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Universality of the Riemann zeta-function

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ABSTRACT

In 1975, S.M. Voronin proved the universality of the Riemann zeta-function $\zeta(s)$. This means that every non-vanishing analytic function can be approximated uniformly on compact subsets of the critical strip by shifts $\zeta(s+i\tau)$. In the paper, we consider the functions $F(\zeta(s))$ which are universal in the Voronin sense.

1. Introduction

Let, as usual, $\zeta(s)$, $s=\sigma+it$, denote the Riemann zeta-function defined, for $\sigma>1$, by

$$\zeta(s)=\sum_{m=1}^{\infty}\frac{1}{m^s},$$

and by analytic continuation elsewhere, except for a simple pole at $s=1$ with residue 1.

In [9], see also [3,10], Voronin discovered a remarkable universality property of the function $\zeta(s)$. Roughly speaking, the universality of $\zeta(s)$ means that every analytic function can be approximated uniformly on some sets by translations $\zeta(s+i\tau)$. The original version of the Voronin theorem is the following.

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Theorem 1. *Let $0 < r < \frac{1}{4}$. Suppose that the function $f(s)$ is continuous on the disc $|s| \leqslant r$ and analytic in interior of this disc. If $f(s)$ has no zeros in the interior of the disc $|s| \leqslant r$, then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leqslant r}\left|f(s) - \zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

In [9], Voronin gives a direct proof of Theorem 1, while in [3] the theorem is deduced from the universality of $\log\zeta(s)$. We remind that $\log\zeta(\sigma + it)$, $\frac{1}{2} < \sigma < 1$, is defined from $\log\zeta(2) \in \mathbb{R}$ by continuous variation along the line segments $[2, 2 + it]$ and $[2 + it, \sigma + it]$, provided that the path does not pass a zero or pole of $\zeta(s)$. If it does, then we take $\log\zeta(s+it) = \lim_{\varepsilon\to+0}\log\zeta(\sigma + i(t+\varepsilon))$.

Theorem 2. *(See [3].) Let $0 < r < \frac{1}{4}$. Suppose that the function $g(s)$ is continuous on the disc $|s| \leqslant r$ and analytic in interior of this disc. Then, for every $\varepsilon > 0$, there exists a real number $\tau = \tau(\varepsilon)$ such that*

$$\max_{|s|\leqslant r}\left|g(s) - \log\zeta\left(s + \frac{3}{4} + i\tau\right)\right| < \varepsilon.$$

The modern version of the Voronin theorem has a more general form. Denote by $\mathrm{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\,\mathrm{meas}\big\{\tau \in [0, T]:\ \ldots\big\},$$

where in place of dots a condition satisfied by τ is to be written. Moreover, let $D = \{s \in \mathbb{C}: \frac{1}{2} < \sigma < 1\}$.

Theorem 3. *Suppose that K is a compact subset of the strip D with connected complement, and let $f(s)$ be a non-vanishing continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\nu_T\Big(\sup_{s\in K}\big|\zeta(s + i\tau) - f(s)\big| < \varepsilon\Big) > 0.$$

Proof of Theorem 3 is given, for example, in [5], see also [1,6–8].

It is known [1,4] that the derivative of $\zeta(s)$ is also universal.

Theorem 4. *Let K be the same as in Theorem 3, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\nu_T\Big(\sup_{s\in K}\big|\zeta'(s + i\tau) - g(s)\big| < \varepsilon\Big) > 0.$$

Note that in Theorems 2 and 4 the approximated function $g(s)$ is not necessarily non-vanishing.

Theorems 2 and 4 show that certain functions of $\zeta(s)$ are also universal. Therefore, a problem arises to describe a set of functions F such that $F(\zeta(s))$ should be universal in the above sense.

Let G be a region on the complex plane. Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. A sufficiently wide class of functions $F : H(D) \to H(D)$ with the universality property of $F(\zeta)$ is described as follows. Suppose that $F^{-1}g \in H(D)$ for each $g \in H(D)$, and that F is of the Lipschitz type, i.e., for all $g_1, g_2 \in H(D)$, there exist positive constants c and $\alpha \leqslant 1$, and a compact subset $K_1 \subset D$ with connected complement such that, for every compact subset $K \subset D$ with connected complement,

$$\sup_{s\in K}\big|F\big(g_1(s)\big) - F\big(g_2(s)\big)\big| \leqslant c\sup_{s\in A}\big|g_1(s) - g_2(s)\big|^{\alpha}$$

for some $A \subset K_1$. Clearly, the universality of $F(\zeta)$ for the function F of the above type is a simple consequence of the universality of $\zeta(s)$ itself. For example, an application of the Cauchy integral formula shows that function $F(g(s)) = g'(s)$, $g \in H(D)$, is of the Lipschitz type with $\alpha = 1$. Thus, this gives an alternative proof of the universality for $\zeta'(s)$.

Our aim is to present more general results. Let

$$S_\zeta = \{g \in H(D)\colon\ g(s) \neq 0\ \forall s \in D,\ \text{or}\ g(s) \equiv 0\}.$$

Denote by U the class of continuous functions $F : H(D) \to H(D)$ such that, for any open set $G \subset H(D)$,

$$\left(F^{-1}G\right) \cap S_\zeta \neq \emptyset.$$

Theorem 5. *Suppose that $F \in U$. Let K and $g(s)$ be the same as in Theorem* 4. *Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\Bigl(\sup_{s\in K}\bigl|F\bigl(\zeta(s+i\tau)\bigr) - g(s)\bigr| < \varepsilon\Bigr) > 0.$$

It is difficult to check the hypotheses of Theorem 5. The next theorem gives more convenient conditions for the universality of $F(\zeta(s))$.

For arbitrary $V > 0$, let $D_V = \{s \in \mathbb{C}\colon\ \frac{1}{2} < \sigma < 1,\ |t| < V\}$, and $S_{\zeta,V} = \{g \in H(D_V)\colon\ g(s) \neq 0\ \forall s \in D,\ \text{or}\ g(s) \equiv 0\}$. Denote by U_V the class of continuous functions $F : H(D_V) \to H(D_V)$ such that, for each polynomial $p = p(s)$,

$$\left(F^{-1}\{p\}\right) \cap S_{\zeta,V} \neq \emptyset.$$

Theorem 6. *Suppose that $F \in U_V$. Let K and $g(s)$ be the same as in Theorem* 4. *Then the assertion of Theorem* 5 *is true.*

For example, for $f \in H(D_V)$, let

$$F(f) = c_1 f'(s) + c_2 f''(s), \quad c_1, c_2 \in \mathbb{C},\ c_1c_2 \neq 0.$$

Then the function F is continuous. Moreover, for each polynomial $p(s)$, there exists a polynomial $q(s)$ such that $q \in F^{-1}\{p\}$ and $q(s) \neq 0$ for $s \in D_V$ in view of the definition of D_V. Therefore, by Theorem 6, the function $F(\zeta(s))$ is universal.

Approximation by shifts $F(\zeta(s+i\tau))$ can be realized on a subset of $H(D)$, for example, on S_ζ. Let a and b be two complex numbers, and denote by $U_{a,b}$ the class of continuous functions $F : H(D) \to H(D)$ such that $F(S_\zeta) = H_{a,b}(D)$, where

$$H_{a,b}(D) = \{g \in H(D)\colon\ g(s) \neq a,\ g(s) \neq b\ \forall s \in D,\ \text{or}\ g(s) \equiv F(0)\}.$$

Theorem 7. *Suppose that $F \in U_{a,b}$, and K is the same as in Theorem* 3. *Let $g(s)$ be a continuous on K function, $g(s) \neq a$, $g(s) \neq b$, on K, which is analytic in the interior of K. Then the assertion of Theorem* 5 *is true.*

Remark. The set $H_{a,b}(D)$ can be replaced by

$$H_{a_1,\dots,a_k}(D) = \{g \in H(D)\colon\ g(s) \neq a_j,\ j = 1,\dots,k,\ \forall s \in D,\ \text{or}\ g(s) \equiv F(0)\},$$

where $a_1,\dots,a_k$ are complex numbers, $k \in \mathbb{N}$.

Suppose that $a = b = 0$. Then Theorem 7 implies the universality of $F(\zeta(s)) = (\zeta(s))^N$, $N \in \mathbb{N}$. In this case,

$$F(S_\zeta) = \{g \in H(D)\colon\ g(s) \neq 0\ \forall s \in D,\ \text{or}\ g(s) \equiv 0\}.$$

If $a = 0$ and $b = 1$, then, by Theorem 7, the function $F(\zeta(s)) = e^{\zeta(s)}$ is also universal because

$$F(S_\zeta) = \{g \in H(D)\colon\ g(s) \neq 0,\ g(s) \neq 1,\ \forall s \in D,\ \text{or}\ g(s) \equiv 1\}.$$

2. Limit theorems

The principal ingredient for the proof of universality for $F(\zeta(s))$ is a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions. Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and define the probability measure

$$P_{T,F}(A) = \nu_T\big(F\big(\zeta(s+i\tau)\big) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big).$$

We will derive a limit theorem with explicitly given limit measure for the measure $P_{T,F}$ as $T \to \infty$ from a limit theorem for the measure

$$P_T(A) = \nu_T\big(\zeta(s+i\tau) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big).$$

For this, we will apply the following property of the weak convergence of probability measures. Let S and S_1 be two metric spaces, and let $h : S \to S_1$ be a Borelian function. Then every probability measure P on $(S, \mathcal{B}(S))$ induces on $(S_1, \mathcal{B}(S_1))$ the unique probability measure Ph^{-1} defined by the equality $Ph^{-1}(A) = P(h^{-1}A)$, $A \in \mathcal{B}(S_1)$. Denote by D_h the set of discontinuity points of the function h. If the space S is separable, then $D_h \in \mathcal{B}(S)$ [2].

Lemma 8. *Suppose that P and P_n, $n \in \mathbb{N}$, are probability measures on $(S, \mathcal{B}(S))$, and $P(D_h) = 0$. If P_n converges weakly to P as $n \to \infty$, then also $P_n h^{-1}$ converges weakly to Ph^{-1} as $n \to \infty$.*

Proof of the lemma is given, for example, in [2, Theorem 5.1].

Let $\gamma = \{s \in \mathbb{C}\colon\ |s| = 1\}$ be the unit circle on the complex plane. Define

$$\Omega = \prod_p \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and this gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p, and on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D)$-valued random element $\zeta(s, \omega)$ by the formula

$$\zeta(s,\omega) = \prod_p \left(1 - \frac{\omega(p)}{p^s}\right)^{-1}.$$

Note that the latter infinite product, for almost all $\omega \in \Omega$, converges uniformly on compact subsets of the strip D. Denote by P_ζ the distribution of the random element $\zeta(s, \omega)$, i.e.,

$$P_\zeta(A) = m_H\big(\omega \in \Omega\colon\ \zeta(s,\omega) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big).$$

Lemma 9. *The probability measure P_T converges weakly to the measure P_ζ as $T \to \infty$.*

Proof of the lemma can be found in [5].

Let $P_{T,V}$ and $P_{\zeta,V}$ be the restrictions to $(H(D_V), \mathcal{B}(H(D_V)))$ of the measures P_T and P_ζ, respectively.

Corollary 10. *The probability measure $P_{T,V}$ converges weakly to $P_{\zeta,V}$ as $T \to \infty$.*

Proof. The corollary is a consequence of Lemmas 9 and 8. □

Theorem 11. *Suppose that $F : H(D) \to H(D)$ is a continuous function. Then the probability measure $P_{T,F}$ converges weakly to the distribution of the random element $F(\zeta(s,\omega))$ as $T \to \infty$.*

Proof. We have that $P_{T,F} = P_T F^{-1}$. Therefore, the continuity of F, and Lemmas 9 and 8 show that the measure $P_{T,F}$ converges weakly to $P_\zeta F^{-1}$ as $T \to \infty$. Thus the definition of $P_\zeta F^{-1}$ gives the assertion of the theorem. □

Denote by $P_{T,F,V}$ and $\zeta_V(s,\omega)$ the restrictions to $H(D_V)$ of probability measure $P_{T,F}$ and the random element $\zeta(s,\omega)$, respectively.

Theorem 12. *Suppose that $F : H(D_V) \to H(D_V)$ be a continuous function. Then the probability measure $P_{T,F,V}$ converges weakly to the distribution of the random element $F(\zeta_V(s,\omega))$ as $T \to \infty$.*

Proof. We use the same arguments as in the proof of Theorem 11, and Corollary 10. □

3. Supports

Let S be a separable metric space, and P be a probability measure on $(S, \mathcal{B}(S))$. We remind that a minimal closed set $S_P \subseteq S$ such that $P(S_P) = 1$ is called a support of P. The set S_P consists of all $x \in S$ such that, for every open neighbourhood G of x, the inequality $P(G) > 0$ is satisfied. Moreover, the support of the distribution of a random element is called a support of this element.

In this section, we consider the support of the random element $F(\zeta(s,\omega))$. For this, we will apply the following statement.

Lemma 13. *The support of the random element $\zeta(s,\omega)$ is the set S_ζ.*

Proof of the lemma is given in [5, Lemma 6.5.5].

Theorem 14. *Suppose that $F \in U$. Then the support of the random element $F(\zeta(s,\omega))$ is the whole of $H(D)$.*

Proof. Since the function F is continuous, for any open set $G \subset H(D)$, we have that $F^{-1}G$ is an open set, too. Moreover, by the definition of the class U, there exists an element x of S_ζ such that $x \in F^{-1}G$, i.e., $F^{-1}G$ is an open neighbourhood of x. Hence, by Lemma 13,

$$m_H\big(\omega \in \Omega\colon\ F\big(\zeta(s,\omega)\big) \in G\big) = m_H\big(\omega \in \Omega\colon\ \zeta(s,\omega) \in F^{-1}G\big) > 0,$$

and this proves the theorem. □

For the support of $F(\zeta_V(s,\omega))$, $F \in U_V$, we need the Mergelyan theorem on the approximation of analytic functions by polynomials.

Lemma 15. *Let $K \subset \mathbb{C}$ be a compact subset with connected complement, and let $g(s)$ be a function continuous on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$, there exists a polynomial $p(s)$ such that*

$$\sup_{s \in K} \big| g(s) - p(s) \big| < \varepsilon.$$

Proof of the lemma is given, for example, in [11].

Also, for every open G, we remind a metric in $H(G)$ that induces its topology of uniform convergence on compacta. It is well known that there exists a sequence $\{K_n\}$ of compact subsets of G such that

$$G = \bigcup_{n=1}^{\infty} K_n,$$

$K_n \subset K_{n+1}$, $n \in \mathbb{N}$, and if $K \subset G$ is a compact subset, then $K \subset K_n$ for some n. For $f, g \in H(G)$, define

$$\rho(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(f, g)}{1 + \rho_n(f, g)},$$

where

$$\rho_n(f, g) = \sup_{s \in K_n} \left| f(s) - g(s) \right|.$$

Then $\rho(f, g)$ is desired metric.

Lemma 16. *The support of the random element $\zeta_V(s, \omega)$ is the whole of $H(D_V)$.*

Proof of the lemma completely coincides with that of Lemma 13, see Lemma 6.5.5 of [5].

Theorem 17. *Suppose that $F \in U_V$. Then the support of the random element $F(\zeta_V(s, \omega))$ is the whole of $H(D_V)$.*

Proof. Let G be an open set of $H(D_V)$. Then $F^{-1}G$ also is an open set. We will prove that $S_{\zeta,V} \cap F^{-1}G \neq \emptyset$.

Suppose that $\{K_n\}$ is a sequence of compact subsets of D_V whose occur in the definition of the metric on $H(D_V)$. Obviously, we can choose K_n to be with connected complement, $n \in \mathbb{N}$. Hence, we have that g approximates f with given accuracy in $H(D_V)$ if g approximates f with a suitable accuracy uniformly on K_n for sufficiently large n. Therefore, in $H(D_V)$, it suffices to consider an approximation on a compact subsets of D_V.

If K is a compact subset of D_V with connected complement, then, by Lemma 15, there exists a polynomial $p = p(s)$ which approximate $f(s) \in H(D_V)$ with a given accuracy uniformly on K. Therefore, if $f \in G$, then we may assume that $p \in G$, too. Hence, by the definition of the class U_V, we obtain that the set $(F^{-1}G) \cap S_{\zeta,V} \neq \emptyset$. This and Lemma 16 show, as in the proof of Theorem 14, that the support of $F(\zeta_V(s, \omega))$ is the whole of $H(D_V)$. □

Theorem 18. *Suppose that $F \in U_{a,b}$. Then the support of the random element $F(\zeta(s, \omega))$ is the set $H_{a,b}(D)$.*

Proof. By the definition of the class $U_{a,b}$, we have that, for each $f \in H_{a,b}(D)$, there exists $g \in S_\zeta$ such that $F(g) = f$. This shows that every open neighbourhood G of $f \in H_{a,b}(D)$ has a positive measure:

$$m_H\big(\omega \in \Omega\colon\ F\big(\zeta(s, \omega)\big) \in G\big) > 0.$$

Moreover,

$$m_H\big(\omega \in \Omega\colon\ F\big(\zeta(s, \omega)\big) \in H_{a,b}(D)\big) = m_H\big(\omega \in \Omega\colon\ \zeta(s, \omega) \in S_\zeta\big) = 1,$$

by Lemma 13. □

4. Main theorems

We will use the following property of the weak convergence of probability measures.

Lemma 19. *Let P and P_n, $n \in \mathbb{N}$, be probability measures on $(S, \mathcal{B}(S))$, and P_n converges weakly to P as $n \to \infty$. Then, for every open set G of $\mathbb{E}$,*

$$\liminf_{n\to\infty} P_n(G) \geqslant P(G).$$

The lemma is a part of Theorem 2.1 from [2].

Proof of Theorem 5. By Lemma 15, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\bigl|g(s) - p(s)\bigr| < \frac{\varepsilon}{2}. \tag{1}$$

Define

$$\mathcal{G} = \left\{h \in H(D)\colon \sup_{s\in K}\bigl|p(s) - h(s)\bigr| < \frac{\varepsilon}{2}\right\}.$$

Then $\mathcal{G}$ is an open set. In view of Theorem 14, $p(s)$ is an element of the support of the distribution $P_{\zeta,F}$ of the random element $F(\zeta(s,\omega))$. Since $\mathcal{G}$ is an open neighbourhood of $p(s)$, this shows that

$$P_{\zeta,F}(\mathcal{G}) > 0. \tag{2}$$

Theorem 11 together with Lemma 19 implies

$$\liminf_{T\to\infty} \nu_T\bigl(F\bigl(\zeta(s+i\tau)\bigr) \in \mathcal{G}\bigr) \geqslant P_{\zeta,F}(\mathcal{G}).$$

Therefore, the definition of $\mathcal{G}$ and (2) yield the inequality

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\bigl|F\bigl(\zeta(s+i\tau)\bigr) - p(s)\bigr| < \frac{\varepsilon}{2}\right) > 0.$$

Hence and from (1) the theorem follows. □

Proof of Theorem 6. There exists $V > 0$ such that $K \subset D_V$. We fix such a number V. The next part of the proof uses Theorems 12 and 17, and completely coincides with the proof of Theorem 5. □

Proof of Theorem 7. By Lemma 15, there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\bigl|g(s) - p(s)\bigr| < \frac{\varepsilon}{6}. \tag{3}$$

Since $g(s) \neq a$ and $g(s) \neq b$ on K, we have that $p(s) \neq a$, $p(s) \neq b$ on K as well if ε is small enough. Therefore, we can define a branch of $\log(p(s) - a)$ which will be analytic function in the interior of K. Again, by Lemma 15, there exists a polynomial $p_1(s)$ such that

$$\sup_{s\in K}\bigl|p(s) - a - \mathrm{e}^{p_1(s)}\bigr| < \frac{\varepsilon}{6}.$$

Moreover, $p_1(s) \neq \log(b-a)$ on K, where the principal value of logarithm is taken, if ε is small enough. Hence,

$$\sup_{s\in K}\left|p(s) - \left(e^{p_1(s)} + a\right)\right| < \frac{\varepsilon}{6}. \tag{4}$$

Similarly, Lemma 15 shows that there exists a polynomial $p_2(s)$ such that

$$\sup_{s\in K}\left|e^{p_1(s)-\log(b-a)} - e^{e^{p_2(s)}}\right| < \frac{\varepsilon}{6(b-a)}.$$

Thus,

$$\sup_{s\in K}\left|e^{p_1(s)} - e^{e^{p_2(s)}}(b-a)\right| < \frac{\varepsilon}{6}. \tag{5}$$

We have that the function

$$h_{a,b}(s) = e^{e^{p_2(s)}}(b-a) + a$$

is analytic on D, and $h_{a,b}(s) \neq a$, $h_{a,b}(s) \neq b$. Therefore, in view of Theorem 18, $h_{a,b}(s)$ is an element of the support of the random element $F(\zeta(s,\omega))$. Moreover, combining inequalities (3)–(5), we find that

$$\sup_{s\in K}\left|g(s) - h_{a,b}(s)\right| < \frac{\varepsilon}{2}. \tag{6}$$

Define

$$\mathcal{G} = \left\{f \in H(D)\colon \sup_{s\in K}\left|h_{a,b}(s) - f(s)\right| < \frac{\varepsilon}{2}\right\}.$$

Then, as in the proof of Theorem 5, we have that

$$P_{\zeta,F}(\mathcal{G}) > 0,$$

and, by Theorem 11 and Lemma 19, we obtain that

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\left|F\left(\zeta(s+i\tau)\right) - h_{a,b}(s)\right| < \frac{\varepsilon}{2}\right) > 0.$$

This together with (6) proves the theorem.

If $a = b$, then we similarly obtain that there exists a polynomial $p(s)$ such that

$$\sup_{s\in K}\left|g(s) - h_a(s)\right| < \frac{\varepsilon}{2}$$

with

$$h_a(s) = e^{p(s)} + a.$$

Thus, from this we deduce the theorem in the above way. □

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References

[1] B. Bagchi, A joint universality theorem for Dirichlet L-functions, Math. Z. 181 (1982) 319–334.
[2] P. Billingsley, Convergence of Probability Measures, Willey, New York, 1968, second ed., Willey–Interscience, 1999.
[3] A.A. Karatsuba, S.M. Voronin, The Riemann Zeta-Function, Walter de Gruyter, Berlin, 1982.
[4] A. Laurinčikas, Zeros of the derivative of the Riemann zeta-function, Liet. Mat. Rink. 25 (1985) 111–118 (in Russian); English translation: Lithuanian Math. J. 25 (1985) 255–260.
[5] A. Laurinčikas, Limit Theorems for the Riemann Zeta-Function, Kluwer, Dordrecht–Boston–London, 1996.
[6] A. Laurinčikas, The universality of zeta-functions, Acta Appl. Math. 78 (1–3) (2003) 251–271.
[7] K. Matsumoto, Probabilistic value-distribution theory of zeta-functions, Sugaku 53 (2001) 279–296 (in Japanese); English translation: Sugaku Expositions 17 (2004) 51–71.
[8] J. Steuding, Value Distribution of L-Functions, Lecture Notes in Math., vol. 1877, Springer-Verlag, Berlin–Heidelberg, 2007.
[9] S.M. Voronin, Theorem of the "universality" of the Riemann zeta-function, Izv. Akad. Nauk SSSR. Ser. Mat. 39 (1975) 475–486 (in Russian); English translation: Math. USSR Izv. 9 (1975) 443–453.
[10] S.M. Voronin, in: A.A. Karatsuba (Ed.), Collected Works: Mathematics, Publishing House MGTU im. N.E. Baumana, Moscow, 2006 (in Russian).
[11] J.L. Walsh, Interpolation and Approximation by Rational Functions in the Complex Domain, Amer. Math. Soc. Collq. Publ., vol. 20, 1960.