Journal of Number Theory 130 (2010) 828–851

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On Li's criterion for the Riemann hypothesis for the Selberg class

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ARTICLE INFO

Article history:
Received 8 October 2007
Revised 20 October 2009
Available online 29 January 2010
Communicated by David Goss

MSC:
11M41
11M36

Keywords:
Li's criterion
Selberg class

ABSTRACT

Text. In this paper, we shall prove a generalization of Li's positivity criterion for the Riemann hypothesis for the extended Selberg class with an Euler sum. We shall also obtain two arithmetic expressions for Li's constants $\lambda_F(n) = \sum_{\rho}^{*}(1-(1-\frac{1}{\rho})^n)$, where the sum is taken over all non-trivial zeros of the function F and the $*$ indicates that the sum is taken in the sense of the limit as $T \to \infty$ of the sum over ρ with $|\mathrm{Im}\rho| \leqslant T$. The first expression of $\lambda_F(n)$, for functions in the extended Selberg class, having an Euler sum is given terms of analogues of Stieltjes constants (up to some gamma factors). The second expression, for functions in the Selberg class, non-vanishing on the line $\mathrm{Re}\, s = 1$, is given in terms of a certain limit of the sum over primes.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=EwDtXrkuwxA.

1. Introduction

In 1997, Xian-Jin Li has discovered a new positivity criterion for the Riemann hypothesis. In [17] he proved that the Riemann hypothesis is equivalent with the non-negativity of numbers

$$\lambda_n = \sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^n\right] \tag{1}$$

for all $n \in \mathbb{N}$, where the sum is taken over all non-trivial zeros of the Riemann zeta function.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.10.012

Later, E. Bombieri and J. C. Lagarias [6] observed that Li's criterion can be generalized to a multiset of complex numbers satisfying certain conditions, and gave an arithmetic formula for numbers λ_n. They have also proved that one-sided tempered growth of coefficients (1) is enough for the Riemann hypothesis (cf. [6, Corollary 1(c)]).

A. Voros has proved that the Riemann hypothesis true is equivalent to the growth of λ_n as $\frac{1}{2}n\log n$, determined by its archimedean part, while the Riemann hypothesis false is equivalent to the oscillations of λ_n with exponentially growing amplitude, determined by its finite part (for details, see [34, Section 3.3]).

J.C. Lagarias [16] has defined the generalized Li coefficient $\lambda_n(\pi)$ attached to an irreducible cuspidal unitary automorphic representation π of $GL_m(\mathbb{Q})$ and proved that the Generalized Riemann Hypothesis for the corresponding automorphic L-function is equivalent to the non-negativity of $\operatorname{Re}\lambda_n(\pi)$, for all $n \in \mathbb{N}$. Lagarias has obtained the arithmetic expression of $\lambda_n(\pi)$ and determined the asymptotic behavior of both the archimedean and the finite part of $\lambda_n(\pi)$.

The Li coefficients have also been generalized to L-functions defined by Hecke operators for the congruence subgroup $\Gamma_0(N)$, in [18] and [20]. F.C.S. Brown [7] has determined zero-free regions of Dirichlet and Artin L-functions (under the Artin hypothesis) in terms of sizes of the corresponding generalized Li coefficients.

The Selberg class of functions is an axiomatic class of functions, that conjecturally contains all L-functions having an Euler product representation. The Generalized Riemann Hypothesis conjectures that all such L-functions have zeros lying on the critical line $\operatorname{Re} s = \frac{1}{2}$. Therefore, a problem of formulating a Generalized Riemann Hypothesis criterion arises naturally in the context of the Selberg class $\mathcal{S}$.

There are a few results even on the location of zeros of functions in the Selberg class inside the critical strip $0 \leqslant \operatorname{Re} s \leqslant 1$. For example, axioms of the Selberg class allow a function in $\mathcal{S}$ to have both a trivial and a non-trivial zero at $\rho = 0$.

The classical analytic arguments (as in [33, Chapter 9]) enable one to obtain the number of zeros up to a height T. Namely, if $N_F^+(T)$ (resp. $N_F^-(T)$) denotes the number of non-trivial zeros ρ of a function F in the Selberg class, such that $0 \leqslant \operatorname{Im}\rho \leqslant T$ (resp. $-T \leqslant \operatorname{Im}\rho \leqslant 0$), then

$$N_F^-(T) = N_F^+(T) = \frac{d_F}{2\pi} T\log T + c_1 T + O(\log T), \tag{2}$$

where d_F is the degree of F, and c_1 is a constant that depends only on F. The proof of (2) and the explicit expression for the constant c_1 is given in Section 5.

On the other hand, even the non-vanishing of functions belonging to the Selberg class on the line $\operatorname{Re} s = 1$ is still an open problem, as well as the existence of the zero-free regions. Sufficient conditions for the non-vanishing of the function $F \in \mathcal{S}$ on the line $\operatorname{Re} s = 1$ are given by S. Narayanan in [22] and by J. Kaczorowski and A. Perelli in [15] (the Normality conjecture).

In Section 5 we will discuss further on the distribution of non-trivial zeros of the function in the Selberg class and the class $\mathcal{S}^{\#\flat}$ defined below.

The extended Selberg class $\mathcal{S}^{\#}$, though larger than $\mathcal{S}$ and believed to contain all functions of number-theoretical interest is not a suitable class for the formulation of the criterion for the Generalized Riemann Hypothesis, since some functions in $\mathcal{S}^{\#}$ may have infinitely many non-trivial zeros in the half-plane $\operatorname{Re} s > 1$. A nice example of such function is the Davenport–Helibronn zeta function, see [13, p. 136]. Actually, the Euler product axiom is of the crucial importance in order to have a notion of the critical strip (once we exploit the functional equation). The Ramanujan hypothesis axiom (axiom (iv) below) as well is of a crucial importance for the Riemann hypothesis, as proved by the example of a function (constructed by J. Kaczorowski) satisfying all axioms of the Selberg class, except the Ramanujan hypothesis and violating the Generalized Riemann Hypothesis, see [28, pp. 27–28].

Since the Ramanujan hypothesis is the most difficult to prove in important special classes of L-functions, in order to make our results widely applicable, in this paper, we consider two classes of functions to be defined below: a class $\mathcal{S}^{\#\flat} \supseteq \mathcal{S}$ of functions from the extended Selberg class $\mathcal{S}^{\#}$, having an Euler sum and the class $\mathcal{S}^{\flat} \subseteq \mathcal{S}$ of functions from $\mathcal{S}$, non-vanishing on the line $\operatorname{Re} s = 1$, whose zeros satisfy a certain growth condition, defined in Section 5.

The first problem we consider is the convergence of the analogue of the series (1) for a function $F \in \mathcal{S}^{\#\flat}$. This is not a trivial problem, since, for example, in the case of automorphic L-functions, the series defining the Li coefficient is only conditionally convergent. This is proved by J.C. Lagarias in [16], using the information on the distribution of zeros of the automorphic L-function in the upper and the lower half-plane. In Section 4 and Section 5, we shall give two different proofs of the conditional convergence of the series defining the Li coefficient of the function $F \in \mathcal{S}^{\#\flat}$. The first proof exploits the explicit formula with a certain test function (as in [4]) that yields to a product formula for the function $\xi_F(s)$ (the Selberg class analogue of the completed zeta function). The second proof is based on the asymptotic formula for a number of non-trivial zeros up to a height T.

Li's original definition of the coefficient λ_n for positive integers n was given by

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left(s^{n-1} \log \xi(s) \right) \Big|_{s=1}, \tag{3}$$

where

$$\xi(s) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

is the completed zeta function. As pointed out in [6], this definition is equivalent with the definition of numbers λ_n as coefficients in the expansion

$$\frac{d}{ds} \log \xi\left(\frac{1}{s-1}\right) = \sum_{n=0}^{\infty} \lambda_{-n-1} s^n. \tag{4}$$

In this paper we shall define the Selberg class analogue of the nth Li coefficient for a function in the extended Selberg class, having an Euler product as the (conditionally convergent) sum analogous to (1). In Appendix A, we introduce the (extended) Selberg class analogues of the definitions (3) and (4) of the nth Li coefficient and prove that they coincide.

The Selberg class analogue of (4) was considered in [26]. However, the Hadamard product factorization of the complete function $\xi_F(s)$ in [26, Formula (2)] is not completely correct (the product does not converge absolutely).

In Section 4 we prove an analogue of Li's criterion for the Generalized Riemann Hypothesis in $\mathcal{S}^{\#\flat}$. In Section 6 we give a formula for the evaluation of the nth Li coefficient in this class. The key difficulty to be overcome is the possibility that $F \in \mathcal{S}^{\#\flat}$ be zero on the line $\operatorname{Re} s = 1$, that excludes any zero-free region and the use of the prime number theorem. This is overcome by the application of the explicit formula with a test function different from the one used in [6] and [25].

Actually, the explicit formula for functions in $\mathcal{S}^{\#\flat}$, proved in Section 3 is the main tool used in the proof of our theorems. For its proof, we rely on results of J. Jorgenson and S. Lang on explicit formulas in the fundamental class of functions [12] and results on expanding the Jorgenson–Lang class of test functions to which the explicit formula applies, obtained in [2] and [3]. For the sake of completeness, in Appendix B we recall necessary background material from [12] and prove that the class $\mathcal{S}^{\#\flat}$ is the subclass of (much larger) fundamental class.

The class $\mathcal{S}^{\flat}$ provides more arithmetic information. In Section 8 we prove an arithmetic formula for the evaluation of the Li coefficients (in terms of a limit of certain sums over primes) of functions in $\mathcal{S}^{\flat}$ and deduce a result concerning generalized Stieltjes constant.

S. Omar and K. Mazhouda have stated a similar results in [25] and [26]. The statement and the proof of the main theorem of [25] are corrected in [27]. The corrected statement of [25, Theorem 2] now agrees with our Theorem 6.2 and holds for all functions $F \in \mathcal{S}$ having a Landau type zero free region. Our Theorem 6.2 is proved independently for a larger class $\mathcal{S}^{\flat}$, as proved in Lemma 5.8 below. A result similar to Theorem 6.1 appears in [26, Formula (9)] (the formula is not completely correct, as explained in Section 6 below), for functions $F \in \mathcal{S}$, non-vanishing on the line $\operatorname{Re} s = 1$. The proof of [26, Theorem 2.3] is incorrect since it is based on the result of A. Ivić [8] that assumes the error term in the prime number theorem for the Selberg class of the form $O(x^{1-\delta})$ for some $\delta > 0$. At the present state of knowledge, even much weaker error term $o(x/\log x)$ is very difficult to obtain for functions $F \in \mathcal{S}$, non-vanishing on the line $\operatorname{Re} s = 1$ (see example in Remark 5.6 below). In Theorem 6.1 we obtain an arithmetic expression of the Li coefficients for functions F belonging to a very broad class $\mathcal{S}^{\#\flat} \supseteq \mathcal{S}$ and give two different new proofs in Section 7 and Section 9. Furthermore, we give two different proofs of the existence of the Li coefficients and prove that three definitions of the Li coefficients attached to the function $F \in \mathcal{S}^{\#\flat}$ analogous to (1), (3) and (4) coincide.

Finally, let us note here that the results by Omar and Mazhouda presented in their papers [25–27] hold true only for functions $F \in \mathcal{S}$ having a Landau type zero free region, while our main results are valid for much larger class $\mathcal{S}^{\#\flat}$.

2. The Selberg class of functions

The Selberg class of functions $\mathcal{S}$, introduced by A. Selberg in [31] is a general class of Dirichlet series F satisfying the following conditions:

(i) F possesses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

that converges absolutely for $\operatorname{Re} s > 1$.

(ii) There exists an integer $m \geqslant 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and called a polar order of F.

(iii) The function F satisfies the functional equation

$$F(s) w Q_F^{2s-1} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j s + \mu_j)}{\Gamma(\lambda_j(1-s) + \overline{\mu}_j)} = \overline{F(1-\overline{s})} = \overline{F}(1-s),$$

or, equivalently

$$\xi_F(s) = \overline{\xi_F}(1-s)$$

where $Q_F > 0$, $r \geqslant 0$, $\lambda_j > 0$, $|w| = 1$, $\operatorname{Re} \mu_j \geqslant 0$, $j = 1, \dots, r$ and

$$\xi_F(s) = s^{m_F}(1-s)^{m_F} F(s) Q_F^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j). \tag{5}$$

Though numbers $\lambda_1, \dots, \lambda_r$ are not unique, it can be shown (see, e.g. [28]) that the number $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant of the functional equation, called a degree of F.

(iv) (Ramanujan conjecture) For every $\epsilon > 0$ $a_F(n) \ll n^{\epsilon}$.

(v) (Euler product)

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s}, \tag{6}$$

where $b_F(n) = 0$, for all $n \neq p^m$ with $m \geqslant 1$ (p is a prime) and $b_F(n) \ll n^{\theta}$, for some $\theta < 1/2$.

The extended Selberg class $\mathcal{S}^{\#}$ is a class of functions satisfying conditions (i), (ii) and (iii).

The condition (v) is called Euler product since it implies that coefficients $a_F(n)$ are multiplicative, i.e. that the function F can be represented as a product $F(s) = \prod_p F_p(s)$, where $F_p(s) = \sum_{m=1}^{\infty} a_F(p^m) p^{-ms}$. This also implies that $b_F(p) = a_F(p)$, for all (but finitely many) primes p.

A very nice introduction into the theory of the Selberg class and extended Selberg class can be found in surveys [28] and [29].

The first three axioms of the Selberg class describe the analytic nature of a general zeta or L-function, while the Ramanujan conjecture and the Euler product give us more arithmetic information about the function F. As pointed out in Section 1, both axioms (iv) and (v) are crucial for the Generalized Riemann Hypothesis. The Ramanujan hypothesis and the boundedness of coefficients $b_F(n)$ in the representation (6) are very difficult to prove for some very important classes of L-functions. For example, an automorphic L-function $L(s,\pi)$ attached to irreducible, cuspidal unitary automorphic representation π of $GL_n(\mathbb{Q})$ satisfy axioms (i) and (ii). The axiom (iii) is satisfied with a small adjustment allowing $\operatorname{Re}\mu_j > -1/4$ and $\operatorname{Re}(\lambda_j + 2\mu_j) > 0$ instead of $\operatorname{Re}\mu_j \geqslant 0$. Namely, it is believed that archimedean Langlands parameters $\kappa_j(\pi)$ are such that $\operatorname{Re}\kappa_j(\pi) \geqslant 0$. This hypothesis is called the Ramanujan hypothesis for the archimedean Langlands parameters. (Note that the parameters μ_j in the axiom (iii) are equal to $\kappa_j(\pi)/2$ once the functional equation for $L(s,\pi)$ is written in the form (iii).) However, all that is unconditionally known at present is the bound $\operatorname{Re}\kappa_j(\pi) > -1/2$, proved by Z. Rudnick and P. Sarnak in [30, Appendix], and the bound $|\operatorname{Re}\kappa_j(\pi)| \leqslant 1/2 - 1/(n^2+1)$ proved in [19, Theorem 2], for the representation π unramified at the archimedean place. Under the Generalized Riemann Hypothesis, exploiting properties of the Li coefficients attached to the Rankin–Selberg L-functions, the last bound is improved to the bound $|\operatorname{Re}\kappa_j(\pi)| \leqslant 1/4$ (for π that is unramified at the archimedean place) in [23, Corollary 3].

The axioms (iv) and (v) are not fully proved for the class of automorphic L-functions (they are widely believed to hold true). However, as a consequence of the more general result (on the Rankin–Selberg L-functions) obtained by H. Jacquet and J.A. Shalika [10, Theorem 5.3], the function $L(s,\pi)$ has an Euler product representation that converges absolutely in the half-plane $\operatorname{Re}s > 1$.

Furthermore, in order to prove the existence of the Li coefficients and find the formula for their evaluation, we shall use "explicit formulas". A very general class of functions for which the explicit formulas are proved to hold is a Jorgenson–Lang fundamental class of functions (see Appendix B). Functions in this class need not satisfy the Ramanujan conjecture and bounds on coefficients in their Euler product.

For these reasons, throughout this paper we shall consider the class $\mathcal{S}^{\#\flat}$ that consists of functions $F \in \mathcal{S}^{\#}$, satisfying the following condition

(v′) (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=2}^{\infty} \frac{c_F(n)}{n^s},$$

converging absolutely for $\operatorname{Re}s > 1$.

The axiom (v′) is narrowing the Jorgenson–Lang's Euler sum condition (defined in Appendix B below) since the sum in (v′) is taken over natural numbers and the absolute convergence is required in the half-plane $\operatorname{Re}s > 1$. This is done because of two major reasons:

The first reason is related to the conditions posed on the test function in the explicit formula. Namely, if the condition (v′) were assumed to hold for $\operatorname{Re}s > \sigma_0 > 1$, then one of the conditions posed the test function would be that it decays as $e^{-(\sigma_0/2+\epsilon)|x|}$, for some $\epsilon > 0$, as $x \to \infty$. In order to insert the test function (20) defined in Section 7 (actually, its perturbed version (21)) one needs to have $\sigma_0 = 1$.

The second reason is that (v′) implies non-vanishing of F in the half-plane $\operatorname{Re}s > 1$. This, together with (iii) implies that the non-trivial zeros of F, i.e. the zeros of the complete function $\xi_F(s)$, lie in the strip $0 \leqslant \operatorname{Re}s \leqslant 1$. This is a natural notion of a critical strip in order to obtain a criterion for Generalized Riemann Hypothesis. We shall denote by $Z(F)$ the set of all non-trivial zeros of $F \in \mathcal{S}^{\#\flat}$. Other zeros of F (i.e. those arising from the poles of the gamma factors) will be called trivial zeros.

The axiom (v′) seems stronger than (v), since the bound on coefficients $b_F(n)$ in (6) implies the convergence of $\log F(s)$ only in the half-plane $\operatorname{Re}s > 3/2$. However, the following theorem holds true.

Theorem 2.1. $\mathcal{S}^{\#\flat} \supseteq \mathcal{S}$.

Proof. It is sufficient to prove that for $F \in \mathcal{S}$ the series (6) converges absolutely in the half-plane $\operatorname{Re} s > 1$.

The Ramanujan conjecture implies that for an arbitrary $\epsilon > 0$ one has $b_F(p) = a_F(p) \ll p^{\epsilon}$, hence the series $\sum_p \frac{b_F(p)}{p^s}$ converges absolutely in the half-plane $\operatorname{Re} s > 1$ (see, e.g. [28, p. 27], for a discussion in the more general case). The bound $b_F(n) \ll n^{\theta}$, for some $\theta < 1/2$ and the fact that $b_F(n) = 0$, unless $n = p^m$ with $m \geqslant 1$, p being a prime yields the absolute convergence of the series

$$\sum_{m=2}^{\infty} \sum_p \frac{b_F(p^m)}{p^{ms}}$$

in the half-plane $\operatorname{Re} s > 1$. Since

$$\log F(s) = \sum_p \frac{b_F(p)}{p^s} + \sum_{m=2}^{\infty} \sum_p \frac{b_F(p^m)}{p^{ms}},$$

the proof is complete. □

3. An explicit formula for the class $\mathcal{S}^{\#\flat}$

In this section we shall prove an explicit formula for functions in the class $\mathcal{S}^{\#\flat}$, with a class of regularized, rapidly decaying test functions of bounded generalized variation (ϕ-variation) that need not necessarily be continuous at zero. Broadening of the class of smooth, rapidly decaying and compactly supported test functions to which the classical explicit formulas apply to is necessary in order to use Li's cut-off test functions (23), introduced by Bombieri and Lagarias (defined below), as well as test functions (9) and (21). (Actually, our test functions are in the class $BV(\mathbb{R}) \subseteq \phi BV(\mathbb{R})$, for all functions ϕ defined below.)

As K. Barner noted in [4] broadening the class of test functions to which the explicit formula applies is significant for many applications in number theory. Therefore, we shall prove an explicit formula for the class $\mathcal{S}^{\#\flat}$ (Theorem 3.1 below) for a broad class of test functions of generalized bounded variation (that contains all test functions considered by Barner in [4]).

Let us recall that a real function f is called *regularized* if it possesses one-sided limits $f(x-0)$ and $f(x+0)$ at each point $x \in \mathbb{R}$ and if

$$f(x) = \frac{1}{2}\big(f(x-0) + f(x+0)\big).$$

Let ϕ be a continuous, increasing function defined on $[0, \infty)$ such that $\phi(0) = 0$ and $\phi(\infty) = \infty$. A function f is said to be of ϕ *bounded variation* on some interval I if

$$V_{\phi}(f, I) = \sup \sum_n \phi\big(\big|f(b_n) - f(a_n)\big|\big) < \infty,$$

where the supremum is taken over all systems $\{(a_n, b_n)\}_n$ of nonoverlapping subintervals of I (cf. [36]).

Example. $\phi(u) = u$ gives us Jordan variation, and $\phi(u) = u^p$, $p > 1$, corresponds to Wiener p-variation.

In the sequel, we shall assume that the function ϕ is a continuous, strictly increasing convex function on $[0, \infty)$ satisfying three asymptotic conditions

(0_1) $\lim_{x \to 0^+} \frac{\phi(x)}{x} = 0$,

(∞_1) $\lim_{x\to\infty}\frac{\phi(x)}{x}=\infty$, and
(p) $\sum\phi^{-1}(\frac{1}{n})(\frac{1}{n})^{\frac{1}{p}}<\infty$, for some $p>1$.

The first two conditions ensure that the Young's complementary function of ϕ is well defined and that the Young's inequality is satisfied (cf. [36] for more details). The last condition is concerned with the convergence of some Stieltjes integrals necessary for the evaluation of the Weil functional defined below using the generalized Parseval formula (see [1, Section 4 and Section 6]).

Let us note that the function $\phi(u)=u^q$ $(q>1)$ satisfies all the conditions stated above, hence, obviously $BV(\mathbb{R})\subseteq\phi BV(\mathbb{R})$. Actually, it is easy to see that $BV(\mathbb{R})\subset\phi BV(\mathbb{R})$. Therefore, the class of test functions in Theorem 3.1 below is broader than the class of test functions considered by K. Barner and A. Weil in [4] and [35]. Namely, the class BV is replaced by the broader class ϕBV and the condition at zero (the third condition in Theorem 3.1) is further relaxed.

As an application of Theorem 3.1 below, we shall prove that the complete function ξ_F, defined by (5) can be represented as a *-convergent (Hadamard) product over its zeros.

In the sequel, the *-convergence will denote the convergence (of a series or an infinite product over complex numbers ρ) in the sense of the limit as $T\to\infty$ of the finite sum or product over all ρ with $|\mathrm{Im}\,\rho|\leqslant T$.

In Appendix B, it is proved that the class $\mathcal{S}^{\#\flat}$ when considered as a set of triples $(F,\overline{F},\Psi_F)$, where

$$\Psi_F(s)=wQ_F^{2s-1}\prod_{j=1}^{r}\frac{\Gamma(\lambda_j s+\mu_j)}{\Gamma(\lambda_j(1-s)+\overline{\mu}_j)}$$

is a factor of the functional equation (iii), is an important subclass of the fundamental class of functions. In this case, the factor Ψ_F of the functional equation is of a regularized product type of a reduced order $(M,m)=(0,0)$. (The definition of a function of a regularized product type and its reduced order is explained in Appendix B.)

Therefore, it is possible to apply results on explicit formulas, proved in [2] (with $M=0$) and [3] to obtain the following theorem.

Theorem 3.1. *Let a regularized function G satisfy the following conditions*:

1. $G\in\phi BV(\mathbb{R})\cap L^1(\mathbb{R})$.
2. $G(x)e^{(1/2+\epsilon)|x|}\in\phi BV(\mathbb{R})\cap L^1(\mathbb{R})$, *for some* $\epsilon>0$.
3. $G(x)+G(-x)-2G(0)=O(|\log|x||^{-\alpha})$, *as* $x\to 0$, *for some* $\alpha>2$.

Let $g(x)=G(-\log x)$, *for* $x>0$ *and* $G_j(x)=G(x)\exp(\frac{ix\,\mathrm{Im}\,\mu_j}{\lambda_j})$. *Then, the formula*

$$\begin{aligned}\lim_{T\to\infty}&\sum_{\substack{\rho\in Z(F)\\|\mathrm{Im}\,\rho|\leqslant T}}ord(\rho)M_{\frac12}g(\rho)\\&=m_F M_{\frac12}g(0)+m_F M_{\frac12}g(1)\\&\quad-\sum_n\frac{c_F(n)}{n^{1/2}}g(n)-\sum_n\frac{\overline{c_F}(n)}{n^{1/2}}g(1/n)+2G(0)\log Q_F\\&\quad+\sum_{j=1}^{r}\int_0^\infty\left[\frac{2\lambda_j G_j(0)}{x}-\frac{\exp\left(\left(1-\frac{\lambda_j}{2}-\mathrm{Re}\,\mu_j\right)\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\left(G_j(x)+G_j(-x)\right)\right]e^{-\frac{x}{\lambda_j}}\,dx\end{aligned}\tag{7}$$

holds true for an arbitrary function $F\in\mathcal{S}^{\#\flat}$, *where* $M_{\frac12}g$ *denotes the translate by* $1/2$ *of the Mellin transform of the function* g.

Proof. We shall not give a proof here, since it closely follows the lines of the proof of the explicit formula proved in [2, Theorem 6.1]. The only difference is the evaluation of the Weil functional

$$W_{\Psi_F}(G)=\frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\int_{-T}^{T}\hat{G}(t)\frac{\Psi_F'}{\Psi_F}\left(\frac{1}{2}+it\right)dt,$$

in such a way to enable a test function G to have a discontinuity at zero. This is done in the same way as in the proof of [3, Theorem 1] and [4, Theorem 1]. □

Remark 3.2. A possible zero of the function $F\in\mathcal{S}^{\#\flat}$ at $s=0$ requires a special attention, since it may arise both as a trivial and a non-trivial zero. If $m_F\geqslant 1$, due to the axiom (iii) and the fact that a function F has a pole at $s=1$ of order m_F, the function F may have only a trivial zero at $s=0$ if the number (say, n_F) of coefficients μ_j that are equal to zero is greater than m_F. In that case $s=0$ is a trivial zero of F of order n_F-m_F, and it is not a zero of ξ_F. If $m_F=0$, then F may have both a non-trivial zero at $s=0$ and a trivial zero at $s=0$ of order n_F. The functional equation axiom implies that, in this case $0\notin Z(F)$ if and only if $1\notin Z(F)$.

These arguments are also used in the proof of formula (7) since the only possible zero or pole of the factor $\Psi_F(s)$ (due to bounds on the parameters λ_j and μ_j in the axiom (iii)) in the (half) strip $0\leqslant \operatorname{Re} s\leqslant 1/2$ is a pole at $s=0$ of order n_F, if $n_F\geqslant 1$. That is why only the sum over non-trivial zeros of F appears on the left-hand side of (7).

Lemma 3.3. *Let $F\in S^{\#\flat}$. Then, ξ_F is an entire function of order* 1.

Proof. Since the analogous proofs appear in many books on analytic number theory, we shall only give a sketch here.

Clearly, axioms (ii) and (iii) imply that ξ_F is entire of some finite order. By the Stirling formula, the gamma factors in the function ξ_F are of order 1. The Dirichlet series axiom implies that $\xi_F(s)$ is bounded by $e^{cR\log R}$ for $|s|<R$ in the half-plane $\operatorname{Re} s>1$. The functional equation yields the same bound for $\operatorname{Re} s<0$. The application of Phragmén–Lindelöf principle (e.g. [32, Section 5.6]) in the critical strip implies that the maximum modulus of $\xi_F(s)$ in the disc $|s|<R$ is bounded by $e^{cR\log R}$ and the proof is complete. □

Let us note here that the axiom (v′) was not used in the above proof. This means that Lemma 3.3 holds true for functions $F\in S^{\#}$. Therefore, if $\xi_F(s)\neq 0$ (or, equivalently, if $s=0$ is not a non-trivial zero of $F\in S^{\#}$), the Hadamard factorization theorem implies that the function ξ_F possesses a representation as the product over its zeros:

$$\xi_F(s)=\xi_F(0)e^{b_F s}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}, \tag{8}$$

where $b_F=\frac{\xi_F'}{\xi_F}(0)$. The following theorem shows that, due to the arithmetic information contained in (v′), we may obtain more information about the function ξ_F, for $F\in S^{\#\flat}$.

Theorem 3.4. *Let $F\in S^{\#\flat}$ be such that $0\notin Z(F)$. Then, for all $s\in\mathbb{C}$ different from zeros of ξ_F one has*

a)
$$\frac{\xi_F'}{\xi_F}(s)=\lim_{T\to\infty}\sum_{\substack{\rho\in Z(F)\\ |\operatorname{Im}\rho|\leqslant T}}\frac{ord\,\rho}{s-\rho}={\sum_{\rho}}^{*}\frac{1}{s-\rho},$$

b)
$$\xi_F(s)=\xi_F(0){\prod_{\rho\in Z(F)}}^{*}\left(1-\frac{s}{\rho}\right),$$

and specially

$$b_F = \frac{\xi_F'}{\xi_F}(0) = -\sum_{\rho\in Z(F)}{}^{*} \frac{1}{\rho}.$$

Proof. a) The test function

$$G_s(x) = \begin{cases} 0, & x>0, \\ 1/2, & x=0, \\ \exp(s-1/2)x, & x<0, \end{cases} \quad s\in\mathbb{C},\ \operatorname{Re} s>1, \tag{9}$$

firstly considered by K. Barner in [4], obviously satisfies conditions of Theorem 3.1 (it has a bounded variation on $\mathbb{R}$). Inserting the function (9) into the explicit formula (7), having in mind that

$$M_{\frac{1}{2}} g_s(\rho) = \int_{-\infty}^{\infty} G_s(x) e^{-(\rho-1/2)x}\,dx = \frac{1}{s-\rho}$$

for $\rho\in Z(F)$, and the fact that

$$\int_0^{\infty}\left[\frac{\lambda_j}{x} - \frac{\exp\left(\left(1-\frac{\lambda_j}{2}-\operatorname{Re}\mu_j\right)\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}} e^{-(s-1/2)x} e^{-\frac{ix\operatorname{Im}\mu_j}{\lambda_j}}\right] e^{-\frac{x}{\lambda_j}}\,dx = \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j),$$

by the Gauss formula, one obtains the equation

$$\frac{F'}{F}(s) = \lim_{T\to\infty} \sum_{\substack{\rho\in Z(F)\\ |\operatorname{Im}\rho|\leqslant T}} \frac{ord\,\rho}{s-\rho} - m_F\left(\frac{1}{s}+\frac{1}{s-1}\right) - \sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}(\lambda_j s+\mu_j) - \log Q_F, \tag{10}$$

valid for $\operatorname{Re} s>1$. The definition of the function ξ_F implies that (10) can be written in the form

$$\frac{\xi_F'}{\xi_F}(s) = \lim_{T\to\infty} \sum_{\substack{\rho\in Z(F)\\ |\operatorname{Im}\rho|\leqslant T}} \frac{ord\,\rho}{s-\rho}, \quad \text{for } \operatorname{Re} s>1. \tag{11}$$

On the other hand, (8) implies that

$$\frac{\xi_F'}{\xi_F}(s) = b_F + \sum_{\rho\in Z(F)} ord(\rho)\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{12}$$

for all complex $s\notin Z(F)$. Comparing (11) and (12) with $s=2$ one immediately obtains that $b_F = -\sum^{*}_{\rho\in Z(F)}\frac{1}{\rho}$. The statement now follows by the uniqueness of analytic continuation.

b) Direct consequence of a). □

4. Li's criterion

In this section we shall define an analogue of the Li coefficients for functions in the class $\mathcal{S}^{\#\flat}$, prove that these coefficients are well defined and formulate an analogue of Li's criterion for the Riemann hypothesis. In Appendix A we shall discuss more about other two equivalent definitions of the Li coefficients.

Theorem 4.1. *Let $F \in \mathcal{S}^{\#\flat}$ be a function such that $0 \notin Z(F)$. Then, the series*

$$\lambda_F(n) = \sum_{\rho \in Z(F)} \left[1 - \left(1 - \frac{1}{\rho}\right)^n\right] \tag{13}$$

is *-convergent for every integer n. Moreover, the series

$$\operatorname{Re} \lambda_F(n) = \sum_{\rho \in Z(F)} \operatorname{Re}\left[1 - \left(1 - \frac{1}{\rho}\right)^n\right]$$

converges absolutely for all integers n

Proof. The function ξ_F is an entire function of order one, hence the series

$$\sigma_F(k) = \sum_{\rho \in Z(F)} \frac{1}{\rho^k}$$

converges absolutely for every integer $k \geqslant 2$. Theorem 3.4 implies that the series $\sigma_F(1)$ is *-convergent, hence the series

$$\lambda_F(n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \sum_{\rho \in Z(F)} \frac{1}{\rho^k}$$

is *-convergent for every positive integer n. Furthermore, the axiom (iii) implies that set $Z(F)$ is invariant under transformation $\rho \mapsto 1 - \overline{\rho}$, hence the equality

$$1 - \left(1 - \frac{1}{\rho}\right)^{-n} = 1 - \left(1 - \frac{1}{1-\rho}\right)^n = 1 - \overline{\left(1 - \frac{1}{1-\overline{\rho}}\right)^n}$$

yields that for positive n one has $\lambda_F(-n) = \overline{\lambda_F(n)}$. This proves that the series (13) is *-convergent for all integers n.

Since ξ_F is an entire function of order 1, and its zeros lie in the critical strip $0 \leqslant \operatorname{Re} s \leqslant 1$, we also obtain that the series

$$\sum_{\rho \in Z(F)} \frac{1 + |\operatorname{Re} \rho|}{(1 + |\rho|)^2}$$

is convergent. Now, the application of [6, Lemma 1, p. 276] to the multiset $Z(F)$ of non-trivial zeros of F completes the proof. □

The numbers $\lambda_F(n)$ defined by (13) are called the *Li coefficients of the function* $F \in \mathcal{S}^{\#\flat}$ (such that $0 \notin Z(F)$).

Remark 4.2. A different proof of the *-convergence of the series $\sigma_F(1)$, for $F \in \mathcal{S}^{\#\flat}$ such that $0 \notin Z(F)$, and hence, a different proof of Theorem 4.1, may be obtained following the ideas of J.C. Lagarias [16, Lemma 2.1]. This proof uses the results on the distribution of zeros of $F \in \mathcal{S}^{\#\flat}$ that will be given in the next section.

The next theorem is the analogue of Li's criterion for the Selberg class.

Theorem 4.3 *(Generalized Riemann Hypothesis criterion). Let $F \in \mathcal{S}^{\#\flat}$ be a function such that $0 \notin Z(F)$. Then, all non-trivial zeros of F lie on the line $\operatorname{Re} s = \frac{1}{2}$ if and only if $\operatorname{Re} \lambda_F(n) \geqslant 0$ for all $n \in \mathbb{N}$.*

Proof. Since $\lambda_F(-n) = \overline{\lambda_F(n)}$, one has $\operatorname{Re} \lambda_F(-n) = \operatorname{Re} \lambda_F(n)$, for all $n \in \mathbb{N}$. Application of [6, Theorem 1] to the multiset $Z(F)$ yields that $\operatorname{Re} \rho \leqslant 1/2$ if and only if $\operatorname{Re} \lambda_F(-n) \geqslant 0$, for all $n \in \mathbb{N}$. The application of the same theorem to the multiset $1 - \overline{Z(F)} = Z(F)$ yields that $\operatorname{Re} \rho \geqslant 1/2$ if and only if $\operatorname{Re} \lambda_F(n) \geqslant 0$. This completes the proof. □

5. Distribution of zeros and the prime number theorem

In this section, we shall prove that zeros of the function $F \in \mathcal{S}^{\#\flat}$ are distributed as (2). Then, we shall discuss further on the distribution of zeros of $F \in \mathcal{S}$ and the Selberg class analogue of the prime number theorem.

5.1. Distribution of zeros of a function $F \in \mathcal{S}^{\#\flat}$

In surveys [13] and [28] on the Selberg class by J. Kaczorowski and A. Perelli it is stated (without proof) that the counting function of non-trivial zeros of $F \in \mathcal{S}$ satisfies (2). Actually, we could not find any reference with the correct and the complete proof of (2). The proof is standard and based on the application of the argument principle and the functional equation. However, in our opinion, the issue of obtaining an upper bound $O(\log T)$ for the variation of $\arg F(1/2 + iT)$ along the straight lines joining 2, $2 + iT$ and $1/2 + iT$, starting at $T = 0$ is not so trivial. Therefore, for the sake of completeness, we give a sketch of the proof of (2), for $F \in \mathcal{S}^{\#\flat}$. Another reason for stating and proving this result is the fact that $\mathcal{S}^{\#\flat} \supseteq \mathcal{S}$, hence the proof of (2) is surely not deduced for functions $F \in \mathcal{S}^{\#\flat}$.

Lemma 5.1. *Let $F \in S^{\#\flat}$. Then,* (2) *holds true with*

$$c_1 = \frac{1}{2\pi}\big(\log q_F - d_F(\log 2\pi + 1)\big),$$

where $q_F = (2\pi)^{d_F} Q_F^2 \prod_{j=1}^r \lambda_j^{2\lambda_j}$ is the conductor of F.

Proof. The standard argument, based on the application of the argument principle, the functional equation axiom (iii) and the Stirling formula for the gamma function implies that, for $T > 1$

$$N_F^+(T) = \frac{d_F}{2\pi} T \log T + c_1 T + c_2 \log T + O(1/T) + S_F(T),$$

where $c_2 = \frac{1}{\pi}\sum_{j=1}^r \operatorname{Im} \mu_j$. Namely, the first four summands arise as the variation of the argument of the function $Q_F^{1/2+iT} \prod_{j=1}^r \Gamma(\lambda_j(1/2 + iT) + \mu_j)$, while $S_F(T)$ is a variation of the function $\frac{1}{\pi} \arg F(1/2 + iT)$ both taken along the straight lines joining 2, $2 + iT$ and $1/2 + iT$, starting at $T = 0$.

In order to prove that $S_F(T) = O(\log T)$, as $T \to \infty$, following standard methods (as in [21]) one needs an approximation formula for the function $\frac{F'}{F}$ in terms of the sum over zeros, analogous to formula obtained in [33, Theorem 9.6] for $\frac{\zeta'}{\zeta}$. Namely, one should start with formula (10) and deduce that

$$\frac{F'}{F}(s) = \sum_{\rho\in Z(F)}^{*} \frac{1}{s-\rho} + O_F\bigl(\log|T|\bigr),$$

for $s=\sigma+iT$, uniformly in $-1\leqslant\sigma\leqslant 2$. Then, applying the same equation with $s=2+iT$, after subtraction and some analytic considerations (analogous to the second proof of Theorem 9.2 in [33]) we end up with the equation

$$\frac{F'}{F}(z) = \sum_{|T-\gamma|\leqslant 1} \frac{1}{z-\rho} + O_F(\log T),$$

where the sum is taken over all non-trivial zeros $\rho=\beta+i\gamma$ of F such that $|T-\gamma|\leqslant 1$. In a similar way we also deduce that the number of zeros ρ such that $|T-\gamma|\leqslant 1$ is $O(\log T)$. (Complete details are given in [24, Lemma 2.2].) Now, by the argument principle, we conclude that $S_F(T)=O(\log T)$, as $T\to\infty$.

The distribution of zeros in the lower half-plane is obtained in the same way. □

Remark 5.2. If a function $F\in\mathcal{S}$ has a polynomial Euler product (more about this subclass of the Selberg class can be found in [14]), it is possible to obtain the approximation formula for $N_F^{\pm}(T)$ by a direct application of the results of Bombieri and Hejhal [5].

Remark 5.3. The second proof of the $*$-convergence of the series $\sigma_F(1)$ can be deduced from Lemma 5.1 repeating the arguments given in the proof of Lemma 2.1 in [16].

5.2. The prime number theorem in the Selberg class

For the function $F\in\mathcal{S}$ we shall denote by $\Lambda_F(n)$ the analogue of the von Mangoldt function, i.e.

$$\Lambda_F(n) = \begin{cases} b_F(p^k)\log p^k, & \text{if } n=p^k,\\ 0, & \text{if } n\neq p^k, \end{cases}$$

and by

$$\psi_F(x) = \sum_{n<x} \Lambda_F(n)$$

the analogue of the Tchebyshev ψ-function.

The Selberg class analogue of the prime number theorem is a theorem that explains the asymptotic behavior of the function $\psi_F(x)$, as $x\to\infty$.

J. Kaczorowski and A. Perelli [15] have proved that the non-vanishing of the function $F\in\mathcal{S}$ on the line $\operatorname{Re} s=1$ is equivalent with the prime number theorem for the Selberg class. They proved the following theorem

Proposition 5.4. *(See [15, Theorem 1].) Let $F\in\mathcal{S}$. Then, $\psi_F(x)=m_F x+o(x)$ if and only if $F(1+it)\neq 0$ for every $t\in\mathbb{R}$.*

The proof of Proposition 5.4 is based on the classical contour integration methods and the following zero density lemma

Lemma 5.5. *([15, Lemma 3].) Let $F \in \mathcal{S}$ and $\epsilon > 0$. Then, there exists a constant $C > 0$ such that*

$$N_F(\sigma, T) \ll_\epsilon T^{C(1-\sigma)+\epsilon}, \tag{14}$$

where $N_F(\sigma, T)$ denotes the number of non-trivial zeros $\rho = \beta + i\gamma$ of a function $F \in \mathcal{S}$, such that $\beta > \sigma$ and $0 < \gamma \leqslant T$.

Lemma 5.5 implies the existence of the number $\theta < 1$, sufficiently close to 1 such that

$$\sum_{\substack{\rho \in Z(F) \\ \operatorname{Re}\rho > \theta}} \frac{x^\rho}{\rho} = o(x), \quad \text{as } x \to \infty,$$

for $F \in \mathcal{S}$, non-vanishing on the line $\operatorname{Re} s = 1$ and this is an error term the prime number theorem.

Remark 5.6. The above error term $o(x)$ is difficult to improve, assuming non-vanishing of the function F on $\operatorname{Re} s = 1$ alone, since the zeros of F can approach arbitrary close to the line $\operatorname{Re} s = 1$. We shall illustrate this statement with the following example.

Example. By the present state of knowledge on the distribution of zeros of a function $F \in \mathcal{S}$, non-vanishing $\operatorname{Re} s = 1$, the set of possible non-trivial zeros could be $Z(F) = Z_1(F) \cup (1 - \overline{Z_1(F)}) \cup Z_2(F)$, where $Z_1(F) = \{1 - e^{-k} + 2\pi i \lfloor e^k \rfloor\}_{k \in \mathbb{N}}$ and $Z_2(F)$ is the set of zeros $\rho = \frac{1}{2} \pm i\gamma$, such that the number of elements of the set $\{\gamma \mid 0 < \gamma < T\}$ is approximately $\frac{d_F}{2\pi} T \log T + c_1 T + O(\log T)$. In this case, $N_F(\sigma, T) = O(\log T)$, for $\sigma > 1/2$, so the estimate (14) is also satisfied.

Furthermore, it is easy to see that, for any $1/2 < \theta < 1$ and k_0 such that $1 - e^{-k_0} > \theta$ one has, for a sequence $x_n = e^n$

$$-\operatorname{Im}\Bigg(\sum_{\substack{\rho_k \in Z_1(F) \\ \operatorname{Re}\rho > \theta}} \frac{x_n^{\rho_k - 1}}{\rho_k}\Bigg) \sim \sum_{k=k_0}^{\infty} e^{-k} e^{-ne^{-k}} \sim \int_{k_0}^{\infty} e^{-ne^{-u}} e^{-u}\, du \sim \int_0^{e^{-k_0}} e^{-nt}\, dt \gg \frac{1}{n} = \log^{-1} x_n,$$

hence

$$\sum_{\substack{\rho \in Z(F) \\ \operatorname{Re}\rho > \theta}} \frac{x^\rho}{\rho} \neq o\left(\frac{x}{\log x}\right), \quad \text{as } x \to \infty$$

regardless of a choice of θ.

Therefore, in order to obtain a bound $o(\frac{x}{\log x})$ in the prime number theorem, an additional assumptions on $F \in \mathcal{S}$ (besides non-vanishing $\operatorname{Re} s = 1$) should be imposed.

On the other hand, the following lemma holds

Lemma 5.7. *Let $F \in \mathcal{S}$ be non-vanishing at $\operatorname{Re} s = 1$. Then, for any positive integer l*

$$\lim_{x \to \infty} \log^l x \left(m_F - \frac{\psi_F(x)}{x}\right) = 0 \tag{15}$$

if and only if

$$\lim_{x \to \infty} \log^l x \sum_\rho \frac{x^{\rho - 1}}{\rho} = 0. \tag{16}$$

Proof. A direct application of the Perron formula, e.g. [21, p. 67] to Dirichlet series $-\frac{F'}{F}$ implies that for any $x > 1$ (not an integer) and $T > 1$ one has

$$\psi_F(x) = m_F x - \sum_{|\mathrm{Im}\rho| \leqslant T} \frac{x^\rho}{\rho} + O\left(\frac{x^{1+\epsilon}}{T}\right), \tag{17}$$

for some small $\epsilon > 0$. The equivalence of formulas (15) and (16) is easily seen, once we let $T \to \infty$ and multiply the above formula with $\frac{\log^l x}{x}$. □

Formula (16) is essential part of the proof of [25, Theorem 2] and the proof of our Theorem 6.2 below. The example in Remark 5.6 shows that some additional assumptions need to be imposed on $F \in \mathcal{S}$, non-vanishing on the line $\mathrm{Re}\, s = 1$ in order that (16) holds true for positive integers l.

Furthermore, if $\psi_F(x) = m_F x + g_F(x)$, then, after representing the sum $\sum_{k<x} \frac{\Lambda_F(k)}{k} \log^{l-1} k$ as a Stieltjes integral and integrating by parts we get

$$\sum_{k<x} \frac{\Lambda_F(k)}{k} \log^{l-1} k - \frac{m_F}{l} \log^l x = \frac{g_F(x)}{x} \log^{l-1} x - \int_1^x \frac{g_F(t)(\log^{l-1} t - (l-1)\log^{l-2} t)}{t^2}\, dt.$$

Hence, the limit

$$\lim_{x\to\infty} \left(\frac{m_F}{l} \log^l x - \sum_{k<x} \frac{\Lambda_F(k)}{k} \log^{l-1} k \right) \tag{18}$$

that appears on the right-hand side of the expression for $\lambda_F(n)$ in [25, Theorem 2] and [26, Theorem 2.3] exists for all positive integers l, only if we assume that (15) holds true for all l. Therefore, the assumption of non-vanishing of a function $F \in \mathcal{S}$ on the line $\mathrm{Re}\, s = 1$, with the present state of knowledge on the distribution of non-trivial zeros of $F \in \mathcal{S}$ is not enough to deduce the existence of the limit (18). (This obstacle was overcome in [27] by assuming the Landau type zero free region for functions in $\mathcal{S}$.) However, this assumption can be weakened.

This justifies the following definition.

Definition. We shall denote by $\mathcal{S}^\flat$ the set of all functions $F \in \mathcal{S}$, non-vanishing on the line $\mathrm{Re}\, s = 1$ and such that (16) holds true for all positive integers l. Obviously, $\mathcal{S}^\flat \subseteq \mathcal{S}$.

Basic properties of the class $\mathcal{S}^\flat$ are given in the following lemma

Lemma 5.8.

a) *$F \in \mathcal{S}^\flat$ if and only if $F \in \mathcal{S}$ and*

$$\psi_F(x) = m_F x + o\left(\frac{x}{\log^l x}\right), \quad \text{for any } l \in \mathbb{N}.$$

b) *If a function $F \in \mathcal{S}$ has a Landau type zero free region, i.e. if non-trivial zeros $\rho = \sigma + iT$, ($|T| > 1$) of $F \in \mathcal{S}$ are such that*

$$\sigma \geqslant 1 - \frac{c}{\log q_F(1 + |T|)},$$

then $F \in \mathcal{S}^\flat$.

Proof. a) Follows directly from Lemma 5.7.

b) The Landau type zero free region implies, by standard analytic arguments (cf. [9, Chapter 5]) that the error term in the prime number theorem is $O(-c\sqrt{\log x})$, hence (16) holds true. □

Finally, let us note that, due to the functional equation, $\rho = 0$ cannot be a non-trivial zero of a function $F \in \mathcal{S}^{\flat}$.

6. Arithmetic formulas for Li's coefficients: Statement of results

The main result of the paper are two theorems that give arithmetic formulas for Li's coefficients.

The first theorem gives us a formula for the coefficients of the function $F \in \mathcal{S}^{\#\flat}$, while the second theorem gives us the formula for $F \in \mathcal{S}^{\flat}$, in terms of a limit of a certain sum over primes.

For $F \in \mathcal{S}^{\#\flat}$ let us denote by $\gamma_F(n)$ coefficients appearing in the Laurent (or Taylor) series expansion of the function $\frac{F'}{F}$ around $s = 1$, i.e. let

$$\frac{F'}{F}(s) = -\frac{m_F}{s-1} + \sum_{n=0}^{\infty} \gamma_F(n)(s-1)^n,$$

for s close to 1. We shall call the constants $\gamma_F(n)$ generalized Stieltjes constants, since they are the Selberg class analogues of the Stieltjes constants appearing in the Laurent series expansion of $\frac{\zeta'}{\zeta}$ around $s = 1$. The Li coefficients can be expressed in terms constants $\gamma_F(n)$.

Theorem 6.1. *Let $F \in \mathcal{S}^{\#\flat}$ be a function such that $0 \notin Z(F)$. Then, for all $n \in \mathbb{N}$*

$$\lambda_F(-n) = m_F + n \log Q_F + \sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) + \sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1), \tag{19}$$

where

$$\eta_F(0) = \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) \quad \text{and} \quad \eta_F(l-1) = \sum_{j=1}^{r} (-\lambda_j)^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \mu_j + k)^l},$$

for $l \geqslant 2$.

In the next section we shall give the proof of Theorem 6.1 using the explicit formula (Theorem 3.1). In Appendix A we shall give another, completely independent proof of this theorem, based on the product formula for the function ξ_F.

The arithmetic formula similar to (19) appears in [26, Formula (9)] (the correct formula should have $\lambda_F(-n)$ on the left-hand side and the sum on the right-hand side of [26, Formula (10)] should start at $l = 0$). However, its proof is not correct, as pointed out in Section 1.

Our next theorem applies only to functions $F \in \mathcal{S}$ such that $\psi_F(x) = m_F x + o(\frac{x}{\log^l x})$, for any $l \in \mathbb{N}$ and gives the formula for the generalized Stieltjes constants.

Theorem 6.2. *Let $F \in \mathcal{S}^{\flat}$. Then*

$$\lambda_F(n) = n \log Q_F + \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \lim_{x\to\infty} \left(\frac{m_F}{l} \log^l x - \sum_{k<x} \frac{\overline{\Lambda_F}(k)}{k} \log^{l-1} k \right)$$
$$+ m_F + n \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) - \sum_{l=2}^{n} \binom{n}{l} (-1)^l \sum_{j=1}^{r} \lambda_j^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \overline{\mu_j} + k)^l},$$

for all $n \in \mathbb{N}$.

Comparing Theorems 6.1 and 6.2 we can deduce the expression for generalized Stieltjes constants $\gamma_F(n)$.

Corollary 6.3. *For $F \in \mathcal{S}^\flat$, one has*

$$\gamma_F(n) = \frac{(-1)^n}{n!} \lim_{x\to\infty}\left(\frac{m_F}{n+1}\log^{n+1} x - \sum_{k<x}\frac{\Lambda_F(k)}{k}\log^n k\right).$$

7. Proof of Theorem 6.1

Proofs of our theorems are based on the application of the explicit formula to a suitable test function.

Bombieri and Lagarias have noticed that if

$$G_n(x) = \begin{cases} e^{-x/2}\sum_{l=1}^n \binom{n}{l}\frac{(-1)^{l-1}x^{l-1}}{(l-1)!}, & \text{if } x>0,\\ n/2, & \text{if } x=0,\\ 0, & \text{if } x<0, \end{cases} \tag{20}$$

then

$$H_n(s) = M_{\frac{1}{2}} g_n(s) = \int_{-\infty}^{\infty} G_n(x)e^{-(s-1/2)x}\,dx = 1-\left(1-\frac{1}{s}\right)^n,$$

for all $n \in \mathbb{N}$, where $g_n(x) = G_n(-\log x)$, $x>0$. Let us note here that the function $H_n(s)$ has a pole at $s=1$ for negative integers n, and a pole at $s=0$ for positive n. The assumption $0 \notin Z(F)$, or, equivalently $1 \notin Z(F)$, implies that

$$\lambda_F(n) = \sum_{\rho\in Z(F)} H_n(\rho)$$

is well defined *-convergent sum (according to Theorem 4.1) for all integers n.

Throughout this proof and the proof of Theorem 6.2, we assume n to be a positive integer.

The function G_n does not satisfy the second growth condition of Theorem 3.1 (it does not decay as $e^{-(\epsilon+1/2)x}$, for some $\epsilon>0$) hence, to prove Theorem 6.1 we shall apply the explicit formula to the test function that is a small perturbation of the function G_n. Namely, we shall consider the test function

$$G_{n,z}(x) = \begin{cases} e^{-(z+1/2)x}\sum_{l=1}^n \binom{n}{l}\frac{(-1)^{l-1}x^{l-1}}{(l-1)!}, & \text{if } x>0,\\ n/2, & \text{if } x=0,\\ 0, & \text{if } x<0 \end{cases} \tag{21}$$

for some positive constant z. It is easy to conclude that $G_{n,z}(x)$ satisfies all assumptions posed on the test function in Theorem 3.1. Now,

$$H_{n,z}(s) = M_{\frac{1}{2}} g_{n,z}(s) = \int_{-\infty}^{\infty} G_{n,z}(x)e^{-(s-1/2)x}\,dx = 1-\left(1-\frac{1}{s+z}\right)^n.$$

Since

$$\sum_{\rho\in Z(F)}\big(H_{n,z}(\rho)-H_n(\rho)\big)=nz\sum_{\rho\in Z(F)}\frac{1}{\rho(\rho+z)}+\sum_{k=2}^{n}\binom{n}{k}(-1)^k\sum_{\rho\in Z(F)}\left(\frac{1}{(\rho+z)^k}-\frac{1}{\rho^k}\right),$$

and all series on the right-hand side converge absolutely and uniformly in z in the interval $(0,\delta)$, for a small $\delta>0$, passing to the limit as $z\to 0^+$, we obtain

$$\lim_{z\to 0^+}\sum_{\rho\in Z(F)}H_{n,z}(\rho)=\sum_{\rho\in Z(F)}H_n(\rho).$$

It is left to evaluate the right-hand side of the explicit formula with the test function $G_{n,z}$. Simple calculations, analogous to the ones used in the proof of Theorem 3.4 yield

$$\begin{aligned}\sum_{\rho\in Z(F)}H_{n,z}(\rho)&=\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{l-1}}{(l-1)!}\left(m_F\frac{(l-1)!}{z^l}-\sum_m\frac{\overline{c_F}(m)}{m^{(1+z)}}(\log m)^{l-1}\right)\\&\quad+m_F H_n(z+1)+n\log Q_F+n\sum_{j=1}^{r}\lambda_j\frac{\Gamma'}{\Gamma}\big(\lambda_j(1+z)+\overline{\mu_j}\big)\\&\quad+\sum_{l=2}^{n}\binom{n}{l}\sum_{j=1}^{r}(-\lambda_j)^l\sum_{k=0}^{\infty}\frac{1}{(\lambda_j(1+z)+\overline{\mu_j}+k)^l}.\end{aligned}\tag{22}$$

On the other hand, since

$$\frac{\overline{F}'}{\overline{F}}(z+1)=-\sum_m\frac{\overline{c_F}(m)}{m^{(1+z)}},$$

and $\frac{\overline{F}'}{\overline{F}}$ is a holomorphic function in the half-plane $\operatorname{Re}s>1$, one has

$$\left(\frac{\overline{F}'}{\overline{F}}\right)^{(l-1)}(z+1)=(-1)^{l-1}\left(-\sum_m\frac{\overline{c_F}(m)}{m^{(1+z)}}(\log m)^{l-1}\right),\quad\text{for } l\geqslant 1.$$

(Let us recall that $\overline{F}(s)=\overline{F(\bar{s})}$.) Therefore

$$\begin{aligned}&\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{l-1}}{(l-1)!}\left(m_F\frac{(l-1)!}{z^l}-\sum_m\frac{\overline{c_F}(m)}{m^{(1+z)}}(\log m)^{l-1}\right)\\&\quad=\sum_{l=1}^{n}\binom{n}{l}\frac{1}{(l-1)!}\left(\left(\frac{\overline{F}'}{\overline{F}}\right)^{(l-1)}(z+1)+(-1)^{l-1}m_F\frac{(l-1)!}{z^l}\right).\end{aligned}$$

The Laurent series expansion of $\overline{F}$ implies that, for $l\geqslant 1$

$$(l-1)!\overline{\gamma_F(l-1)}=\lim_{z\to 0}\left(\left(\frac{\overline{F}'}{\overline{F}}\right)^{(l-1)}(z+1)+(-1)^{l-1}m_F\frac{(l-1)!}{z^l}\right).$$

Passing to the limit as $z \to 0^+$ on both sides of (22) (having in mind that $\lim_{z\to 0} H_n(z+1) = 1$, for $n \in \mathbb{N}$) yields

$$\sum_{\rho\in Z(F)} H_n(\rho) = \sum_{l=1}^{n} \binom{n}{l} \overline{\gamma_F(l-1)} + m_F + n\log Q_F + n\sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}) + \sum_{l=2}^{n} \binom{n}{l} \sum_{j=1}^{r} (-\lambda_j)^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \overline{\mu_j} + k)^l}.$$

The fact that $\lambda_F(-n) = \overline{\lambda_F(n)}$ completes the proof of Theorem 6.1.

8. Proof of Theorem 6.2

To prove Theorem 6.2 we shall consider the cut-off test function

$$G_{n,X}(x) = \begin{cases} G_n(x), & \text{if } 0 < x < \log X, \\ n/2, & \text{if } x = 0, \\ \frac{1}{2}G_n(\log X), & \text{if } x = \log X, \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

and denote by $H_{n,X}(s)$ the translate by $1/2$ of the Mellin transform of $g_{n.X}$. It is easy to see that $G_{n,X}(x)$ satisfies all conditions posed on the test function in Theorem 3.1.

Firstly, we shall prove that

$$\lim_{X\to\infty} \sum_{\rho\in Z(F)} H_{n,X}(\rho) = \sum_{\rho\in Z(F)} H_n(\rho).$$

A direct calculation shows that

$$\sum_{\rho\in Z(F)} \big(H_{n,X}(\rho) - H_n(\rho)\big) = \sum_{l=1}^{n} \binom{n}{l} (-1)^{l-1} \log^{l-1} X \sum_{\rho\in Z(F)} \frac{X^{-\rho}}{\rho} + O\bigg(\sum_{\rho\in Z(F)} \frac{X^{-\operatorname{Re}\rho}\log^{n-2} X}{|\rho|^2}\bigg). \tag{24}$$

Since zeros of F come in pairs ρ and $1-\overline{\rho}$, we see that

$$\overline{\sum_{\rho\in Z(F)} \frac{X^{-\rho}}{\rho}} = \sum_{\rho\in Z(F)} \frac{X^{-1+\rho}}{1-\rho} = \sum_{\rho\in Z(F)} \frac{X^{-1+\rho}}{\rho(1-\rho)} - \sum_{\rho\in Z(F)} \frac{X^{-1+\rho}}{\rho}.$$

The assumption (16) on $F \in \mathcal{S}^\flat$ implies that

$$\lim_{X\to\infty} \log^{n-1} X \sum_{\rho\in Z(F)} \frac{X^{-\rho}}{\rho} = -\lim_{X\to\infty} \bigg(\log^{n-1} X \overline{\sum_{\rho\in Z(F)} \frac{X^{\rho-1}}{\rho}} - \sum_{\rho\in Z(F)} \frac{X^{-1+\rho}}{\rho(1-\rho)}\bigg) = 0.$$

Similarly, we conclude that

$$\lim_{X\to\infty} \sum_{\rho\in Z(F)} \frac{X^{-\operatorname{Re}\rho}\log^{n-2} X}{|\rho|^2} = 0.$$

Therefore, the left-hand side of (24) tends to zero, as $X \to \infty$, hence $\lim_{X\to\infty} \sum_{\rho\in Z(F)} H_{n,X}(\rho) = \sum_{\rho\in Z(F)} H_n(\rho)$.

Inserting the test function $G_{n,X}$ into the explicit formula we obtain

$$\sum_{\rho\in Z(F)} H_{n,X}(\rho) = n\log Q_F + m_F \sum_{l=1}^{n} \binom{n}{l} (-1)^{l-1} \left(1 - \sum_{j=1}^{l} \frac{1}{(l-j)!} \frac{\log^{l-j} X}{X}\right)$$
$$- \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \left(m_F \frac{\log^l X}{l} - \sum_{\substack{p,m \\ p^m<X}} \frac{\overline{\Lambda_F}(p^m)}{p^m} \left(\log p^m\right)^{l-1}\right)$$
$$+ \sum_{j=1}^{r} n \int_0^{\log X} \left[\frac{\lambda_j}{x} - \frac{\exp\left((1-\lambda_j-\overline{\mu_j})\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\right] e^{-\frac{x}{\lambda_j}}\, dx$$
$$- \sum_{j=1}^{r} \sum_{l=1}^{n} \binom{n}{l} \frac{(-1)^{l-1}}{(l-1)!} \int_0^{\log X} x^{l-1} \frac{\exp\left((-\lambda_j-\overline{\mu_j})\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\, dx. \tag{25}$$

Since the left-hand side of (25) tends to $\lambda_F(n)$, as $X \to \infty$, it is left to evaluate the limit of the first sum and integrals on the right-hand side of (25).

Obviously,

$$\lim_{X\to\infty} \sum_{l=1}^{n} \binom{n}{l} (-1)^{l-1} \left(1 - \sum_{j=1}^{l} \frac{1}{(l-j)!} \frac{\log^{l-j} X}{X}\right) = 1.$$

The application of the Gauss formula for the logarithmic derivative of the gamma function yields

$$\lim_{X\to\infty} \int_0^{\log X} \left[\frac{\lambda_j}{x} - \frac{\exp\left((1-\lambda_j-\overline{\mu_j})\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\right] e^{-\frac{x}{\lambda_j}}\, dx = \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \overline{\mu_j}).$$

Finally, since

$$\frac{1}{1-e^{-\frac{x}{\lambda_j}}} = \sum_{k=0}^{\infty} e^{-\frac{kx}{\lambda_j}}$$

we obtain that

$$\lim_{X\to\infty} \int_0^{\log X} x^{l-1} \frac{\exp\left((-\lambda_j-\overline{\mu_j})\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\, dx = (l-1)! \cdot \lambda_j^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j+\overline{\mu_j}+k)^l}.$$

This proves the theorem.

Acknowledgment

The author thanks the referee for many valuable suggestions that improved the clarity of the presentation. The questions asked by the referee led the author to prove Theorem A.1 and give the second proof of Theorem 6.1 below.

Appendix A. Definitions of Li coefficients in the Selberg class

In this appendix we will discuss various definitions of the Selberg class analogue of the Li coefficients and prove that they are equivalent for functions $F \in \mathcal{S}^{\#\flat}$, such that $0 \notin Z(F)$ (or $1 \notin Z(F)$).

Following Li, Bombieri and Lagarias we shall introduce two definitions of the Li coefficients, analogous to (3) and (4). Namely, for a positive integer n, we define coefficients $\lambda_{1,F}(n)$ by the formula

$$\lambda_{1,F}(n) = \frac{1}{(n-1)!}\frac{d^n}{ds^n}\left(s^{n-1}\log\xi_F(s)\right)\Big|_{s=1}, \tag{26}$$

and coefficients $\lambda_{2,F}(n)$ to be the coefficients in the Taylor series expansion

$$\frac{d}{ds}\log\xi_F\left(\frac{1}{1-s}\right) = \frac{1}{(1-s)^2}\frac{\xi_F'}{\xi_F}\left(\frac{1}{1-s}\right) = \sum_{n=0}^{\infty}\lambda_{2,F}(n+1)s^n. \tag{27}$$

Theorem A.1. *Let $F \in S^{\#\flat}$, such that $0 \notin Z(F)$. Then, $\lambda_F(-n) = \lambda_{1,F}(n) = \lambda_{2,F}(n)$, for all positive integers n.*

Proof. Firstly, we shall prove that $\lambda_F(-n) = \lambda_{1,F}(n)$. The functional equation axiom yields

$$\overline{\frac{d^n}{ds^n}\left(s^{n-1}\log\xi_F(s)\right)\Big|_{s=1}} = (-1)^n\frac{d^n}{ds^n}\left((1-\bar{s})^{n-1}\log\xi_F(\bar{s})\right)\Big|_{s=0}.$$

Theorem 3.4(b) implies that

$$\log\xi_F(\bar{s}) = \sum_{\rho\in Z(F)}^{*}\log\left(1-\frac{\bar{s}}{\rho}\right) = -\sum_{\rho\in Z(F)}^{*}\sum_{m=1}^{\infty}\frac{\rho^{-m}}{m}\bar{s}^m,$$

for s in a small enough neighborhood of zero (where the power series on the right-hand side of the above formula converges uniformly). Hence

$$\log\xi_F(\bar{s}) = -\sum_{m=1}^{\infty}\frac{\bar{s}^m}{m}\sigma_F(m). \tag{28}$$

The uniform convergence enable us to differentiate the above power series term by term and obtain

$$\overline{\lambda_{1,F}(n)} = \frac{(-1)^n}{(n-1)!}\sum_{k=0}^{n}\binom{n}{k}\left((1-\bar{s})^{n-1}\right)^{(n-k)}\left(\log\xi_F(\bar{s})\right)^{(k)}\Big|_{s=0} =$$

$$= \sum_{k=1}^{n}\binom{n}{k}(-1)^{2n-k+1}\sigma_F(k) = \sum_{\rho\in Z(F)}^{*}\left(-\sum_{k=1}^{n}\binom{n}{k}\left(-\frac{1}{\rho}\right)^k\right) = \lambda_F(n).$$

This proves the first part of theorem (since $\lambda_F(-n) = \overline{\lambda_F(n)}$).

Theorem 3.4(a) yields

$$\frac{1}{(1-s)^2}\frac{\xi_F'}{\xi_F}\left(\frac{1}{1-s}\right) = \sum_{\rho\in Z(F)}^{*}\frac{1}{1-\rho}\cdot\frac{1}{1-s}\cdot\frac{1}{1-(\frac{\rho}{\rho-1})s}.$$

In the small enough neighborhood of zero, one has

$$\frac{1}{1-\rho}\cdot\frac{1}{1-s}\cdot\frac{1}{1-(\frac{\rho}{\rho-1})s}=\frac{1}{1-\rho}\sum_{k=0}^{\infty}s^k\cdot\sum_{j=0}^{\infty}\left(\frac{\rho}{\rho-1}s\right)^j$$
$$=\sum_{m=0}^{\infty}\left[1-\left(\frac{\rho}{\rho-1}\right)^{m+1}\right]s^m.$$

Therefore, in a small enough neighborhood of zero

$$\overline{\frac{1}{(1-s)^2}\frac{\xi_F'}{\xi_F}\left(\frac{1}{1-s}\right)}=\sum_{m=0}^{\infty}\left(\sum_{\rho\in Z(F)}^{*}\left[1-\left(1-\frac{1}{1-\bar{\rho}}\right)^{m+1}\right]\bar{s}^m\right)$$
$$=\sum_{m=0}^{\infty}\lambda_F(m+1)\bar{s}^m,$$

since $Z(F)$ is invariant under transformation $\rho\mapsto 1-\bar{\rho}$. Hence, $\lambda_{2,F}(m+1)=\overline{\lambda_F(m+1)}$, for $m\geqslant 0$ and the proof is complete. □

Now, we shall give another proof of Theorem 6.1, following ideas of Lagarias [16, Section 4].

Second proof of Theorem 6.1. We start with formula (28) together with the functional equation for the function ξ_F in order to obtain that, in the neighborhood of $s=0$ one has

$$\frac{\xi_F'}{\xi_F}(s+1)=-\overline{\frac{\xi_F'}{\xi_F}(-\bar{s})}=\sum_{m=0}^{\infty}(-1)^m\overline{\sigma_F(m+1)}\cdot s^m. \tag{29}$$

On the other hand, by the definition of ξ_F one has

$$\frac{\xi_F'}{\xi_F}(s+1)=\frac{m_F}{s+1}+\frac{F'}{F}(s+1)+\frac{m_F}{s}+\log Q_F+\sum_{j=1}^{r}\frac{\Gamma'}{\Gamma}\big(\lambda_j(s+1)+\mu_j\big). \tag{30}$$

Let us define numbers $\tau_F(n)$ to be coefficients in the Taylor series expansion of the function $\log Q_F+\sum_{j=1}^{r}\frac{\Gamma'}{\Gamma}(\lambda_j(s+1)\lambda+\mu_j)$ at $s=0$, i.e. let us put

$$\log Q_F+\sum_{j=1}^{r}\frac{\Gamma'}{\Gamma}\big(\lambda_j(s+1)+\mu_j\big)=\sum_{n=0}^{\infty}\tau_F(n)s^n.$$

Since

$$\frac{F'}{F}(s+1)+\frac{m_F}{s}=\sum_{n=0}^{\infty}\gamma_F(n)s^n\quad\text{and}\quad\frac{m_F}{s+1}=m_F\sum_{n=0}^{\infty}(-1)^n s^n$$

(around $s=0$) comparing (29) and (30) we obtain that

$$(-1)^m\overline{\sigma_F(m+1)}=(-1)^m m_F+\gamma_F(m)+\tau_F(m),$$

for $m \geqslant 0$. By the definition of $\lambda_F(n)$ we obtain immediately

$$\lambda_F(-n) = m_F + \sum_{m=1}^{n} \binom{n}{m} \gamma_F(m-1) + \sum_{m=1}^{n} \binom{n}{m} \tau_F(m-1). \tag{31}$$

Following the lines of the proof of Lemma 4.3 in [16] it is easy to deduce that $\tau_F(m-1) = \eta_F(m-1)$, for $m \geqslant 2$, while $\tau_F(0) = n \log Q_F + \eta_F(0)$ and the proof of formula (19) is complete. □

Appendix B. The Jorgenson–Lang fundamental class of functions

The explicit formula for the class $\mathcal{S}^{\#\flat}$, with a very broad class of test functions was the main tool used in proofs of our main Theorems 4.1, 6.1 and 6.2. Its proof follows from the fact that $S^{\#\flat}$ is contained in a much wider class, a fundamental class of functions, introduced by J. Jorgenson and S. Lang in [12]. In order to make this paper self-contained, in this appendix we recall definition of the fundamental class of functions and show that $S^{\#\flat}$ is the subclass of this class.

The fundamental class of functions is a class of triples $(Z, \widetilde{Z}, \Phi)$ satisfying following three conditions [12, pp. 45–46]:

1. (Meromorphy) Functions Z and $\widetilde{Z}$ are meromorphic functions of a finite order.

2. (Euler sum) There are sequences $\{q\}$ and $\{\widetilde{q}\}$ of real numbers greater than one, depending on Z and $\widetilde{Z}$ such that for every q and $\widetilde{q}$ there exist complex numbers $c(q)$ and $c(\widetilde{q})$ and $\sigma_0' \geqslant 0$ such that for all $\operatorname{Re} s > \sigma_0'$

$$\log Z(s) = \sum_{q} \frac{c(q)}{q^s} \quad \text{and} \quad \log \widetilde{Z}(s) = \sum_{\widetilde{q}} \frac{c(\widetilde{q})}{\widetilde{q}^s}.$$

The series are assumed to converge uniformly and absolutely in any half-plane of the form $\operatorname{Re} s \geqslant \sigma_0' + \epsilon > \sigma_0'$.

3. (Functional equation) There exist a meromorphic function Φ of finite order and σ_0 with $0 \leqslant \sigma_0 \leqslant \sigma_0'$ such that

$$Z(s)\Phi(s) = \widetilde{Z}(\sigma_0 - s)$$

and the factor Φ of the functional equation is of a regularized product type.

The function Φ is of a *regularized product type* [12, Definition 6.1] if it can be written as

$$\Phi(s) = e^{P(s)} Q(s) \prod_{j=1}^{m} D_j(\alpha_j s + \beta_j)^{k_j}, \tag{32}$$

where $Q(s)$ is a rational function, $P(s)$ is a polynomial, k_j is an integer, D_j is a regularized product and complex numbers α_j and β_j are chosen such that the zeros and poles of D_j lie in the union of vertical strips and sectors $\{z \in \mathbb{C}\colon -\pi/2 + \epsilon < \arg(z) < \pi/2 + \epsilon\}$ and $\{z \in \mathbb{C}\colon \pi/2 + \epsilon < \arg(z) < 3\pi/2 - \epsilon\}$, for some $\epsilon > 0$.

The definition of a *regularized product* associated to some sequences of complex numbers is fully described in [11, Part I, Section 2]. Since the definition is rather long, let us note here that a regularized product can be viewed as a generalization of a Weierstrass product. Therefore, the (classical) gamma function is a regularized product. A *reduced order of a regularized product* D_j is defined as a pair of numbers (M_j, m_j) depending on D_j in a way that is fully described in [12, pp. 18–19]. For our purposes it is sufficient to know that a reduced order controls the growth of D_j in vertical strips. Namely, if (M_j, m_j) is a reduced order of D_j, then, $\frac{D_j'}{D_j}(\sigma \pm iT)$ grows at most as $T^{M_j} \log^{m_j} |T|$.

Specially, a gamma function is a regularized product of reduced order $(0,0)$, as proved in [12, Ex. 1 on p. 39].

The notion of a reduced order of a function that is of a regularized product type is important in the proof of the explicit formula. Namely, this order controls the growth of functions $\frac{Z'}{Z}$ and $\frac{\Phi'}{\Phi}$ in vertical strips, hence affects the conditions posed on the test function. A *reduced order of a function* $\Phi(s)$ *of a regularized product type* defined by (32) is (M,m) where $M = \max\{\deg P - 1, M_j\}$ and m is the largest of numbers m_j such that $M_j = M$.

Lemma B.1. *The family of triples* $(F, \overline{F}, \Psi_F)$, *where* $F \in \mathcal{S}^{\#\flat}$ *is contained in the fundamental class.*

Proof. The first axiom of the fundamental class is satisfied, since, by axiom (ii) (of the Selberg class) F and $\overline{F}$ are meromorphic functions of finite order. The second axiom of the fundamental class is satisfied with sequences $\{q\}$ and $\{\widetilde{q}\}$ taken to be the sequence of positive integers $n \geqslant 2$, $c(q) = \frac{c_F(n)}{\log n}$, $c(\widetilde{q}) = \frac{\overline{c_F(n)}}{\log n}$ and $\sigma_0' = 1$. Namely, the function $G(s) = \sum_{n=2}^{\infty} \frac{c_F(n)}{\log n \cdot n^s}$ is defined by the absolutely convergent Dirichlet series and, hence, holomorphic in the half-plane $\operatorname{Re} s > 1$. Furthermore, $G'(s) = -\sum_{n=2}^{\infty} \frac{c_F(n)}{n^s} = \frac{F'}{F}(s) = (\log F)'(s)$. Therefore, $\log F(s)$ also has a Dirichlet series representation converging absolutely in the half-plane $\operatorname{Re} s > 1$.

Finally, the functional equation axiom of the fundamental class is satisfied with $\sigma_0 = \sigma_0' = 1$ and

$$\Phi(s) = \Psi_F(s) = w Q_F^{2s-1} \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)\big(\Gamma\big(\lambda_j(1-s) + \overline{\mu}_j\big)\big)^{-1}.$$

Function $\Psi_F(s)$ is of a regularized product type since gamma function is a regularized product and numbers λ_j and μ_j are such that poles and zeros gamma factors lie in the union of vertical strips and sectors, described above. Obviously, the reduced order of Ψ_F is $(0,0)$. □

Remark B.2. The class $S^{\#\flat}$ is unchanged if the axiom (v′) be replaced with the assumption that

$$\log F(s) = \sum_{n=2}^{\infty} \frac{c_F(n)}{\log n \cdot n^s},$$

where the series on the right converges absolutely in the half-plane $\operatorname{Re} s > 1$. The reason for stating the axiom (v′) in the form of the logarithmic derivative instead of logarithm (of a function F) is the proof of the explicit formula (Theorem 3.1), where only the logarithmic derivative appears.

Finally, let us note that triples $(F, \overline{F}, \Psi_F)$, where $F \in \mathcal{S}^{\#}$ need not belong to the fundamental class since $\mathcal{S}^{\#}$ may contain functions which have zeros in all half-planes of the form $\operatorname{Re} s > \sigma$, with $\sigma > 0$. Such functions do not have an Euler sum, since the Euler sum axiom in the fundamental class implies non-vanishing of the function in the half-plane $\operatorname{Re} s > \sigma_0'$.

Supplementary material

The online version of this article contains additional supplementary material. Please visit doi:10.1016/j.jnt.2009.10.012.

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