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A q-analogue of the Riordan group ☆

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ABSTRACT

The Riordan group consisting of Riordan matrices shows up naturally in a variety of combinatorial settings. In this paper, we define a q-Riordan matrix to be a q-analogue of the (exponential) Riordan matrix by using the Eulerian generating functions of the form $\sum_{n\geqslant 0} f_n z^n/n!_q$. We first prove that the set of q-Riordan matrices forms a loop (a quasigroup with an identity element) and find its loop structures. Next, it is shown that q-Riordan matrices associated to the counting functions may be applied to the enumeration problem on set partitions by block inversions. This notion leads us to find q-analogues of the composition formula and the exponential formula, respectively.

1. Introduction

The *Riordan group* [10] is the set of infinite lower triangular matrices in which their kth columns have the generating functions of the form $g(z)f(z)^k$ for $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ where $g, f \in \mathbb{C}[\![z]\!]$ with $g(0) \neq 0$, $f(0) = 0$ and $f'(0) \neq 0$. Such a matrix is called Riordan matrix and denoted as $(g(z), f(z))$ or (g, f). The group multiplication is given by $(g, f)(h, \ell) = (g(h \circ f), \ell \circ f)$ where $\circ$ denotes the composition of the functions.

Riordan matrices naturally appear in a formulation of the *Umbral Calculus* as introduced by Roman [9]. If we regard power series g and f as exponential generating functions we obtain exponential Riordan matrices and the exponential Riordan group. Many properties of the (exponential) Riordan

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matrices have been studied in connection with counting problems, specially for combinatorial sums, identities and inversions (see [10,11]).

The concept of Riordan group is strictly related to the Lagrange inversion formula (LIF) which is the natural device for inverting the elements of the group in terms of coefficients of certain other power series. There have also been many generalizations of the LIF as well as a q-analogue [5]. It is known [4] that the q-analogues appear in the diverse subjects of combinatorics, quantum group theory, group representation theory, number theory, statistical mechanics and q-deformed super algebras, to cite only a few. The Riordan group also appears in the new domain of combinatorial quantum physics, namely in the problem of normal ordering of boson strings [2,3]. Thus it is natural to study a q-analogue of the Riordan group.

In this paper, we consider *Eulerian* generating functions of the form

$$\sum_{n\geqslant 0} f_n \frac{z^n}{n!_q},$$

where $n!_q = 1\cdot(1+q)\cdots(1+q+\cdots+q^{n-1})$, $0!_q = 1$ and f_n is a polynomial in q. This function is a q-analogue of the exponential generating function. It arises in several combinatorial applications [6,7] such as finite vector spaces, partitions and counting permutations by inversions.

The purpose of this paper is to introduce a q-analogue of the Riordan group and to find its combinatorial significance. Specifically, in Section 2 we define a q-Riordan matrix to be a q-analogue of the (exponential) Riordan matrix by using the Eulerian generating functions. In Section 3, we prove that the set of q-Riordan matrices forms a loop and find its algebraic structures. In Section 4, we demonstrate its combinatorial significance by showing that a q-Riordan matrix associated to the counting functions may be applied to the enumeration problem on set partitions by block inversions. This notion leads us to find q-analogues of the composition formula and the exponential formula addressed in the two text books by Aigner [1] and Stanley [12]. Further, a new combinatorial interpretation for q-Stirling numbers of the second kind is obtained.

2. q-Riordan matrix

We begin this section by describing notations and definitions following Johnson [7] (also see [6]), which leads to a new method for computations and classifications of q-Riordan matrices.

Using $[n]_q = (1-q^n)/(1-q) = 1+q+\cdots+q^{n-1}$, the q-factorial $n!_q$ may be written as $n!_q = [1]_q[2]_q\cdots[n]_q$. Then the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q(n-k)!_q}.$$

The *q-derivative* $D_q f(z)$ of $f(z)$ is defined as

$$D_q f(z) = \frac{f(z)-f(qz)}{z-qz}$$

and it is often denoted as $f'(z)$. The q-product rule follows that

$$D_q f(z)g(z) = f(z)g'(z) + f'(z)g(qz).$$

For $k \in \mathbb{N}_0$, the *kth symbolic power* $f^{[k]}(z)$ of $f(z)$ with $f(0)=0$ is inductively given by

$$D_q f^{[k]}(z) = [k]_q f^{[k-1]}(z) f'(z) \quad \text{for } k \geqslant 1 \text{ and } f^{[0]}(z) = 1$$

where $f^{[k]}(0) = 0$ for $k \geqslant 1$. In particular, $f^{[1]}(z) = f(z)$. If $g(z) = \sum_{n\geqslant 0} g_n \frac{z^n}{n!_q}$ then the *q-composition* $\circ_q$ of g with f is defined as

$$(g \circ_q f)(z) = g\big[f(z)\big] = \sum_{n\geqslant 0} g_n \frac{f^{[n]}(z)}{n!_q}.$$

Since $z^{[n]} = z^n$, we have $g[z] = g(z)$.

Throughout this paper, for $n \in \mathbb{N}_0$ let $\mathcal{E}_q(n)$ be the set of the Eulerian generating functions of the form

$$a_n \frac{z^n}{n!_q} + a_{n+1}\frac{z^{n+1}}{(n+1)!_q} + a_{n+2}\frac{z^{n+2}}{(n+2)!_q} + \cdots, \quad a_n = 1.$$

To simplify expressions, we adopt the convention that the coefficients of $z^n/n!_q$ of Eulerian generating functions $g(z), f(z), h(z), \ell(z)$, etc. are always denoted by the g_n, f_n, h_n, ℓ_n, etc.

With a pair of functions $g(z) \in \mathcal{E}_q(0)$ and $f(z) \in \mathcal{E}_q(1)$, let us define a *q-Riordan matrix* $L = (\ell_{n,k})_{n,k\in\mathbb{N}_0}$ as

$$\sum_{n\geqslant k} \ell_{n,k}\frac{z^n}{n!q} = g(z)\frac{f^{[k]}(z)}{k!_q} \tag{1}$$

and it is denoted as $(g(z), f(z))_q$ or $(g, f)_q$ i.e., its k-column has the Eulerian generating function $g(z)f^{[k]}(z)/k!_q \in \mathcal{E}_q(k)$. The product is convolution of two functions so that

$$\ell_{n,k} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q g_{n-j} f_{j,k}, \tag{2}$$

where

$$\frac{f^{[k]}(z)}{k!_q} = \sum_{n\geqslant k} f_{n,k}\frac{z^n}{n!_q}. \tag{3}$$

Since $\ell_{n,k} = 0$ for $n < k$ and $\ell_{n,n} = 1$ for $n \in \mathbb{N}_0$, every q-Riordan matrix is an infinite lower triangular matrix with unit diagonal elements. We note that if $q = 0$ and $q = 1$ then $(g(z), f(z))_q$ reduces to the usual Riordan matrix and the exponential Riordan matrix, respectively.

For instance, let $L = (e_q(z), z)_q$ where $e_q(z) = 1 + z + \frac{z^2}{2!_q} + \frac{z^3}{3!_q} + \cdots$. Since $z^{[k]} = z^k$, it follows from (2) that $\ell_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$. Thus L is a q-Riordan matrix displayed in the matrix form:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1+q & 1 & 0 & 0 \\ 1 & 1+q+q^2 & 1+q+q^2 & 1 & 0 \\ 1 & 1+q+q^2+q^3 & (1+q+q^2)(1+q^2) & 1+q+q^2+q^3 & 1 \\ & & \cdots & & \end{pmatrix}.$$

As observed in [6], the coefficients $f_{n,k}$ of $\frac{f^{[k]}(z)}{k!_q}$ in (3) may be recursively expressed as

$$f_{n,k} = \sum_{j=k-1}^{n-1} \begin{bmatrix} n-1 \\ j \end{bmatrix}_q f_{n-j} f_{j,k-1} \quad \text{for } k \geqslant 1 \text{ and } f_{0,0} = 1 \tag{4}$$

where $f_{j,0} = 0$ for $j \geqslant 1$.

The following theorem is useful for deriving the q-analogues of the composition formula and the exponential formula, see Corollaries 4.4 and 4.5.

Theorem 2.1. *Let $h, \ell \in \mathcal{E}_q(0)$. Then*

$$(g, f)_q(h_0, h_1, \ldots)^T = (\ell_0, \ell_1, \ldots)^T \quad \textit{if and only if} \quad gh[f] = \ell.$$

Proof. Multiplication over to the generating functions yields

$$\ell(z)=\sum_{k\geqslant 0} h_k g(z)\frac{f^{[k]}}{k!_q}=g(z)\sum_{k\geqslant 0} h_k\frac{f^{[k]}}{k!_q}=g(z)h\bigl[f(z)\bigr],$$

as required. □

We call Theorem 2.1 the *fundamental theorem of q-Riordan matrix*, and write as

$$(g,f)_q h=gh[f].$$

This simple observation reduces many computations to merely q-composition of functions.

3. Loop structures of q-Riordan matrices

Let us denote the set of q-Riordan matrices by $\mathcal{R}_q$. An algebraic structure of $\mathcal{R}_q$ is of our interest. Indeed, we shall prove that $\mathcal{R}_q$ forms a loop (a quasigroup with an identity element).

We recall some basic notions of loops. A set S with a binary operation $S\times S\to S$; $(a,b)\mapsto a*b$ is called a *groupoid* if it contains an identity 1, i.e., $a*1=1*a=a$ for all $a\in S$. For $a\in S$, an element $b\in S$ is called a left (resp. right) inverse of a if $b*a=1$ (resp. $a*b=1$). A groupoid S is called a *left* (*resp. right*) loop if there is a unique solution $x\in S$ of the equation $a*x=b$ (resp. $x*a=b$) for all $a,b\in S$. If S is a left and a right loop, then S is called a *loop*.

Let $f(z)=\sum_{k\geqslant 1} f_k z^k/k!_q$. An Eulerian generating function g is called a *left* (*resp. right*) *q-compositional inverse* of f if $(g\circ_q f)(z)=z$ (resp. $(f\circ_q g)(z)=z$). We will use $\overline{f}_L$ and $\overline{f}_R$ to denote the left and right q-compositional inverses of f respectively.

Theorem 3.1. *Let $f(z)=\sum_{k\geqslant 1} f_k z^k/k!_q$. Then there exist unique left and right q-compositional inverses of f if and only if $f_1\neq 0$.*

Proof. Assume that $g(z)=\sum_{k\geqslant 1} g_k z^k/k!_q$ satisfies $g[f(z)]=z$. Since

$$g\bigl[f(z)\bigr]=\sum_{k\geqslant 1} g_k\frac{f^{[k]}(z)}{k!_q}=\sum_{n\geqslant 1}\left(\sum_{k=1}^{n} g_k f_{n,k}\right)\frac{z^n}{n!_q}=g_1 f_1 z+\sum_{n\geqslant 2}\left(g_n f_1^n+\sum_{k=1}^{n-1} g_k f_{n,k}\right)\frac{z^n}{n!_q}=z,$$

we have

$$g_1 f_1=1,\qquad g_n f_1^n+\sum_{k=1}^{n-1} g_k f_{n,k}=0,\quad n\geqslant 2.$$

We can solve the first equation uniquely for g_1 if and only if $f_1\neq 0$. We can then solve the second equation uniquely for g_2, etc. Hence g exists if and only if $f_1\neq 0$.

In a similar way, an analogues result holds for the right q-compositional inverse of f. □

We now define the *q-multiplication* $\ltimes$ on the set $\mathcal{R}_q$ as

$$(g,f)_q\ltimes(h,\ell)_q=\bigl(gh[f],\ell[f]\bigr)_q.$$

We note that $(\mathcal{R}_q,\ltimes)$ is a groupoid with the identity $(1,z)_q$, the usual infinite identity matrix I_∞.

Theorem 3.2. *The set $\mathcal{R}_q$ of all q-Riordan matrices is a loop under the q-multiplication $\ltimes$.*

Proof. First we show that $\mathcal{R}_q$ is a left loop. For any $A,B\in\mathcal{R}_q$, consider the equation $A\ltimes X=B$. Let $A=(g,f)_q$, $B=(h,\ell)_q$ and $X=(x,y)_q$. Since

$$(h,\ell)_q=(g,f)_q\ltimes(x,y)_q=\big(gx[f],y[f]\big)_q,$$

we have $h=gx[f]$ and $\ell=y[f]$. Let $\frac{f^{[k]}}{k!_q}=\sum_{n\geqslant k}f_{n,k}\frac{z^n}{n!_q}$. Since

$$x[f]=\sum_{k\geqslant 0}x_k\frac{f^{[k]}(z)}{k!_q}=\sum_{k\geqslant 0}x_k\left(\sum_{n\geqslant k}f_{n,k}\frac{z^n}{n!_q}\right)=\sum_{n\geqslant 0}\left(\sum_{k=0}^{n}x_kf_{n,k}\right)\frac{z^n}{n!_q},$$

it follows from $h=gx[f]$ that for $n\geqslant 0$

$$h_n=\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}g_{n-j}\left(\sum_{k=0}^{j}x_kf_{j,k}\right).$$

Solving the above equation for x_n, $n=0,1,\dots$, we obtain uniquely $x_0=1$ and for $n\geqslant 1$

$$x_n=h_n-\sum_{j=0}^{n-1}\left\{\begin{bmatrix}n\\j\end{bmatrix}_q g_{n-j}\left(\sum_{k=0}^{j}x_kf_{j,k}\right)+x_jf_{n,j}\right\}. \tag{5}$$

In a similar way, from $\ell=y[f]$ we obtain uniquely $y_1=1$ and for $n\geqslant 2$

$$y_n=\ell_n-\sum_{k=1}^{n-1}y_kf_{n,k}. \tag{6}$$

Since $x\in\mathcal{E}_q(0)$ and $y\in\mathcal{E}_q(1)$ by (5) and (6), $X=(x,y)_q\in\mathcal{R}_q$ and it is a unique solution of $A\ltimes X=B$. Hence $\mathcal{R}_q$ is a left loop.

In a similar way, one can show that $\mathcal{R}_q$ is a right loop. Hence the proof is completed. □

A non-empty subset H of a loop $(S,*)$ is a subloop if H is closed under $*$. We have the following subloops of $\mathcal{R}_q$ under the q-multiplication $\ltimes$:

(i) $\{(1,f)_q\colon f\in\mathcal{E}_q(1)\}$,
(ii) $\{(g,z)_q\colon g\in\mathcal{E}_q(0)\}$,
(iii) $\{(f',f)_q\colon f\in\mathcal{E}_q(1)\}$.

One can directly see that the subloop $\{(g,z)_q\colon g\in\mathcal{E}_q(0)\}$ forms a group with $(g,z)_q^{-1}=(1/g,z)_q$. For instance,

$$\big(e_q(z),z\big)_q^{-1}=\big(E_q(-z),z\big)_q$$

since $e_q(z)E_q(-z)=1$ where $E_q(z)=\sum_{n\geqslant 0}q^{\binom{n}{2}}z^n/n!_q$.

Next, we investigate the left (right) inverse of a q-Riordan matrix. We note by definition that a q-Riordan matrix B is the left (resp. right) *q-inverse* of A if $B\ltimes A=(1,z)_q$ (resp. $A\ltimes B=(1,z)_q$).

Theorem 3.3. *For all $A\in\mathcal{R}_q$, there are unique left and right q-inverses of A.*

Proof. Let $A=(g,f)\in\mathcal{R}_q$. Since

$$\left(\frac{1}{g[\overline{f}_R]},\overline{f}_R\right)_q\ltimes(g,f)_q=(1,z)_q,$$

$(1/g[\overline{f}_R],\overline{f}_R)_q$ is the left q-inverse of $(g(z),f(z))_q$.

Since there is the function v such that $v[f]=1/g$, we have

$$(g,f)_q\ltimes(v,\overline{f}_L)_q=\big(gv[f],\overline{f}_L[f]\big)_q=(1,z)_q.$$

Thus $(v,\overline{f}_L)_q$ is the right q-inverse of $(g,f)_q$. □

We denote the left q-inverse and right q-inverse of A by A_L^{-1} and A_R^{-1} respectively.

Example 1. Let $A=(e_q(z),e_q(z)-1)_q$. Since $e_q[\mathrm{loq}(1+z)]=1+z$ where $\mathrm{loq}(1+z)=\sum_{n\geqslant 1}(-1)^{n-1}\times(n-1)!_q z^n/n!_q$, we have

$$\left(\frac{1}{1+z},\mathrm{loq}(1+z)\right)_q \ltimes \left(e_q(z),e_q(z)-1\right)_q=\left(\frac{1}{1+z}e_q\big[\mathrm{loq}(1+z)\big],e_q\big[\mathrm{loq}(1+z)\big]-1\right)_q$$
$$=(1,z)_q.$$

Thus $A_L^{-1}=(1/(1+z),\mathrm{loq}(1+z))_q$.

Theorem 3.4. *In the loop $\mathcal{R}_q$, the equation $A\ltimes X=B$ (resp. $Y\ltimes A=B$) has a unique solution $X=A_L^{-1}\ltimes B$ (resp. $Y=B\ltimes A_R^{-1}$) if and only if $A\in\{(g,z)_q\colon g\in\mathcal{E}_q(0)\}$.*

Proof. Let $A=(g,f)_q$ and $B=(h,\ell)_q$. Assume that $X=A_L^{-1}\ltimes B$ is the unique solution of $A\ltimes X=B$. Since $A_L^{-1}=(\frac{1}{g[\bar{f}_R]},\bar{f}_R)_q$, from $A\ltimes(A_L^{-1}\ltimes B)=B$ we have $\ell[\bar{f}_R]\circ_q f=\ell$. The uniqueness of X follows $f=z$. Hence $A\in\{(g,z)_q\colon g\in\mathcal{E}_q(0)\}$.

Conversely, let $A=(g,z)$. Consider the equation $A\ltimes X=B$ where $B=(h,\ell)_q$. Since $A_L^{-1}=(1/g,z)_q$, we have

$$A\ltimes\left(A_L^{-1}\ltimes B\right)=(g,z)_q\ltimes(h/g,\ell)=(h,\ell)=B.$$

Hence $X=A_L^{-1}\ltimes B$ is a unique solution of $A\ltimes X=B$ which completes the proof. □

We note that left q-composition inverse of f and right q-composition inverse of f are usually different. Thus the left inverse A_L^{-1} and the right inverse A_R^{-1} of a q-Riordan matrix A are different. Further, the associative law on the set $\mathcal{R}_q$ does not hold since if it holds then

$$A_L^{-1}=A_L^{-1}\ltimes I_\infty=A_L^{-1}\ltimes\left(A\ltimes A_R^{-1}\right)=\left(A_L^{-1}\ltimes A\right)\ltimes A_R^{-1}=I_\infty\ltimes A_R^{-1}=A_R^{-1}.$$

Hence $\mathcal{R}_q$ does not satisfy the associativity.

Theorem 3.5. *The loop $\mathcal{R}_q$ is a group if and only if $q=0$ or $q=1$.*

Proof. Let $A=(g,f)_q$, $B=(h,\ell)_q$ and $C=(u,v)_q\in\mathcal{R}_q$. Assume that both $q\neq 0$ and $q\neq 1$. If $(A\ltimes B)\ltimes C=A\ltimes(B\ltimes C)$ holds in $\mathcal{R}_q$ then

$$(v\circ_q\ell)\circ_q f=v\circ_q(\ell\circ_q f).$$

Let $(v\circ_q\ell)\circ_q f=\sum_{n\geqslant 1}x_nz^n/n!_q$ and $v\circ_q(\ell\circ_q f)=\sum_{n\geqslant 1}y_nz^n/n!_q$. Using (4), it can be shown that $x_4-y_4=f_2\ell_2v_2q(q-1)$. Since $q\neq 0$ and $q\neq 1$, $x_4=y_4$, this leads a contradiction to the associative law in $\mathcal{R}_q$. Hence $\mathcal{R}_q$ is not a group.

Conversely, if q is 0 or 1, then $\mathcal{R}_q$ forms a group. Hence the result follows. □

For a loop S and $a\in S$, define $\lambda_a\colon S\to S$ by $\lambda_a(x)=ax$. Since each map λ_a is bijective, $\lambda(S):=\{\lambda_a\mid a\in S\}$ generates a group $\mathcal{M}(S):=\langle\lambda_a;a\in S\rangle$. Now we define the *precession map* $\delta_{a,b}$ for all $a,b\in S$ by

$$\delta_{a,b}:=\lambda_{ab}^{-1}\lambda_a\lambda_b.$$

Obviously, these mappings are characterized by the property

$$a(bx)=(ab)\delta_{a,b}(x)\quad\text{for all } x\in S.$$

Further, the set $\langle\delta_{a,b};a,b\in S\rangle$ is a subset of $\mathcal{M}(S)$ generated by all precession maps $\delta_{a,b}$, see [8] for more details of precession maps.

Theorem 3.6. *If $A, B \in \{(g,z)_q\colon\ g \in \mathcal{E}_q(0)\}$, then $(A \ltimes B) \ltimes X = A \ltimes (B \ltimes X)$ for all $X \in \mathcal{R}_q$.*

Proof. Let $A = (g,z)_q$, $B = (h,z)_q$ and $X = (u,v)_q$. Then we have

$$(A \ltimes B) \ltimes X = (gh,z)_q \ltimes (u,v)_q = (ghu,v)_q = (g,z)_q \ltimes (hu,v)_q = A \ltimes (B \ltimes X)$$

which completes the proof. □

From Theorem 3.6, we obtain the following result.

Corollary 3.7. *Let $\mathcal{A} = \{(g,z)_q : g \in \mathcal{E}_q(0)\}$. Then $\langle \delta_{A,B};\, A, B \in \mathcal{A}\rangle$ is trivial.*

4. Combinatorial significance of the q-Riordan matrix

In this section, we prove that if G and F are the Eulerian generating functions whose coefficients are associated to the counting functions $\mathbb{N}_0 \to \mathbb{C}[\![q]\!]$, then the q-Riordan matrix $(G,F)_q$ can be applied to the enumeration problem of set partitions by block inversions.

Let $\mathbf{n}$ denote the n-set $\{1,2,\ldots,n\}$. By a *k-partition* of $\mathbf{n}$ we mean a partition of the set $\mathbf{n}$ into k non-empty disjoint sets. Each set is called a block of the partition. We denote the family of all k-partitions of $\mathbf{n}$ by $\Pi_{n,k}$.

Consider a k-partition $\pi = \{B_1,\ldots,B_k\} \in \Pi_{n,k}$. Let $\ell(B_i)$ denote the least element in the block B_i. By an *inversion of a pair of blocks* (B_i,B_j) we shall mean a pair $(a,b) \in B_i \times B_j$ such that $a > b$ but $\ell(B_i) < \ell(B_j)$. From now on, we will assume that $1 = \ell(B_1) < \cdots < \ell(B_k)$. We note that $1 \in B_1$. If the elements of blocks are arranged in increasing order then the block inversions of (B_i,B_j) coincide with inversions of the permutation $\sigma := B_i \cup B_j$ in which the braces are ignored and the elements are treated as a linear array preserving the order. We denote the number of block inversions of (B_i,B_j) by $\operatorname{inv}(B_i,B_j)$. For example, if $B_1 = \{1,2,5\}$ and $B_2 = \{3,4,6,7\}$ then the block inversions are $(5,3)$, $(5,4)$, which are the same as inversions of the permutation $\sigma = 1\,2\,5\,3\,4\,6\,7$.

The number of *inversions of a k-partition* $\pi = \{B_1,\ldots,B_k\} \in \Pi_{n,k}$ is now denoted by $\operatorname{inv}(\pi)$ and is defined by

$$\operatorname{inv}(\pi) = \sum_{1 \leqslant i < j \leqslant k} \operatorname{inv}(B_i,B_j) \quad (k \geqslant 2), \tag{7}$$

where $\operatorname{inv}(\pi) = 0$ if $k = 1$.

In particular, if $\mathcal{B}_X$ is a collection of bi-partitions $\{A,B\}$ of $X \subset \mathbf{n}$ with $|X| \geqslant 2$ in which A contains the least element of X and $|B| = j$ then it follows from [6, Lemma 5.1] that

$$\sum_{\{A,B\} \in \mathcal{B}_X} q^{\operatorname{inv}(A,B)} = \begin{bmatrix} |X|-1 \\ j \end{bmatrix}_q. \tag{8}$$

Let us associate to the counting function $f : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ the Eulerian generating function

$$F(z) = \sum_{n \geqslant 0} f(n) \frac{z^n}{n!_q}.$$

To simplify expressions, we adopt the convention that the coefficients of $z^n/n!_q$ of the Eulerian generating functions $F(z), G(z), H(z), L(z)$ associated to the counting functions $f, g, h, \ell : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ are denoted by the $g(n)$, $f(n)$, $h(n)$, $\ell(n)$, respectively.

We are now able to give the main theorem of this section, which leads to the combinatorial significance of the q-Riordan matrix. First we observe that Eq. (4) may be also written as

$$f_{n,k} = \sum_{0=j_0<j_1<\cdots<j_k=n} \begin{bmatrix} j_k-1 \\ j_{k-1} \end{bmatrix}_q \begin{bmatrix} j_{k-1}-1 \\ j_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} j_2-1 \\ j_1 \end{bmatrix}_q f_{j_k-j_{k-1}} f_{j_{k-1}-j_{k-2}} \cdots f_{j_1-j_0}. \tag{9}$$

Theorem 4.1. *Let $g, f : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ be counting functions with $g(0) = 1$, $f(0) = 0$ and $f(1) = 1$. If $h_k : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ for fixed k is defined by*

$$h_k(n) = \sum_{\pi=\{B_1,\ldots,B_{k+1}\}\in\Pi_{n+1,k+1}} g\big(|B_1|-1\big) f\big(|B_2|\big)\cdots f\big(|B_{k+1}|\big) q^{\operatorname{inv}(\pi)}, \tag{10}$$

then the array $(h_k(n))_{n,k\in\mathbb{N}_0}$ may be expressed as the q-Riordan matrix given by $(G, F)_q$.

Proof. For fixed positive integers $b_1, \ldots, b_{k+1}$ such that $b_1 + \cdots + b_{k+1} = n + 1$, let

$$\widetilde{\Pi}_{n+1,k+1} = \big\{\{B_1, \ldots, B_{k+1}\} \in \Pi_{n+1,k+1} \colon |B_1| = b_1, \ldots, |B_{k+1}| = b_{k+1}\big\}.$$

For $\alpha = \{B_1, \ldots, B_{k+1}\} \in \widetilde{\Pi}_{n+1,k+1}$, consider the bi-partition $\{B_i, C_i\}$ of $B_i \cup C_i$ where $C_i = (\mathbf{n+1}) - \bigcup_{j=1}^{i} B_j$ and $|C_i| = n + 1 - b_1 - \cdots - b_i$. Since

$$\operatorname{inv}(\alpha) = \sum_{i=1}^{k} \operatorname{inv}(B_i, C_i), \tag{11}$$

it follows from (8) and (11) that

$$\begin{aligned}\sum_{\alpha\in\widetilde{\Pi}_{n+1,k+1}} q^{\operatorname{inv}(\alpha)} &= \sum_{\alpha\in\widetilde{\Pi}_{n+1,k+1}} q^{\sum_{i=1}^{k}\operatorname{inv}(B_i,C_i)} \\ &= \prod_{i=1}^{k}\Bigg(\sum_{\{B_i,(\mathbf{n+1})-\bigcup_{j=1}^{i}B_j\}} q^{\operatorname{inv}(B_i,C_i)}\Bigg) \\ &= \begin{bmatrix} n \\ n+1-b_1\end{bmatrix}_q \begin{bmatrix} n-b_1 \\ n+1-b_1-b_2\end{bmatrix}_q \cdots \begin{bmatrix} n-b_1-\cdots-b_{k-1} \\ n+1-b_1-\cdots-b_k\end{bmatrix}_q.\end{aligned} \tag{12}$$

For brevity, let $j_{k-i+1} = n + 1 - \sum_{\ell=1}^{i} b_\ell$ for $i = 1, 2, \ldots, k$. Since $j_1 < j_2 < \cdots < j_k$, it follows from (12) that the formula (10) can be expressed as

$$\begin{aligned} h_k(n) &= \sum_{\pi=\{B_1,\ldots,B_{k+1}\}\in\Pi_{n+1,k+1}} g\big(|B_1|-1\big) f\big(|B_2|\big)\cdots f\big(|B_{k+1}|\big) q^{\operatorname{inv}(\pi)} \\ &= \sum_{b_1+\cdots+b_{k+1}=n+1}\Bigg(g(b_1-1)f(b_2)\cdots f(b_{k+1}) \sum_{\alpha\in\widetilde{\Pi}_{n+1,k+1}} q^{\operatorname{inv}(\alpha)}\Bigg) \\ &= \sum_{0<j_1<\cdots<j_k<n+1} g(n-j_k) f(j_k-j_{k-1})\cdots f(j_1)\begin{bmatrix} n \\ j_k\end{bmatrix}_q \begin{bmatrix} j_k-1 \\ j_{k-1}\end{bmatrix}_q \cdots \begin{bmatrix} j_2-1 \\ j_1\end{bmatrix}_q. \end{aligned}$$

Letting $F^{[k]}(z)/k!_q = \sum_{n\geqslant k} f(n,k) z^n/n!_q$, from (2) and (9) we obtain

$$h_k(n) = \sum_{j=k}^{n} \begin{bmatrix} n \\ j\end{bmatrix}_q g(n-j) f(j,k),$$

which is the (n,k)-entry of the q-Riordan matrix $(G(z), F(z))_q$. Hence the result follows as required. $\square$

Along the similar lines of the opposite direction of the proof in Theorem 4.1 we have the following corollary.

Corollary 4.2. *Let $F(z)$ be an Eulerian generating functions associated to the counting functions $f:\mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ with $f(0)=0$ and $f(1)=1$. If $(1, F(z))_q = (h_{n,k})_{n,k\in\mathbb{N}_0}$ then*

$$h_{n,k} = \sum_{\pi=\{B_1,\ldots,B_k\}\in\Pi_{n,k}} f\big(|B_1|\big)\cdots f\big(|B_k|\big)q^{\mathrm{inv}(\pi)} \quad (n \geqslant k \geqslant 1)$$

with $h_{0,0}=1$ and $h_{n,0}=0$ for $n \geqslant 1$.

Proof. Let $(1, F(z))_q = (h_{n,k})_{n,k\in\mathbb{N}_0}$ be the corresponding q-Riordan matrix. Since the first column of $(h_{n,k})_{n,k\in\mathbb{N}_0}$ is $(1,0,\ldots)^T$, we clearly obtain $h_{0,0}=1$ and $h_{n,0}=0$ for $n \geqslant 1$. Let $n,k \geqslant 1$. Since $\sum_{n\geqslant k} h_{n,k}z^n/n!_q = F^{[k]}(z)/k!_q$ by (1), it follows from (9) that

$$\begin{aligned} h_{n,k} = &\sum_{0=j_0<j_1<\cdots<j_{k-1}<j_k=n} \begin{bmatrix} j_k-1 \\ j_{k-1} \end{bmatrix}_q \begin{bmatrix} j_{k-1}-1 \\ j_{k-2} \end{bmatrix}_q \cdots \begin{bmatrix} j_2-1 \\ j_1 \end{bmatrix}_q \\ &\times f(j_k-j_{k-1})f(j_{k-1}-j_{k-2})\cdots f(j_1-j_0) \end{aligned} \tag{13}$$

Set $b_i = j_{k-i+1}-j_{k-i}$ for $i=1,2,\ldots,k$. Since $\sum_{i=1}^k b_i = j_k - j_0 = n$ and $b_i > 0$, it follows from (12) and (13) that

$$\begin{aligned} h_{n,k} &= \sum_{b_1+\cdots+b_k=n} f(b_1)f(b_2)\cdots f(b_k)\begin{bmatrix} n-1 \\ n-b_1 \end{bmatrix}_q \begin{bmatrix} n-1-b_1 \\ n-b_1-b_2 \end{bmatrix}_q \cdots \begin{bmatrix} n-1-b_1-\cdots-b_{k-2} \\ n-b_1-\cdots-b_{k-1} \end{bmatrix}_q \\ &= \sum_{b_1+\cdots+b_k=n} f(b_1)f(b_2)\cdots f(b_k)\Bigg(\sum_{\alpha\in\widetilde{\Pi}_{n,k}} q^{\mathrm{inv}(\alpha)}\Bigg) \\ &= \sum_{\pi\in\Pi_{n,k}} f\big(|B_1|\big)f\big(|B_2|\big)\cdots f\big(|B_k|\big)q^{\mathrm{inv}(\pi)} \quad (n,k \geqslant 1). \end{aligned}$$

This completes the proof. □

By using the fundamental theorem of q-Riordan matrix, we obtain the following theorem.

Theorem 4.3. *Let $g, f, \ell : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ be counting functions with $g(0)=1$, $f(0)=0$ and $f(1)=1$, and let $h:\mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ be defined through*

$$h(n) = \sum_{k=0}^{n} \sum_{\pi=\{B_1,\ldots,B_{k+1}\}\in\Pi_{n+1,k+1}} g\big(|B_1|-1\big)f\big(|B_2|\big)\cdots f\big(|B_{k+1}|\big)\ell(k)q^{\mathrm{inv}(\pi)}.$$

Then $H(z) = G(z)L[F(z)]$.

Proof. By Theorems 2.1 and 4.1 we immediately obtain

$$H(z) = \big(G(z), F(z)\big)_q L(z) = G(z)L\big[F(z)\big],$$

as required. □

In particular, if $g(0)=1$ and $g(n)=0$ for all $n \geqslant 1$ i.e., $G(z)=1$ then from Theorem 4.1 we obtain the q-analogues of the *composition formula* and of the *exponential formula* addressed in the famous two text books respectively by Aigner [1, p. 113] and Stanley [12, pp. 3–5].

Corollary 4.4 *(The q-analogue of the composition formula). Let $f, \ell : \mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ be the counting functions with $f(0)=0$ and $f(1)=1$, and let $h:\mathbb{N}_0 \to \mathbb{C}[\![q]\!]$ be defined through*

$$h(n)=\sum_{k=1}^{n}\sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_{n,k}} f\big(|B_1|\big)\cdots f\big(|B_k|\big)\ell(k)q^{\operatorname{inv}(\pi)}, \qquad h(0)=\ell(0).$$

Then $H(z)=L[F(z)]$.

In particular, if $\ell(n)=1$ for all $n\geqslant 0$ i.e., $L(z)=e_q(z)$ then we obtain the following.

Corollary 4.5 *(The q-analogue of the exponential formula). Let* $f:\mathbb{N}_0\to\mathbb{C}[\![q]\!]$ *be the counting function with* $f(0)=0$ *and* $f(1)=1$, *and let* $h:\mathbb{N}_0\to\mathbb{C}[\![q]\!]$ *be defined through*

$$h(n)=\sum_{k=1}^{n}\sum_{\pi=\{B_1,\dots,B_k\}\in\Pi_{n,k}} f\big(|B_1|\big)f\big(|B_2|\big)\cdots f\big(|B_k|\big)q^{\operatorname{inv}(\pi)} \quad (n\geqslant 1), \qquad h(0)=1.$$

Then $H(z)=e_q[F(z)]$.

We end this section by obtaining a new combinatorial interpretation for the q-Stirling numbers $\left\{{n \atop k}\right\}_q$ of the second kind [7] defined as

$$\left\{{n \atop k}\right\}_q=\sum_{\pi\in\Pi_{n,k}} q^{\operatorname{wt}(\pi)} \tag{14}$$

where $\operatorname{wt}(\pi)$ is given by the generic definition of the weight based on the relabeling weights; see [7] for more information about $\operatorname{wt}(\pi)$. If we observe the generating function [7]:

$$\sum_{n\geqslant k}\left\{{n \atop k}\right\}_q\frac{z^n}{n!_q}=\frac{(e_q(z)-1)^{[k]}}{k!_q},$$

then we obtain the q-Riordan matrix:

$$\left(\left\{{n \atop k}\right\}_q\right)_{n,k\in\mathbb{N}_0}=\big(1,e_q(z)-1\big)_q.$$

Since $e_q(z)-1=\sum_{n\geqslant 1} z^n/n!_q$, it follows from Corollary 4.2 that

$$\left\{{n \atop k}\right\}_q=\sum_{\pi\in\Pi_{n,k}} q^{\operatorname{inv}(\pi)}, \tag{15}$$

which gives a new combinatorial interpretation for $\left\{{n \atop k}\right\}_q$.

To illustrate, let us consider the 2-partitions of $\{1,2,3,4\}$. Then we obtain:

$\pi\in\Pi_{4,2}$	$\operatorname{inv}(\pi)$	$\operatorname{wt}(\pi)$	$\pi\in\Pi_{4,2}$	$\operatorname{inv}(\pi)$	$\operatorname{wt}(\pi)$
$\{\{1\},\{2,3,4\}\}$	0	0	$\{\{1,2,4\},\{3\}\}$	1	2
$\{\{1,2\},\{3,4\}\}$	0	0	$\{\{1,4\},\{2,3\}\}$	2	2
$\{\{1,2,3\},\{4\}\}$	0	0	$\{\{1,3,4\},\{2\}\}$	2	1
$\{\{1,3\},\{2,4\}\}$	1	1			

Thus

$$\left\{{4 \atop 2}\right\}_q=\sum_{\pi\in\Pi_{4,2}} q^{\operatorname{inv}(\pi)}=3+2q+2q^2=\sum_{\pi\in\Pi_{4,2}} q^{\operatorname{wt}(\pi)}.$$

We note that if $k\geqslant 3$ then our formula (15) is much simpler than (14) since the relabeling weights are not necessarily.

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