Discrete Applied Mathematics 157 (2009) 1696–1701

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Riordan group involutions and the Δ-sequence[☆]

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ARTICLE INFO

Article history:
Received 13 December 2007
Received in revised form 20 December 2008
Accepted 31 January 2009
Available online 28 February 2009

Keywords:
Riordan matrix
A, Z, and Δ-sequences
Pseudo-involution
Bell subgroup

ABSTRACT

Several important combinatorial arrays, after inserting some minus signs, turn out to be involutions when considered as lower triangular matrices. Among these are the Pascal, RNA, and directed animal matrices. These examples and many others are in the Bell subgroup of the Riordan group. We characterize all such pseudo-involutions by means of a single sequence called the Δ-sequence. Finally we compute the Δ-sequences for the powers of a pseudo-involution in the Bell subgroup.

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1. Introduction

We will use a pair of formal power series $g(z) = g_0 + g_1 z + g_2 z^2 + \cdots$ and $f(z) = f_1 z + f_2 z^2 + \cdots$ with $g_0 \neq 0$. An infinite lower triangular matrix $D = (d_{n,k})_{n,k\geq 0}$ is called a *Riordan matrix* if its column $k \geq 0$ has generating function $g(z)(f(z))^k$, i.e., $d_{n,k} = [z^n]g(z)(f(z))^k$ where $[z^n]$ is the coefficient operator. If in addition $f_1 \neq 0$ the Riordan matrix is called *proper* [10] and will be an element of the Riordan group to be defined later. With little loss of generality we will require that $g_0 = 1$. As is usual, we will write $D = (g(z), f(z))$.

Note that elements in the leftmost column of a Riordan matrix $D = (g(z), f(z))$ are $g_0, g_1, g_2, \ldots$. Since

$$d_{n,k} = [z^n]g(z)(f(z))^k = \sum_{j=0}^{n}[z^j]f(z)[z^{n-j}]g(z)(f(z))^{k-1},$$

every element $d_{n,k}$ for $n, k \geq 1$ can be expressed as a linear combination of the elements in the preceding column together with the sequence $(f_1, f_2, \ldots)$, i.e.,

$$d_{n,k} = \sum_{j=1}^{n} f_j d_{n-j,k-1} = f_1 d_{n-1,k-1} + f_2 d_{n-2,k-1} + \cdots + f_n d_{0,k-1}.$$

Originally, the term Riordan matrix was introduced by Shapiro et al. in [8]. Riordan matrices were later characterized by some other sequences found by Merlini et al. [5] and Sprugnoli [10] but the original idea appeared in Rogers [6]. In fact, every element of a Riordan matrix can be expressed as a linear combination of the elements in the preceding row. That is, if $D = (d_{n,k})_{n,k\geq 0}$ is a Riordan matrix, there exist unique sequences $A = (a_0, a_1, a_2, \ldots)$ and $Z = (z_0, z_1, z_2, \ldots)$ with $a_0 \neq 0$, $z_0 \neq 0$ such that

☆ This paper was supported by Faculty Research Fund, Sungkyunkwan University, 2008.

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doi:10.1016/j.dam.2009.01.012

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$, $(k, n = 0, 1 \ldots)$,
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, $(n = 0, 1, \ldots)$.

Actually, the existence of the sequence A assures that the array is Riordan, and the sequence Z exists for every lower triangular array having no zeros in the main diagonal. The coefficients $a_0, a_1, a_2, \ldots$ and $z_0, z_1, z_2, \ldots$ appearing in (i) and (ii) are called the *A-sequence* and the *Z-sequence* of D, respectively. If $A(z)$ and $Z(z)$ are the generating functions of the corresponding sequences A and Z of a Riordan matrix $D = (g(z), f(z))$ then the functions $g(z), f(z), A(z), Z(z)$ are connected by the relations:

$$f(z) = zA(f(z)), \quad \text{and} \quad g(z) = 1/(1 - zZ(f(z))). \tag{1}$$

The set of proper Riordan matrices forms a group called the *Riordan group* with the operation being matrix multiplication $*$. In terms of the generating functions this works out as

$$(g(z), f(z)) * (h(z), \ell(z)) = (g(z)h(f(z)), \ell(f(z))).$$

It is easy to see that the identity element of the Riordan group is $I = (1, z)$, the usual identity matrix, and the inverse of $(g(z), f(z))$ is $\left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right)$, where $\bar{f}(z)$ is the compositional inverse of $f(z)$, i.e., $f(\bar{f}(z)) = \bar{f}(f(z)) = z$.

Usually, a proper Riordan matrix of combinatorial interest will have all nonnegative entries on and below the main diagonal and cannot itself have order 2. If we restrict all entries to being real numbers then any element of finite order must have order 1 or 2. An element D of the Riordan group is called a *pseudo-involution* if DM has order 2 where $M = (1, -z) = \mathrm{diag}(1, -1, 1, -1, \ldots)$. See [2–4] for more information regarding pseudo-involutions. Equivalent conditions are that MD is an involution or that $D^{-1} = MDM$ which can be rephrased as $D^{-1} = \left((-1)^{n-k} d_{n,k}\right)_{n,k\geq 0}$.

In this paper, we find a new single defining sequence for these Riordan matrices that are essentially self-inverse. We call this new sequence the *Δ-sequence* and tabulate about ten examples at the end of this article. Further we compute the Δ-sequence for powers of a pseudo-involution in the *Bell subgroup* [7] which is the set of all proper Riordan matrices of the form $\left(\frac{f(z)}{z}, f(z)\right)$.

2. Δ-sequence of the Riordan matrix

We begin with an example. The RNA matrix [2] is given by

$$\mathrm{RNA} = \left(\frac{f(z)}{z}, f(z)\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & \boxed{3} & 1 & 0 & 0 & 0 \\ \boxed{4} & \boxed{6} & 6 & 4 & 1 & 0 & 0 \\ 8 & \mathbf{13} & 13 & 10 & 5 & 1 & 0 \\ 17 & 28 & 30 & 24 & 15 & 6 & 1 \\ & & & \cdots & & & \end{bmatrix}$$

where $f(z) = \frac{1-z+z^2-\sqrt{(1-z+z^2)^2-4z^2}}{2z}$. From (1), it is easy to see that the RNA matrix has the A-sequence $(1, 1, 0, 1, -1, 2, \ldots)$ and the Z-sequence $(1, 0, 1, -1, 2, \ldots)$. By way of contrast it will turn out that this is the unique element in the Bell subgroup with the Δ-sequence $(1, 1, 1, \ldots)$. There is also a connection with biology. The reason this is called the RNA matrix is that the elements in the left hand column are the number of possible RNA secondary structures on a chain of length n. The remaining elements of the matrix count such chains with k vertices designated as the start of a yet to be completed link.

Now, we observe that every element of $\mathrm{RNA} = (r_{n,k})_{n,k\geq 0}$ can be expressed as a linear combination of the elements in the diagonal starting just above the element and going up and to the right together with the one element up one row and to the left. For example,

$$r_{5,0} = 8 = 1 \cdot 4 + 1 \cdot 3 + 1 \cdot 1 = 1 \cdot r_{4,0} + 1 \cdot r_{3,1} + 1 \cdot r_{2,2},$$
$$r_{5,1} = 13 = 4 + 1 \cdot 6 + 1 \cdot 3 = r_{4,0} + 1 \cdot r_{4,1} + 1 \cdot r_{3,2}$$

and in general,

$$r_{n+1,k} = r_{n,k-1} + \sum_{j\geq 0} 1 \cdot r_{n-j,k+j} \tag{2}$$

where $r_{n,k-1} = 0$ if $k = 0$.

There are two other examples worth mentioning at this point. One is the Pascal triangle matrix which is well known to be a pseudo-involution in this equivalent form. If $P = \left(\binom{n}{k}\right)_{n,k\geq 0}$ then $P^{-1} = \left((-1)^{n-k}\binom{n}{k}\right)_{n,k\geq 0}$. The Δ-sequence for Pascal's triangle is $(1, 0, 0, 0, \ldots)$. The other example comes from physics and is the directed animals matrix discussed in Section 3.

Motivated by these examples, we focus our attention on the new sequence of Riordan matrices whose entries satisfy a linear combination like (2).

Let $D = (d_{n,k})_{n,k\geq 0}$ be a Riordan matrix. We say that D has a *Δ-sequence* $(b_0, b_1, b_2, \ldots)$ if there exist numbers $b_0, b_1, b_2, \ldots$ which are independent of n and k such that

$$d_{n+1,k} = d_{n,k-1} + \sum_{j\geq 0} b_j \cdot d_{n-j,k+j}$$

where $d_{n,k-1} = 0$ if $k = 0$. We denote the corresponding generating function as $\Delta(z)$.

Lemma 2.1. *Let $D = (g(z), f(z))$ be a Riordan matrix. Then D has a Δ-sequence $(b_0, b_1, b_2, \ldots)$ if and only if $g(z)$ and $f(z)$ satisfy the following:*

(i) $g(z) = 1 + zg(z)\Delta(zf(z))$,
(ii) $f(z) = z + zf(z)\Delta(zf(z))$.

Proof. We prove this using generating functions. By the definition, D has a Δ-sequence $(b_0, b_1, b_2, \ldots)$ if and only if for $k \geq 1$ we have

$$g(z)(f(z))^k = z(g(z)(f(z))^{k-1} + b_0 g(z)(f(z))^k + b_1 zg(z)(f(z))^{k+1} + \cdots), \tag{3}$$

and for $k = 0$, recalling that $g(0) = g_0 = 1$, we have

$$g(z) = 1 + zg(z)(b_0 + b_1 zf(z) + b_2(zf(z))^2 + \cdots) = 1 + zg(z)\Delta(zf(z)).$$

Dividing both sides of (3) by $g(z)(f(z))^{k-1}$ yields

$$f(z) = z + zf(z)(b_0 + b_1 zf(z) + b_2(zf(z))^2 + \cdots) = z + zf(z)\Delta(zf(z)).$$

Hence the proof is completed. ■

Note that $zg(z) = f(z)$ in Lemma 2.1 which is precisely the condition that an element is in the Bell subgroup.

Naturally not every Riordan matrix in the Bell subgroup has a Δ-sequence. For instance,

$$\left(\frac{1}{\sqrt{1-4z}}, \frac{z}{\sqrt{1-4z}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 \\ 20 & 16 & 6 & 1 & 0 \\ 70 & 64 & 30 & 8 & 1 \\ & & \cdots & & \end{bmatrix}$$

is in the Bell subgroup but has no Δ-sequence.

We now characterize all Riordan matrices with a Δ-sequence. In this paper, we will abbreviate a pseudo-involution in the Bell subgroup to *PIBell matrix*. We note that the RNA matrix is a PIBell matrix.

From (1), it is easy to show that if $D = (g(z), f(z))$ is a pseudo-involution then

$$A(z) = \frac{-z}{f(-z)} \quad \text{and} \quad Z(z) = \frac{g(-z)-1}{f(-z)}. \tag{4}$$

Theorem 2.2. *Let $D = (g(z), f(z))$ be a Riordan matrix. Then D has a Δ-sequence if and only if D is a PIBell matrix.*

Proof. Suppose that $D = (g(z), f(z))$ has a Δ-sequence with the generating function $\Delta(z) = \sum_{n=0}^{\infty} b_n z^n$. From Lemma 2.1 it is obvious that $g(z) = \frac{f(z)}{z}$ and hence D is an element of the Bell subgroup.

Now, we show that $D = \left(\frac{f(z)}{z}, f(z)\right)$ is a pseudo-involution. It suffices to show that $(DM)^2 = (1, z)$, i.e., $f(-f(z)) = -z$. Let $y = -f(z)$. It follows from (ii) of Lemma 2.1 that $-y = z - yz\Delta(-zy)$. By transposing y and z, we obtain $-z = y - zy\Delta(-yz)$. This implies that y is a compositional inverse of $-f(z)$, i.e. $-f(-f(z)) = z$. Hence $D = \left(\frac{f(z)}{z}, f(z)\right)$ is a pseudo-involution.

Conversely, suppose that $D = \left(\frac{f(z)}{z}, f(z)\right)$ is a pseudo-involution. It follows from (4) and $g(z) = \frac{f(z)}{z}$ that

$$Z(-z) = \frac{f(z)-z}{zf(z)}. \tag{5}$$

We claim that D has the generating function $\Delta(z)$ such that $\Delta(zf(z)) = Z(-z)$. In fact, from (5) we have

$$z + zf(z)\Delta(zf(z)) = z + zf(z)\left(\frac{f(z)-z}{zf(z)}\right) = f(z),$$

which shows (ii) of Lemma 2.1, and clearly (i) holds for $g(z) = \frac{f(z)}{z}$. Hence by Lemma 2.1, D has the Δ-sequence satisfying $\Delta(zf(z)) = Z(-z)$. Thus the proof is completed. ■

Theorem 2.2 allows us to obtain a pseudo-involution associated with that sequence in the Bell subgroup whenever Δ-sequence is given. However the formula (ii) of Lemma 2.1 cannot be inverted because $zf(z)$ does not have a compositional inverse.

It is known [10] that $D = (g(z), f(z))$ is a Riordan matrix in the Bell subgroup if and only if A-sequence and Z-sequence are connected by $A(z) = 1 + zZ(z)$. Hence if $D = \left(\frac{f(z)}{z}, f(z)\right)$ is a pseudo-involution then the functions $f(z), A(z), \Delta(z)$ are connected by the relation

$$A(-z) = 1 - z\Delta(zf(z)).$$

Theorem 2.3. *Let $D = (d_{n,k})_{n,k\geq 0} = (g(z), f(z))$ be a PIBell matrix with a Δ-sequence $(b_0, b_1, b_2, \ldots)$. Then D can be expressed as a linear combination in terms of $S^k\bar{D}S^{k+1}$, i.e.,*

$$D = \bar{D} + \sum_{k=0}^{\infty} b_k S^k \bar{D} S^{k+1} \tag{6}$$

where $\bar{D} = (1, f(z))$, and $S = [s_{n,k}]_{n,k\geq 0}$ is the infinite $(0, 1)$-shifted matrix defined by $s_{n,k} = 1$ if $k = n - 1$ for $n \geq 0$ and 0 otherwise.

Proof. Note that the (n, k)-entry $(\bar{D})_{n,k}$ of $\bar{D}$ is $d_{n-1,k-1}$ and the multiplication of S^j to the left (or right) of D yields the shifted matrix S^jD (or DS^j) whose (n, k)-entry is $(S^jD)_{n,k} = (\bar{D})_{n-j,k}$ (or $(DS^j)_{n,k} = (\bar{D})_{n,k+j}$, resp.). By the definition of the Δ-sequence, we have

$$d_{n,k} = d_{n-1,k-1} + \sum_{j\geq 0} b_j d_{n-j-1,k+j} = (\bar{D})_{n,k} + \sum_{j\geq 0} b_j (S^j \bar{D} S^{j+1})_{n,k},$$

which proves (6). ■

3. Application

Let us consider the directed animal matrix [1,7] given by

$$(1 + zM(z), z(1 + zM(z))) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 4 & 6 & 6 & 4 & 1 & 0 \\ 9 & 13 & 13 & 10 & 5 & 1 \\ & & \cdots & \cdots & & \end{bmatrix},$$

where $M(z) = \sum_{n\geq 0} M_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$ is the generating function for the Motzkin numbers M_n. It is known [7] that the entry $m(n, k)$ of the directed animal matrix counts the number of single-source directed animals consisting of n points of which k are on the x-axis. Further, there is an interesting hook shaped recursion:

$$m(n + 1, k) = m(n, k - 1) + \sum_{j\geq 0} m(n - 1, k - 1 + j)$$

where $m(n, k - 1) = 0$ if $k = 0$.

The directed animal matrix is a PIBell matrix and Δ-sequence is $(1, 1, 2, 5, 14\ldots)$, the Catalan numbers C_n. In fact, if $D = (\frac{f(z)}{z}, f(z))$ is the Riordan matrix with $\Delta(z) = (1 - \sqrt{1 - 4z})/2z = \sum_{n\geq 0} C_n z^n$ then it follows from (ii) of Lemma 2.1 that

$$\frac{1 - \sqrt{1 - 4t}}{2t} = \frac{t - z^2}{zt}, \qquad (zf(z) = t).$$

Solving the above equation for t yields $t = z(1 + z - \sqrt{1 - 2z - 3z^2})/2$. Since $t = zf(z)$ we have

$$f(z) = \frac{1 + z - \sqrt{1 - 2z - 3z^2}}{2} = z(1 + zM(z)),$$

which implies that D is the directed animal matrix and is a PIBell matrix from Theorem 2.2.

Hence we obtain another interesting recursion from the Δ-sequence:

$$m(n + 1, k) = m(n, k - 1) + \sum_{j\geq 0} C_j \cdot m(n - j, k + j)$$

where $C_j = \frac{1}{j+1}\binom{2j}{j}$ is the j-th Catalan number and $m(n, k - 1) = 0$ if $k = 0$.

4. Δ-sequence of an integral power

In this section, we compute the Δ-sequence of an integral power of a PIBell matrix.

Let us define a composition $f^{(n)}(z)$ of $f(z)$ of order $n \geq 2$ by

$$f^{(n)}(z) := (\underbrace{f \circ f \circ \cdots \circ f}_{n})(z).$$

Now let $D = \left(\frac{f(z)}{z}, f(z)\right)$ be a Riordan matrix in the Bell subgroup. Closure in the Bell subgroup implies that for any nonnegative integer n,

$$D^n = \left(\frac{f^{(n)}(z)}{z}, f^{(n)}(z)\right).$$

Lemma 4.1. *If $D = (g(z), f(z))$ is a pseudo-involution then D^n is a pseudo-involution for $n \in Z$.*

Proof. It is obvious for the cases of $n = 0, 1$. Let $n = 2$ and let $D = (g(z), f(z))$ be a pseudo-involution, i.e., $(DM)^2 = I$ or $DMD = M$. Then

$$(D^2M)^2 = D(DMD)DM = DMDM = I,$$

which implies that D^2 is a pseudo-involution. Suppose that D^n is a pseudo-involution for $n \geq 2$. Proceeding by induction we have

$$(D^{n+1}M)^2 = D(D^nMD^n)DM = DMDM = I.$$

Similarly, one can easily show that D^n is a pseudo-involution for each $n < 0$, since $(DM)^2 = I$ if and only if $(D^{-1}M)^2 = I$. ■

Theorem 4.2. *Let $D = \left(\frac{f(z)}{z}, f(z)\right)$ be a pseudo-involution for the Δ-sequence $(b, bd, bd^2, bd^3 \ldots)$ with the generating function $\Delta(z) = \frac{b}{1-dz}$ $(b, d \in R)$. Then D^n has the Δ-sequence with the generating function $n\Delta(z)$.*

Proof. First, we characterize the pseudo-involution $D = \left(\frac{f(z)}{z}, f(z)\right)$ where $\Delta(z) = \frac{b}{1-dz}$. It follows from (ii) of Lemma 2.1 that if $d = 0$ then $f(z) = \frac{z}{1-bz}$, and if $d \neq 0$ then

$$f := f(z) = \frac{1 - bz + dz^2 - \sqrt{(1 - bz + dz^2)^2 - 4dz^2}}{2dz}.$$

Let $d \neq 0$ and $n \geq 2$. Then we have $D^n = \left(\frac{f^{(n)}(z)}{z}, f^{(n)}(z)\right)$. We claim that

$$f^{(n)}(z) = \frac{1 - nbz + dz^2 - \sqrt{(1 - nbz + dz^2)^2 - 4dz^2}}{2dz}. \tag{7}$$

We proceed by induction $n \geq 2$. It is easy to show that (7) holds for $n = 2$. By applying induction, we obtain

$$f^{(n+1)}(z) = f^{(n)}(f(z)) = \frac{1 - nbf + df^2 - \sqrt{(1 - nbf + df^2)^2 - 4df^2}}{2df}.$$

Since $1 + bf + df^2 = \frac{f+dz^2f}{z}$ we have

$$\begin{aligned} f^{(n+1)}(z) &= \frac{\frac{f+dz^2f}{z} - (n+1)bf - \sqrt{(\frac{f+dz^2f}{z} - (n+1)bf)^2 - 4df^2}}{2df} \\ &= \frac{1 - (n+1)bz + dz^2 - \sqrt{(1 - (n+1)bz + dz^2)^2 - 4dz^2}}{2dz}, \end{aligned}$$

which proves (7). Just note that $f^{(n)}(z)$ is the generating function that goes with the generating function $\frac{nb}{1-dz}$ for the Δ-sequence. Hence D^n has the Δ-sequence with the generating function $n\Delta(z)$.

Now let $d = 0$. Then $D^n = \left(\frac{1}{1-nbz}, \frac{z}{1-nbz}\right)$. It is easy to show that D^n has the Δ-sequence $(nb, 0, 0, \ldots)$, i.e., the generating function is $n\Delta(z)$.

For $n < 0$, let $n = -m$ $(m > 0)$. Then $D^n = (D^{-1})^m$. Hence a similar argument yields the theorem. Thus the proof is completed. ■

Table 1
Riordan arrays with the special Δ-sequences.

Δ-sequences	$\Delta(z)$	$f(z)$	Comments
$1, 0, 0, 0, 0, \ldots$	1	$\frac{z}{1-z}$	Pascal triangle
$1, 1, 0, 0, 0, \ldots$	$1+z$	$\frac{1-z-\sqrt{(1-z)^2-4z^3}}{2z^2}$	A023431
$1, 2, 0, 0, 0, \ldots$	$1+2z$	$\frac{1-z-\sqrt{(1-z)^2-8z^3}}{4z^2}$	A025249
$2, 1, 0, 0, 0, \ldots$	$2+z$	$\frac{1-2z-\sqrt{(1-2z)^2-4z^3}}{2z^2}$	A091561
$3, 1, 0, 0, 0, \ldots$	$3+z$	$\frac{1-3z-\sqrt{(1-3z)^2-4z^3}}{2z^2}$	A000245
$1, 1, 1, 1, 1, \ldots$	$\frac{1}{1-z}$	$\frac{1-z+z^2-\sqrt{(1-z+z^2)^2-4z^2}}{2z}$	RNA matrix
$n, n, n, n, n, \ldots$	$\frac{n}{1-z}$	$\frac{1-nz+z^2-\sqrt{(1-nz+z^2)^2-4z^2}}{2z}$	(RNA matrix)^n
$1, 1, 2, 5, 14, \ldots$	$C=\frac{1-\sqrt{1-4z}}{2z}$	$\frac{1+z-\sqrt{1-2z-3z^2}}{2}$	Directed animal matrix
$2, 2, 4, 10, 28, \ldots$	$2C$	$z(2C-1)$	A068875
$1, 2, 6, 22, 90, \ldots$	$\frac{1-z-\sqrt{1-6z+z^2}}{2z}$	$\frac{1+z+z^2-(1+z+z^2)^2-4z(1+z)^2}{2(1+z)}$	A078481
$-1, 1, -1, 1, -1, \ldots$	$\frac{-1}{1+z}$	$-\frac{1+z-z^2-\sqrt{(1+z-z^2)^2+4z^2}}{2z}$	A129509

By a simple computation, we see that the RNA matrix $= (r(n,k))_{n,k\geq 0}$ has the Δ-sequence $(1, 1, 1, \ldots)$ with the generating function $\Delta(z) = \frac{1}{1-z}$. Hence we obtain an interesting recursion:

$$r(n+1,k) = r(n,k-1) + \sum_{j\geq 0} r(n-j,k+j) \quad \text{if } n,k \geq 0$$

where $r_{n,k-1} = 0$ if $k = 0$ for $n \geq 0$.

The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.3. *Let R be the RNA matrix with Δ-sequence $(1, 1, 1, \ldots)$. Then R^n has the Δ-sequence $(n, n, n, \ldots)$ for $n \in \mathbf{Z}$.*

We end this paper describing a list of PIBell matrices with some interesting Δ-sequences in Table 1.

See Sloane's Encyclopedia of Integer Sequences [9] for much more information about these sequences but we briefly note three of these sequences that count Dyck or Motzkin paths with various restrictions. The sequence A023431 counts Motzkin paths with no peaks and no consecutive up steps, A000245 counts Dyck paths that start with two consecutive up steps, and A068875 counts Dyck paths with no consecutive UDUD steps.

If $(f(z)/z, f(z))$ is a PIBell matrix with the Δ-sequence then $\left((f(z)/z)^n, f(z)\right)$ is a pseudo-involution for all integers n with the same Δ-sequence except for the leftmost column. It remains an open problem to find all generating functions $g(z)$ such that $(g(z), f(z))$ is a pseudo-involution. A possible starting point is the observation that for any Riordan group element $(g(z), f(z))$ there is a semidirect product factorization

$$(g(z), f(z)) = \left(\frac{zg(z)}{f(z)}, z\right) * \left(\frac{f(z)}{z}, f(z)\right)$$

into an element in the Appell subgroup (i.e. $f(z) = z$) and an element in the Bell subgroup.

Acknowledgments

We would like to thank the referees for helpful comments that improved both the content and exposition of this article.

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