Riordan group is a tool for solving combinatorial identities, it is also of interest to apply the new group properties and stochastic Riordan arrays developed in this paper to problems from combinatorics and graph theory. Lastly, we note that the properties presented in this paper are suitable for inclusion in university level abstract and linear algebra courses.

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