Combinatorics with the Riordan Group

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The Tennis Ball Problem

You are given *in sequence* tennis balls labeled 1, 2, 3, 4, 5, ...

The Tennis Ball Problem

You are given *in sequence* tennis balls labeled 1, 2, 3, 4, 5, ...

At each turn:

- you receive two balls
- you feed the two balls into a ball machine
- the machine shoots an available ball onto the court

Consider the balls left on the court after n turns.

The Tennis Ball Problem

1. What's the probability that the balls on the court have all even labels?

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2. What's the probability that the balls on the court are consecutively labeled?

The Tennis Ball Problem

1. What's the probability that the balls on the court have all even labels?
2. What's the probability that the balls on the court are consecutively labeled?
3. What's the expected sum of the labels of the balls on the court?

The Tennis Ball Problem

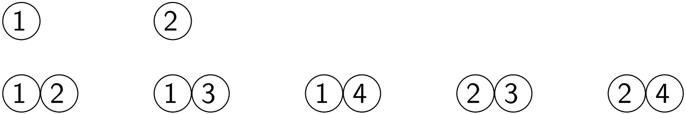
After n turns, how many different combinations of balls on the court are possible?

Let's Count!

Let's Count!

1 2

Let's Count!



Let's Count!

1	2			
1 2	1 3	1 4	2 3	2 4
1 2 3	1 2 4	1 2 5	1 2 6	1 3 4
1 3 5	1 3 6	1 4 5	1 4 6	2 3 4
2 3 5	2 3 6	2 4 5	2 4 6	

The Tennis Ball Problem

Continuing to count, the following sequence emerges:

$$2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$$

The Tennis Ball Problem

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$$2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \ldots$$

Look it up!

The On-Line Encyclopedia of Integer Sequences (OEIS)

The Catalan Numbers

Catalan numbers count **Paths with Bi-Colored Level steps**.

Step set: $U(1,1)$, $D(1,-1)$, $L(1,0)$, $L(1,0)$

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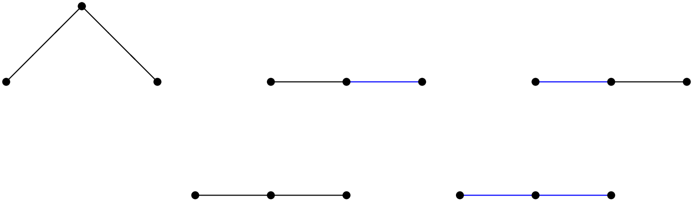

Tennis Balls vs. Bi-Colored Paths

There is a **bijection** between **Tennis Ball Collections** and Paths with Bi-Colored Level Steps.

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only even ball was chosen
only odd ball was chosen
both balls were chosen
neither ball was chosen

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BUT every time we use both balls from one turn, we must choose neither ball from some other turn.

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For each step i along the path we are offered the choice of four steps, U, D, L and L:

Bi-Colored Paths
use L
use L
use U
use D

Tennis Balls vs. Bi-Colored Paths

There is a **bijection** between Tennis Ball Collections and **Paths with Bi-Colored Level Steps.**

For each step i along the path we are offered the choice of four steps, U, D, L and L:

Bi-Colored Paths
use L
use L
use U
use D

BUT every time we use a U for step i, we must choose a D step at some subsequent point along the path.

Tennis Balls vs. Bi-Colored Paths

Balls on Court	Bi-Colored Paths
only even ball was chosen	use L
only odd ball was chosen	use L
both balls were chosen	use U
neither ball was chosen	use D

Tennis Balls vs. Bi-Colored Paths

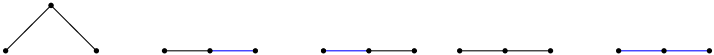
Balls on Court	Bi-Colored Paths
only even ball was chosen	use L
only odd ball was chosen	use L (blue)
both balls were chosen	use U
neither ball was chosen	use D

①② ②③ ①④ ②④ ①③

Tennis Balls vs. Bi-Colored Paths

Balls on Court	Bi-Colored Paths
only even ball was chosen	use L
only odd ball was chosen	use L
both balls were chosen	use U
neither ball was chosen	use D

①② ②③ ①④ ②④ ①③



The Catalan Numbers and Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \cdots & & & & & &
\end{array}
$$

$\binom{n}{k}$ = k-th entry of the n-th row of Pascal's Triangle, $n \geq k \geq 0$

The Catalan Numbers and Pascal's Triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & \color{red}{1} & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & \color{red}{2} & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & \color{red}{6} & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & \color{red}{20} & & 15 & & 6 & & 1
\end{array}
$$

The Catalan Numbers and Pascal's Triangle

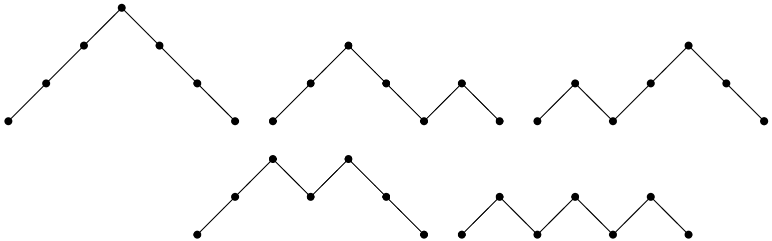
$$1/1 = 1$$
$$1 \qquad 1$$
$$1 \qquad 2/2 = 1 \qquad 1$$
$$1 \qquad 3 \qquad\qquad 3 \qquad 1$$
$$1 \qquad 4 \qquad 6/3 = 2 \qquad 4 \qquad 1$$
$$1 \qquad 5 \qquad 10 \qquad\qquad 10 \qquad 5 \qquad 1$$
$$1 \qquad 6 \qquad 15 \qquad 20/4 = 5 \qquad 15 \qquad 6 \qquad 1$$
$$\vdots$$

The n-th Catalan number is

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

The Catalan Numbers

Catalan numbers count **Ballot Paths** from $(0,0)$ to $(2n,0)$:



Generating Functions

Definition

The **generating function** *for an infinite sequence*

$$a_0, a_1, a_2, a_3, a_4, a_5, \ldots$$

is

$$A(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

Example

$$1, 1, 1, 1, 1, 1, \cdots \to 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

Generating Functions: More Examples

$$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \quad \rightarrow \quad 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

Generating Functions: More Examples

$$1,1,1,1,1,1,1,1,1,\ldots \quad \rightarrow \quad 1+z+z^2+z^3+\cdots = \frac{1}{1-z}$$

$$1,2,4,8,16,32,64,\ldots \quad \rightarrow \quad 1+2z+4z^2+8z^3+\cdots$$

Generating Functions: More Examples

$$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \quad \rightarrow \quad 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

$$1, 2, 4, 8, 16, 32, 64, \ldots \quad \rightarrow \quad 1 + 2z + 4z^2 + 8z^3 + \cdots$$

$$= 1 + (2z) + (2z)^2 + (2z)^3 + (2z)^4 + \cdots$$

Generating Functions: More Examples

$$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \rightarrow 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

$$\begin{aligned} 1, 2, 4, 8, 16, 32, 64, \ldots \rightarrow & \; 1 + 2z + 4z^2 + 8z^3 + \cdots \\ &= 1 + (2z) + (2z)^2 + (2z)^3 + (2z)^4 + \cdots \\ &= \frac{1}{1-(2z)} = \frac{1}{1-2z} \end{aligned}$$

Generating Functions: More Examples

$$1,1,1,1,1,1,1,1,1,\ldots \quad \rightarrow \quad 1+z+z^2+z^3+\cdots = \frac{1}{1-z}$$

$$\begin{aligned} 1,2,4,8,16,32,64,\ldots \quad \rightarrow \quad & 1+2z+4z^2+8z^3+\cdots \\ & = 1+(2z)+(2z)^2+(2z)^3+(2z)^4+\cdots \\ & = \frac{1}{1-(2z)} = \frac{1}{1-2z} \end{aligned}$$

$$1,2,3,4,5,6,7,8,9\ldots \quad \rightarrow \quad 1+2z+3z^2+4z^3+\cdots = \frac{1}{(1-z)^2}$$

Generating Functions: More Examples

$$1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots \rightarrow 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

$$\begin{aligned} 1, 2, 4, 8, 16, 32, 64, \ldots \rightarrow\ & 1 + 2z + 4z^2 + 8z^3 + \cdots \\ &= 1 + (2z) + (2z)^2 + (2z)^3 + (2z)^4 + \cdots \\ &= \frac{1}{1-(2z)} = \frac{1}{1-2z} \end{aligned}$$

$$1, 2, 3, 4, 5, 6, 7, 8, 9 \ldots \rightarrow 1 + 2z + 3z^2 + 4z^3 + \cdots = \frac{1}{(1-z)^2}$$

$$0, 1, 2, 3, 4, 5, 6, \ldots \rightarrow z + 2z^2 + 3z^3 + 4z^4 + \cdots = \frac{z}{(1-z)^2}$$

A Generating Function for the Catalan Numbers

Let $C(z)$ be the generating function for the Catalan numbers.

$$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots = \ ?$$

Is there a closed form? A label for the suitcase?

An Observation

$$\begin{aligned} C^2(z) &= (1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots) \times \\ &\quad (1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots) \end{aligned}$$

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\begin{aligned}
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&= 1 + 2z +
\end{aligned}
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&= 1 + 2z + 5z^2 +
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\end{aligned}
$$

But $C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots,$

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&= 1 + 2z + 5z^2 + 14z^3 + 42z^4 + \cdots
\end{aligned}
$$

But $C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots$,

$$\Rightarrow zC^2(z) + 1 = C(z)$$

An Observation

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$$\Rightarrow zC^2(z) - C(z) + 1 = 0$$

An Observation

$$\begin{aligned} C^2(z) &= (1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots) \times \\ &\quad (1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots) \\ &= 1 + 2z + 5z^2 + 14z^3 + 42z^4 + \cdots \end{aligned}$$

But $C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + \cdots,$

$$\Rightarrow zC^2(z) + 1 = C(z)$$

$$\Rightarrow zC^2(z) - C(z) + 1 = 0$$

$$\Rightarrow \boxed{C(z) = \frac{1 - \sqrt{1 - 4z}}{2z}}$$

Another Formula for the Catalan Numbers

By squaring $C(z)$ we saw that

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-1} C_0$$

or equivalently,

$$\boxed{C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k}}$$

Solutions to the Tennis Ball Problem

- **Even Labels –**

$$\frac{1}{\frac{1}{n+2}\binom{2n+2}{n+1}} = \frac{n+2}{\binom{2n+2}{n+1}}$$

Example

The probability of all even labels after 3 *turns is* $\frac{5}{\binom{8}{4}} = \frac{5}{70} = \frac{1}{14}$

Solutions to the Tennis Ball Problem

- **Expected Sum of Labels –**

Theorem (Mallows-Shapiro, 1999)

The total sum of the labels over all possible combinations of balls on the court is

$$\frac{2n^2+5n+4}{n+2}\binom{2n+1}{n} - 2^{2n+1}$$

and the expected sum of the labels of the balls on the court is

$$\frac{n(4n+5)}{6}$$

Example

The expected sum of the labels after 3 *turns is* $\frac{3(17)}{6} = 8.5$

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- **Consecutive Labels** – Exercise for you!

Sequences of Generating Functions

What if we created a sequence of generating functions?

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Example

$$\frac{1}{1-z}, \ \frac{z}{(1-z)^2}, \ \frac{z^2}{(1-z)^3}, \ \frac{z^3}{(1-z)^4}, \ \frac{z^4}{(1-z)^5}, \ldots$$

Sequences of Generating Functions

What if we created a sequence of generating functions?

Example

$$\frac{1}{1-z}, \frac{z}{(1-z)^2}, \frac{z^2}{(1-z)^3}, \frac{z^3}{(1-z)^4}, \frac{z^4}{(1-z)^5}, \ldots$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

The Riordan Group

An element $R \in \mathcal{R}$ is an infinite lower triangular array whose k-th column has generating function $g(z)f^k(z)$, where $k = 0, 1, 2, \ldots$ and $g(z)$, $f(z)$ are generating functions with $g(0) = 1$, $f(0) = 0$. That is,

$$R = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \cdots \\ g(z) & g(z)f(z) & g(z)f^2(z) & g(z)f^3(z) & \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \cdots \end{bmatrix}$$

We say R is a **Riordan matrix** and write $R = (g(z), f(z))$.

Pascal's Triangle as a Riordan Matrix

Example

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & \cdots & & & \end{bmatrix} = \left(\frac{1}{1-z}, \frac{z}{1-z} \right)$$

A Catalan Triangle

Example

$$(C(z), zC(z)) = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 2 & 2 & 1 & & & & \\ 5 & 5 & 3 & 1 & & & \\ 14 & 14 & 9 & 4 & 1 & & \\ 42 & 42 & 28 & 14 & 5 & 1 & \\ 132 & 132 & 90 & 48 & 20 & 6 & 1 \\ & & & \cdots & & & \end{bmatrix}$$

where

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}$$

The Riordan Group, $(\mathcal{R}, *)$

- **Multiplication:**

$$(g(z), f(z)) * (h(z), l(z)) = (g(z)h(f(z)), l(f(z)))$$

- **Identity:**

$$I = (1, z)$$

- **Inverses:**

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right),$$

where $\bar{f}$ is the compositional inverse of f.

An Identity via Riordan Multiplication

$$\begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & \ldots & & & \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 \\ & & & \ldots & \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 1 & \\ 4 & 3 & 1 \\ 8 & \vdots & \vdots \\ 16 & \vdots & \vdots \\ 32 & \vdots & \\ 64 & & \\ \vdots & & \end{bmatrix}$$

A Proof Via the Riordan Group

Identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof.

$$\begin{aligned}\left(\frac{1}{1-z}, \frac{z}{1-z}\right) * \left(\frac{1}{1-z}, z\right) &= \left(\frac{1}{1-z} \cdot \frac{1}{1-\frac{z}{1-z}}, \frac{z}{1-z}\right) \\ &= \left(\frac{1}{1-2z}, \frac{z}{1-z}\right)\end{aligned}$$

Features of the Riordan Group: Dot Diagrams

Let R be a Riordan matrix with entries $r_{n,k}$ for $n, k \geq 0$

Definition

We say that $[b_1, b_2, b_3, \ldots; a_0, a_1, a_2, \ldots]$ *is the* **dot diagram** *for* R *if*

$$r_{n,0} = b_1 \cdot r_{n-1,0} + b_2 \cdot r_{n-1,1} + b_3 \cdot r_{n-1,2} + \cdots \text{ , for } n \geq 0$$

and

$$r_{n,k} = a_0 \cdot r_{n-1,k-1} + a_1 \cdot r_{n-1,k} + a_2 \cdot r_{n-1,k+1} + \cdots \text{ , for } n, k \geq 1$$

(D. Rogers, 1978)

Dot Diagram for Pascal's Triangle

Example

$$\left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ 1 & 3 & 3 & 1 & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ 1 & 5 & 10 & 10 & 5 & 1 & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & \cdots & & & \end{bmatrix}$$

has dot diagram $[1; 1, 1]$

An Interesting Result

Theorem (Peart-Woodson 1993)

If R has dot diagram

$$[b, \lambda; 1, b, \lambda],$$

then

$$R = \left(\frac{1}{1-bz}, \frac{z}{1-bz} \right) \cdot \left(C(\lambda z^2), zC(\lambda z^2) \right),$$

where $C(z)$ is the generating function for the Catalan numbers.

Furthermore, R represents the number of paths in the upper half plane from $(0, 0)$ to (n, k) using b types of level steps, λ types of down steps, and 1 type of up step.

A Catalan Triangle

Example

$[2, 1; 1, 2, 1]$ *gives*

$$\begin{bmatrix} 1 & & & & & & \\ 2 & 1 & & & & & \\ 5 & 4 & 1 & & & & \\ 14 & 14 & 6 & 1 & & & \\ 42 & 48 & 27 & 8 & 1 & & \\ 132 & 165 & 110 & 44 & 10 & 1 & \\ 429 & 572 & 429 & 208 & 65 & 12 & 1 \\ & & & \cdots & & & \end{bmatrix} = \left(C^2(z), zC^2(z)\right)$$

There are 48 paths from $(0,0)$ *to* $(4,1)$ *using* U, L, L, D.

$48 = 1 \cdot 14 + 2 \cdot 14 + 1 \cdot 6$

Path Counting and the Riordan Group

Now, simple matrix multiplication produces an interesting result. Notice

$$\begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ 42 & 48 & 27 & 8 & 1 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \cdots \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 16 \\ 64 \\ 256 \\ \cdots \end{bmatrix}$$

and we have...

Another Identity!

Translating matrix multiplication into a summation formula, we have

Identity

$$\boxed{\sum_{k=0}^{n} \frac{(k+1)^2}{n+1} \binom{2n+2}{n-k} = 4^n}$$

Proof.

1. Riordan group algebra, OR
2. Path counting argument....

A Combinatorial Proof

$$\sum_{k=0}^{n}(k+1)\cdot\frac{(k+1)}{n+1}\binom{2n+2}{n-k}=4^n$$

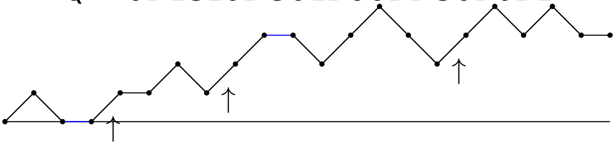
Using only steps of the form U, L, L, D, compute:

- **(RHS)** # of paths using n steps
- **(LHS)** For every $k = 0, 1, \ldots, n$,

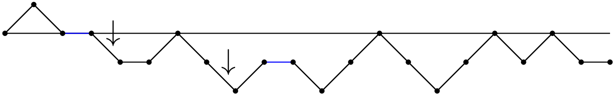
  $(k+1) \times ($ # of paths from $(0,0)$ to (n,k) $)$

Proof of Identity 1

$$Q = UDL\mathbf{U}LUD\mathbf{U}ULDUUDD\mathbf{U}UDUDL$$



$$\phi_2(Q) = UDL\mathbf{D}LUD\mathbf{D}ULDUUDDUUDUDL$$



Elements of Pseudo-Order Two

An nontrivial element B of a group has **order two** if $B^2 = I$, where I is the identity.

An element B of the Riordan group has **pseudo-order two** if BM has order two, where $M = (1, -z)$ is the diagonal matrix with alternating 1's and -1's on the diagonal.

Pascal's Triangle as a Pseudo-Involution

Example

$$\left(\frac{1}{1-z}, \frac{z}{1-z}\right) = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & & \dots & & \end{bmatrix} = P$$

has pseudo-order two.

Pascal's Triangle is a Pseudo-Involution

$$PM = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \end{bmatrix} \begin{bmatrix} 1 & & & & & \\ 0 & -1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & -1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & -1 \\ & & & \cdots & & \end{bmatrix} =$$

$$\begin{bmatrix} 1 & & & & & \\ 1 & -1 & & & & \\ 1 & -2 & 1 & & & \\ 1 & -3 & 3 & -1 & & \\ 1 & -4 & 6 & -4 & 1 & \\ 1 & -5 & 10 & -10 & 5 & -1 \\ & & & \cdots & & \end{bmatrix}$$

and $(PM)^2 =$

$$\begin{bmatrix} 1 \\ 1 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 3 & -1 \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \\ & & & \dots \end{bmatrix} \begin{bmatrix} 1 \\ 1 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 3 & -1 \\ 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \\ & & & \dots \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ & & & \dots \end{bmatrix} = I$$

Another Identity

But we can rewrite the preceding matrix equality as

Identity

$$\sum_{k=0}^{n} (-1)^{k+m} \binom{n}{k} \binom{k}{m} = \delta_{n,m}$$

where $\delta_{n,m} = 1$ if $n = m$ and 0 otherwise.

How can we generalize?

$$\begin{bmatrix} 1 \\ 4 & 1 \\ 16 & 8 & 1 \\ 64 & 48 & 12 & 1 \\ 256 & 256 & 96 & 16 & 1 \\ & & \cdots \end{bmatrix} = \left(\frac{1}{1-4z}, \frac{z}{1-4z}\right)$$

also has pseudo-order two, since

$$\begin{bmatrix} 1 \\ 4 & -1 \\ 16 & -8 & 1 \\ 64 & -48 & 12 & -1 \\ 256 & -256 & 96 & -16 & 1 \\ & & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ 4 & -1 \\ 16 & -8 & 1 \\ 64 & -48 & 12 & -1 \\ 256 & -256 & 96 & -16 & 1 \\ & & \cdots \end{bmatrix}$$

$$= I$$

An Open Question

Question (L. Shapiro, 2001)

Is it true that any element A^* of pseudo-order 2 can be written as $AMA^{-1}M$ for some A?

An Open Question

Example

$$A^* = \begin{bmatrix} 1 & & & & \\ 4 & 1 & & & \\ 16 & 8 & 1 & & \\ 64 & 48 & 12 & 1 & \\ 256 & 256 & 96 & 16 & 1 \\ & & \dots & & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ 42 & 48 & 27 & 8 & 1 \\ & & \dots & & \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 2 & 1 & & & \\ 3 & 4 & 1 & & \\ 4 & 10 & 6 & 1 & \\ 5 & 20 & 21 & 8 & 1 \\ & & \dots & & \end{bmatrix}$$

Yet Another Identity!

Now we are able to extend our previous identity to the following:

Identity

$$\boxed{\binom{n}{m}4^{n-m} = \sum_{k=0}^{n} \frac{k+1}{n+1}\binom{k+m+1}{2m+1}\binom{2n+2}{n-k}}$$

Did we get lucky?
Or is this representative of something more general?

My Contribution to Shapiro's Question

Theorem (N. Cameron, 2002)

The Riordan matrix $R^* =$

$$\left(\frac{1 + \frac{\epsilon z}{1-bz} C\left(\frac{\lambda z^2}{(1-bz)^2}\right) - \frac{\delta z^2}{(1-bz)^2} C^2\left(\frac{\lambda z^2}{(1-bz)^2}\right)}{1 - \frac{\epsilon z}{1-bz} C\left(\frac{\lambda z^2}{(1-bz)^2}\right) - \frac{\delta z^2}{(1-bz)^2} C^2\left(\frac{\lambda z^2}{(1-bz)^2}\right)} \cdot \frac{1}{1-2bz}, \frac{z}{1-2bz} \right)$$

has pseudo-order two. Furthermore, $R^* = RMR^{-1}M$, *where* R *has dot diagram* $[b+\epsilon, \lambda+\delta; 1, b, \lambda]$.

This implies that all “Pascal-type” Riordan matrices have the form

$$\left(\frac{1}{1-2bz}, \frac{z}{1-2bz}\right) = R \cdot (MR^{-1}M)$$

where R has dot diagram $[b, \lambda; 1, b, \lambda]$.

Consider the pseudo-involution

$$S^* = \begin{bmatrix} 1 & & & & \\ 6 & 1 & & & \\ 36 & 12 & 1 & & \\ 216 & 108 & 18 & 1 & \\ 1296 & 864 & 216 & 24 & 1 \\ & & \cdots & & \end{bmatrix} = \left(\frac{1}{1-6z}, \frac{z}{1-6z}\right)$$

$$= \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 11 & 6 & 1 & & \\ 45 & 31 & 9 & 1 & \\ 197 & 156 & 60 & 12 & 1 \\ & & \cdots & & \end{bmatrix} \begin{bmatrix} 1 & & & & \\ 3 & 1 & & & \\ 7 & 6 & 1 & & \\ 15 & 23 & 9 & 1 & \\ 31 & 72 & 48 & 12 & 1 \\ & & \cdots & & \end{bmatrix}$$

Identity

$6^n =$

$$\frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{n-k}(k+1)\binom{n+1}{j}\binom{n+1-j}{n-k-2j}3^{n-k-2j} \cdot \left(2^{k+j+1}-2^j\right)$$

Identity

$6^n =$

$$\frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{n-k}(k+1)\binom{n+1}{j}\binom{n+1-j}{n-k-2j}3^{n-k-2j} \cdot \left(2^{k+j+1}-2^{j}\right)$$

Proof.

(Combinatorial) Proceeds in the same way as before, except there are more choices when changing last ascents to premier descents. □

Other Questions to Consider

- There are interesting elements of pseudo-order two for which Shapiro's question is not answered.
- The s-Tennis Ball Problem has been generalized and resolved, but there are some variations that have not been addressed.
- An interesting open(?) identity:

$$4^n C_n = \sum_{k=0}^{n} C_{2k} C_{2n-2k}$$

Thanks for Listening!