q-BERNOULLI NUMBERS AND POLYNOMIALS ASSOCIATED WITH MULTIPLE q-ZETA FUNCTIONS AND BASIC L-SERIES

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Abstract. By using q-Volkenborn integration and uniform differentiable on $\mathbb{Z}_p$, we construct p-adic q-zeta functions. These functions interpolate the q-Bernoulli numbers and polynomials. The value of p-adic q-zeta functions at negative integers are given explicitly. We also define new generating functions of q-Bernoulli numbers and polynomials. By using these functions, we prove analytic continuation of some basic (or q-) L-series. These generating functions also interpolate Barnes' type Changhee q-Bernoulli numbers with attached to Dirichlet character as well. By applying Mellin transformation, we obtain relations between Barnes' type q-zeta function and new Barnes' type Changhee q-Bernolli numbers. Furthermore, we construct the Dirichlet type Changhee (or q-) L-functions.

1. Introduction, Definition and Notations

For any complex number z, it is well known that the usual Bernoulli polynomials $B_n(z)$ are defined by means of of the generating function (see [60], [51], [2], and [7]):

$$F(t,x)=\frac{te^{zt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(z)\frac{t^n}{n!}\quad (\mid t\mid< 2\pi\ ). \tag{1.1}$$

Note that, substituting $z=0$ into (1.1), $B_n(0)=B_n$ is the usual nth Bernoulli number:

$$F(t)=\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\quad (\mid t\mid< 2\pi\ ). \tag{1.2}$$

Over five decades ago, Carlitz [5] defined q-extensions of these classical Bernoulli numbers and polynomials and proved properties generating those satisfied by B_n and $B_n(z)$. Recently, Koblitz [35] used these properties, especially the so-called *distribution relation* for q-Bernoulli polynomials, in order to construct the corresponding q-extensions of the p-adic measures and to define a q-extension of p-adic Dirichlet L-series.

When one talks of q-extensions, q can be variously considered as an indetermined, a complex number $q\in\mathbb{C}$, or, when p be a prime number, a p-adic number $q\in\mathbb{C}_p$, where $\mathbb{C}_p$ is the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. If $q\in\mathbb{C}$, one normally assumes $\mid q\mid<1$. If $q\in\mathbb{C}_p$, then we assume

$$\mid q-1\mid_p< p^{-\frac{1}{p-1}},$$

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so that

$$q^x = \exp(x \log q) \text{ for } |\ x\ |_p \leq 1.$$

We use the notation (see also [55], p. 346 *et seq.*):

$$[x] = [x : q] = \frac{1-q^x}{1-q}.$$

Thus

$$\lim_{q \to 1}[x : q] = x$$

for any $x \in \mathbb{C}$ in the complex case and any x with $|\ x\ |_p \leq 1$ in the p-adic case (see [35], [36] and [17]).

Carlitz's q-Bernoulli numbers $\beta_n = \beta_n(q)$ can be determined inductively by[5]

$$\beta_0 = 1, q(q\beta+1)^n - \beta_n = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

with the usual convention about replacing β^n by β_n.

The q-Bernoulli polynomials $\beta_n(x : q)$ are given as $(q^x\beta + [x])^n$, i.e., as follows

$$\beta_n(x : q) = \sum_{k=0}^{n} \binom{n}{k} \beta_k q^{kx}[x]^{n-k}.$$

As $q \to 1$, we have $\beta_n(q) \to B_n, \beta_n(x : q) \to B_n(x)$.

Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$, the set of positive integer numbers. Then the generalized Bernoulli numbers $B_{n,\chi}$ are defined by

$$F_\chi(t) = \sum_{a=0}^{f-1} \frac{\chi(a)te^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_{n,\chi}\frac{t^n}{n!} \quad (\ |\ t\ |< 2\pi\ ). \tag{1.3}$$

Then we defined generalized Carlitz's q-Bernoulli number $\beta_{m,\chi}(q) = \beta_{m,\chi}$ as follows[11], [14], [10]

$$\beta_{m,\chi}(q) = [f]^{m-1}\sum_{a=0}^{f-1} \chi(a)q^a\beta_m(\frac{a}{f} : q^f). \tag{1.4}$$

As $q \to 1$,(1.4) is reduced to (1.3).

The Euler numbers E_n are usually defined by means of of the following generating function (see, for example, [54], p. 63, Eq. 1.6 (40); see also [56]) for different definition):

$$\frac{2e^t}{e^{2t}+1} = \sec h(t) = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} \quad (\ |\ t\ |< \frac{\pi}{2}\ ).$$

These numbers are classical and important in number theory. Frobenius extended such numbers as E_n to the so-called Frobenius-Euler numbers $H_n(u)$ belonging to an algebraic number u, with $|\ u\ |> 1$, and many authors have investigated their properties ([17], [22]). Shiratani and Yamamoto [51] constructed a p-adic interpolation $G_p(s,u)$ of the Frobenius-Euler numbers $H_n(u)$ and as its application, they obtained an explicit formula for $L_p^{'}(0,\chi)$ with any Dirichlet character χ. In [56], Tsumura defined the generalized Frobenius-Euler numbers $H_{n,\chi}(u)$ for any Dirichlet character χ, which are analogous to the generalized Bernoulli numbers. He constructed their Shiratani and Yamamoto p-adic interpolation $G_p(s,u)$ of $H_n(u)$.

Let u be an algebraic number. For $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler numbers $H_n(u)$ belonging to u are defined by means of of the generating function

$$\frac{1-u}{e^t-u} = e^{H(u)t}$$

with usual convention of symbolically replacing $H^n(u)$ by $H_n(u)$. Thus for the Frobenius-Euler numbers $H_n(u)$ belonging to u, we have (see[50])

$$\frac{1-u}{e^t-u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}. \tag{1.5}$$

By using (1.5), and following the usual convention of symbolically replacing $H^n(u)$ by $H_n(u)$, we have

$$H_0 = 1 \text{ and } (H(u)+1)^n = uH_n(u) \text{ for } (n \geq 1).$$

We also note that

$$H_n(-1) = \mathfrak{E}_n,$$

where $\mathfrak{E}_n$ denotes the aforementioned Tsumura version (see[50]) of the classical Euler numbers E_n which we recalled above.

For an algebraic number $u \in \mathbb{C}$ with $|u| > 1$, the Frobenius-Euler polynomials belonging to u, that is, the polynomials $H_n(u, x)$ are defined by (see[56])

$$\frac{1-u}{e^t-u}e^{xt} = e^{H(u,x)t} = \sum_{n=0}^{\infty} H_n(u,x)\frac{t^n}{n!} \tag{1.6}$$

with usual convention of symbolically replacing H^n by H_n as before. By using (1.5) and (1.6), we readily have

$$H_n(u,0) = H_n(u) \text{ and } H_n(u,x) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} H_k(u)x^{n-k}.$$

Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We define the nth generalized Euler numbers $H_{n,\chi}(u)$ belonging to u, by[56]

$$\sum_{a=0}^{f-1} \frac{(1-u^f)\chi(a)e^{at}u^{f-a-1}}{e^{ft}-u^f} = \sum_{n=0}^{\infty} H_{n,\chi}(u)\frac{t^n}{n!}. \tag{1.7}$$

By using (1.5) to (1.7), we can easily see that

$$\begin{aligned} H_{n,\chi}(u) &= f^n \sum_{a=0}^{f-1} \chi(a)u^{f-a-1}H_n(u^f, \frac{a}{f}) \\ &= \sum_{a=0}^{f-1} \chi(a)u^{f-a-1} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} H_k(u^f)a^{n-k}f^k. \end{aligned}$$

We note that, when $\chi = 1$, we have

$$H_{n,1}(u) = H_n(u), \text{ for } (n \geq 0).$$

Carlitz[5] also defined q-Euler numbers and polynomials as follows:

$$H_0(u:q) = 1 \text{ and } (qH(u:q)+1)^n - uH_n(u:q) = 0 \; (n \geq 1),$$

where u is a complex number $|u|>1$. For $n\geq 0$, and with the usual convention of replacing H^n by H_n, we have (see [5])

$$H_n(u,x:q)=(q^x H(u,x:q)+[x])^n. \tag{1.8}$$

When $q\to 1$ in (1.8), we obtain the following limit relationship with the Frobenius-Euler numbers $H_n(u)$ (1.5):

$$\lim_{q\to 1} H_n(u,1:q)=H_n(u)$$

(see, for detail [11], [8], [1],[22]).

Consider the finite products $E_n=\prod_{j<n} X_j$ of a sequence $(X_j)_{j\geq 0}$ of sets. We would like to say that these partial products converge to the infinite product $E=\prod_{j\geq 0} X_j$ and thus consider this last product as limit of the sequence (E_n). The *projective limit* $E=\lim_{\leftarrow} E_n$ is defined by (see [46], p. 28):

Definition 1. *A sequence $(E_n,\varphi_n)_{n\geq 0}$ of sets and maps $\varphi_n : E_{n+1}\to E_n$ $(n\geq 0)$ is called a projective system. A set E given to gether with maps $\psi_n : E\to E_n$ such that $\psi_n=\varphi_n o\psi_{n+1}$ $(n\geq 0)$ is called a projective limit of the sequence $(E_n,\varphi_n)_{n\geq 0}$ if the following condition is satisfied:*

For each set X and maps $f_n : X\to E_n$ satisfying $f_n=\varphi_n o f_{n+1}$ $(n\geq 0)$ there is a unique factorization f of f_n through the set E:

$$f_n=\psi_n o f : X\to E\to E_n \quad (n\geq 0).$$

The maps $\varphi_n : E_{n+1}\to E_n$ are usually called transition maps of the projective system. The whole system, represented by

$$E_0\leftarrow E_1\leftarrow \ldots \leftarrow E_n\leftarrow \ldots,$$

is also called an inverse system.

Kim [18] defined the q-Volkenborn integration and gave relations between the q-Bernoulli numbers and the q-Euler numbers. He constructed a new measure as well.

Let p be a fixed prime. For a fixed positive integer d with $(p,d)=1$, we set (see [18])

$$\begin{aligned} X &= X_d=\lim_{\overleftarrow{N}} \mathbb{Z}/\mathbb{Z}dp^N, \\ X_1 &= \mathbb{Z}_p, \\ X^* &= \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a+dp\mathbb{Z}_p \end{aligned}$$

and

$$a+dp^N\mathbb{Z}_p=\left\{x\in X \mid x\equiv a(\operatorname{mod} dp^N)\right\},$$

where $a\in\mathbb{Z}$ satisfies the condition $0\leq a<dp^N$([16]). The p-adic absolute value in is normalized in such a way that

$$|p|_p=\frac{1}{p}.$$

We say that f is a uniformly differentiable function at a point $a\in\mathbb{Z}_p$, and write $f\in UD(\mathbb{Z}_p)$, if the difference quotient

$$F_f(x,y)=\frac{f(x)-f(y)}{x-y}$$

has a limit

$$l = f^{'}(a) \text{ as } (x,y) \to (a,a).$$

For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression:

$$\frac{1}{[p^N]} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j)\mu_q(j + p^N \mathbb{Z}_p),$$

which represents a $q$analogue of Riemann sums for f. The integral of f on $\mathbb{Z}_p$ is defined as the limit of these sums (as $N \to \infty$) if this limit exists. The q-Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]} \sum_{0 \leq j < p^N} q^j f(j), \tag{1.9}$$

where

$$\begin{aligned} \mu_q(j) &= \mu_q(j + p^N \mathbb{Z}_p) = \frac{q^j}{[p^N]} \\ (0 &\leq j < p^N;\ N \in \mathbb{Z}^+) \end{aligned}$$

(see, for detail [16], [17], [15], [19], [20]).

Kurt Hensel (1861-1941) invented the so-called p-adic numbers around the end of the nineteenth century. In sipite of their being already one hundred years old, these numbers are still today enveloped in an aura of mystery within scientific community. Although they have penetrated several mathematical fields, Number Theory, Algebraic Geometry, Algebraic Topology, Analysis, Mathematical Physics, String Theory, Field Theory, Stochastic Differential Equations on real Banach Spaces and Manifolds and other parts of the natural sciences, they have yet to reveal their full potentials in (for example) physics. While solving mathematical and physical problems and while constructing and investigating measures on manifolds, the p-adic numbers are used. There is an unexpected connection of the p-adic Analysis with q-Analysis and Quantum Groups, and thus with Noncommutative Geometry, and q-Analysis is a sort of q-deformation of the ordinary analysis. Spherical functions on Quantum Groups are q-special functions. (see [18], [20], [22], [23], [16], [46], [13], [49], [58], [59], [33], [34]).

Kim[20] defined the Daehee numbers, $D_m(z : q)$ by using an invariant p-adic z-integrals as follows:

$$\mu_z(a + p^N \mathbb{Z}_p) = \frac{z^a}{[p^N : z]}$$

and

$$[x : z] = \frac{1 - z^x}{1 - z},$$

which can be extended to distributions on $\mathbb{Z}_p$,

$$D_m(z : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_z(x), \tag{1.10}$$

$z \in \mathbb{C}_p$. If we take $z = q$ in (1.10), then we observe that

$$D_m(q : q) = \beta_m(q)$$

in terms of Carlitz's q-Bernoulli numbers mentioned above. In the case when $z = u$ in (1.10), the Daehee numbers become the q-Eulerian numbers as follows:

$$D_m(u : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_u(x) = H_m(u^{-1} : q).$$

By the definition of the Daehee numbers, we easily see that

$$D_m(u : q) = \frac{1}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} (-1)^{m-l} \frac{l+1}{[l+1]}.$$

The Daehee polynomials are defined as follows[20], [17]:

$$D_m(z, x : q) = \int_{\mathbb{Z}_p} [x+t]^m d\mu_z(t), \tag{1.11}$$

$z \in \mathbb{C}_p$.

We readily see from (1.11) that

$$D_m(z, x : q) = \sum_{l=0}^{m} \binom{m}{l} q^{lx} [x]^{n-l} D_l(z : q) = (q^x D(z : q) + [x])^m$$

with usual convention of symbolically replacing $D^m(z, x : q)$ by $D_m(z, x : q)$.

We note also that

$$D_m(z, x : q) = H_m(u^{-1}, x : q),$$

and

$$D_m(q, x : q) = \beta_m(x : q),$$

where $H_m(u^{-1}, x : q)$ and $\beta_m(x : q)$ are Carlitz's q-Euler Polynomials and Carlitz's q-Bernoulli Polynomials, respectively.

Ruijsenaars[47] showed how various known results concerning the Barnes multiple zeta and gamma functions can be obtained as specializations of simple features shared by a quite extensive class of functions. The pertinent functions involve Laplace transforms, and their asymptotic was obtained by exploiting this. He demonstrated how Barnes' multiple zeta and gamma functions fit into a recently developed theory of minimal solutions to first-order analytic difference equations. Both of these approaches to the Barnes functions gave rise to novel integral representations.

In one of an impressive series of papers ([4]; see also [54], Chapter 2), Barnes developed the so-called multiple zeta and multiple gamma functions. Barnes' multiple zeta function $\zeta_N(s, w \mid a_1, ..., a_N)$ depends on parameters $a_1, ..., a_N$ that will be taken positive throughout this paper. It defined by the series

$$\zeta_N(s, w \mid a_1, ..., a_N) = \sum_{m_1,...,m_N=0}^{\infty} (w + m_1a_1 + ... + m_Na_N)^{-s}, \tag{1.12}$$

where $\mathrm{Re}(w) > 0, \mathrm{Re}(s) > N$. From the definition (1.12), we immediately obtain the recurrence relation [47]:

(1.13)

$$\zeta_{M+1}(s, w+a_{M+1} \mid a_1, ..., a_{N+1}) - \zeta_{M+1}(s, w \mid a_1, ..., a_{N+1}) = -\zeta_M(s, w \mid a_1, ..., a_N)$$

with

$$\zeta_0(s, w) = w^{-s}.$$

Barnes showed that ζ_N has a meromorphic continuation in s (with simple poles only at $s = 1, ..., N$) and defined his multiple gamma function $\Gamma_N^B(w)$ in terms of the s-derivative at $s = 0$, which may be recalled here as follows[47]:

$$\Psi_N(w \mid a_1, ..., a_N) = \partial_s \zeta_N(s, w \mid a_1, ..., a_N) \mid_{s=0} .$$

Clearly, analytic continuation of (1.13) yields the recurrence relation:

$$\begin{aligned} \Psi_{M+1}(w + a_{M+1} &\mid a_1, ..., a_{N+1}) - \Psi_{M+1}(w \mid a_1, ..., a_{N+1}) \\ &= -\Psi_M(w \mid a_1, ..., a_N), \end{aligned}$$

with

$$\Psi_0(w) = -\ln w.$$

Up to inessential factors, the functions ζ_1 and Ψ_1 are equal to the Hurwitz zeta function and the Ψ (or the digamma) function (cf. e.g., Ref. [60]). For $a_1 = a_2 = 1$, the function

$$\exp(\Psi_2(a_1 + a_2 - w \mid a_1, a_2) - \Psi_2(w \mid a_1, a_2))$$

was already studied by Hölder in 1886 [47]. It was called the double sine function by Kurokawa[38]. In fact, Kurokawa[38] considered multiple sine functions defined in terms of $\Psi_N(w)$ and related these functions to Selberg zeta functions and the determinants of Laplacians occruing in symmetric space theory [38]. Barnes' multiple zeta and gamma functions were also encountered by Shintani within the context of analytic number theory. In recent years, they showed up in the form actor program for integrable field theories and in studies of XXZ model correlation functions[12]. See also recent paper by M. Nishizawa [42], where q-analogues of the multiple gamma functions are studied. Friedman and Ruijsenaars[9] showed that Shintani's work on multiple zeta and gamma functions can be simplified and extended by exploiting difference equations. They re-proved many of Shintani's formulas and prove several new ones. They also relate Barnes' triple gamma function to the elliptic gamma function appearing in connection with certain integrable systems.

The Barnes' multiple Bernoulli polynomials $B_n(x, r \mid a_1, ..., a_r)$, cf. [4], are defined by

$$\frac{t^r e^{xt}}{\prod_{j=1}^{r}(e^{a_j t} - 1)} = \sum_{n=0}^{\infty} B_n(x, r \mid a_1, ..., a_r) \frac{t^n}{n!}, \tag{1.14}$$

for $\mid t \mid < 1$.

By (1.12) and (1.14), it is easy to see that

$$\zeta_N(-m, w \mid a_1, ..., a_N) = \frac{(-1)^N m!}{(N+m)!} B_{N+m}(w, N \mid a_1, ..., a_N),$$

for $w > 0$ and m is a positive integer.

In recent years, many mathematicians and physicians have investigated zeta functions, multiple zeta functions, L-series, and multiple q-Bernoulli numbers and polynomials because mainly of their interest and importance. These functions and numbers are in used not only in Complex Analysis and Mathematical Physics, but also in used in p-adic Analysis and other areas. In particular, multiple zeta functions occur within the context of Knot Theory, Quantum Field Theory, Applied Analysis and Number Theory (see [3], [4], [6], [16], [18], [20], [22],[30], [45], [52], [40], [41], [57], [48], [36], [37]).

Kim[23] studied on the multiple L-series and functional equation of this functions. He found the value of this function at negative integers in terms of generalized Bernoulli numbers.

Russias and Srivastava[45] presented a systematic investigation of several families of infinite series which are associated with the Riemann zeta functions, the digamma functions,the harmonic numbers, and the Stirling numbers of the first kind.

Matsumoto[39] considered general multiple zeta functions of several variables, involving both Barnes multiple zeta functions and Euler-Zagier sums as special cases. He proved the meromorphic continuation to the whole space , asymptotic expansions, and upper bounded estimates. These results were expected to have applications to some arithmetical L-functions. His method was based on the classical Mellin-Barnes integral formula.

Ota[43] studied on Kummer-type congruences for derivatives of Barnes' multiple Bernoulli polynomials. Ota[43] also generalized these congruences to derivatives of Barnes' multiple Bernoulli polynomials by an elementary method and gave a p-adic interpolation of them. Subsequently, Ota [44] defined derivatives of the Dedekind sums and their reciprocity law. They were obtained from values at non-positive integers of the first derivatives of Barnes' double zeta functions. As special cases, they give finite product expressions of the Stirling modular form and the double gamma function at positive rational numbers. The original Dedekind sum appears at various places in mathematics, so the derivative of the Dedekind sums may be expected to be useful as well. It would be very interesting if we could obtain different proofs from Ota's for the reciprocity laws about derivatives, just as the original reciprocity law of Dedekind was obtained from the transformation formulas of $\log \eta(z)$. We note that by considering Barnes' r-ple zeta function [44] or zeta functions with characters Ota obtain reciprocity laws for sums involving derivatives of the Barnes r-ple Bernoulli polynomials or Dedekind sums with character.

Simsek[53] gave relations between zeta functions, trigonometric functions and Dedekind sums. He also found reciprocity law of this sums related to Lambert series and Eisenstein series.

Woon[61] presented a series of diagrams showing the Julia set of the Riemann zeta functions and its related Mandelbrot set. The Julia and Mandelbrot sets of the Riemann zeta function have unique features and are quite unlike those of any elementary functions.

By using non-Archimedean q-integration, Kim [20] introduced multiple Changhee q-Bernoulli polynomials which form a q-extension of Barnes' multiple Bernoulli polynomials. He also constructed the Changhee q-zeta functions (which gives q-analogues of Barnes' multiple zeta functions) and indicated some relationships between the Changhee q-zeta function and Daehee q-zeta function.

A sequence of p-adic rational numbers as multiple Changhee q-Bernoulli numbers and polynomials are defined as follows[20], [27]:

Let $a_1, ..., a_r$ be nonzero elements of the p-adic number field and let $z \in \mathbb{C}_p$.

$$\beta_n^{(r)}(w : q \mid a_1, ..., a_r) = \frac{1}{\prod_{j=1}^{r} a_j} \int_{\mathbb{Z}_p^r} [w + \sum_{j=1}^{r} a_j x_j]^n d\mu_q(x), \tag{1.15}$$

and

$$\beta_n^{(r)}(q \mid a_1, ..., a_r) = \frac{1}{\prod_{j=1}^{r} a_j} \int_{\mathbb{Z}_p^r} [\sum_{j=1}^{r} a_j x_j]^n d\mu_q(x),$$

where

$$\int_{\mathbb{Z}_p^r} f(x)d\mu_q(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}...\int_{\mathbb{Z}_p} f(x)d\mu_q(x_1)d\mu_q(x_2)...d\mu_q(x_r).$$

It is easily observed from (1.15) that

$$(1.16)\ \beta_n^{(r)}(w \quad : \quad q \mid a_1,...,a_r) = \frac{1}{(1-q)^n}\sum_{l=0}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)(-1)^l q^{wl}\prod_{j=1}^{r}\frac{(l+\frac{1}{a_j})}{[la_j+1]}$$

$$= \sum_{l=0}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)[w]^{n-l}q^{wl}\beta_l^{(r)}(q \mid a_1,...,a_r),$$

for every positive integer n.

By (1.14) and (1.16), we note that

$$\lim_{n\to\infty}\beta_n^{(r)}(w : q \mid a_1,...,a_r) = B_n(w, r \mid a_1,...,a_r).$$

In the special case when

$$(a_1,...,a_r) = (1,...,1).$$

we see that

$$\beta_n^{(r)}(w : q \mid 1,1,...,1) = \beta_n^{(r)}(w : q),$$

where $\beta_n^{(r)}(w : q)$ are the q-Bernoulli polynomials of order r (see[18]), which reduces to the ordinary Bernoulli polynomials of higher order $B_n^{(r)}(w)$ if $q = 1$ (see for detail [21], [23], [24])

Kim[22] defined the analytic continuation of multiple zeta functions (the Euler-Barnes multiple zeta functions) depending on parameters $a_1,...,a_r$ in the complex number field as follows:

$$(1.17)\ \ \zeta_r(s, w, u \mid a_1,...,a_r) = \sum_{m_1,...,m_r=0}^{\infty} u^{-(m_1+...+m_r)}(w+m_1a_1+...+m_Na_N)^{-s},$$

where $\Re(w) > 0$ and $u \in \mathbb{C}$ with $\mid u \mid> 1$ ([23], [25], [26], [27], [29]).

We summarize our paper as follows:

In Section 2, of our paper, by using q-Volkenborn integration and uniform differentiable function on $\mathbb{Z}_p$, we will construct p-adic q-zeta functions. This functions interpolate q-Bernoulli numbers. The values of the p-adic q-zeta functions are given explicitly.

In Section 3, our primary aim is to give generating function of q-Bernoulli numbers and polynomials. These numbers and functions can be used to prove analytic continuation of q-L-functions.

In Section 4, we give new generating functions which produce new definition of Barnes' type Changhee q-Bernoulli polynomials and the generalized Barnes' type Changhee q-Bernoulli numbers attached to Dirichlet character. These functions are very important in constructing multiple zeta functions. By using Mellin transformation formula, we also give relations between new Barnes' type Changhee q-zeta functions and Barnes' type new Changhee q-Bernoulli numbers.

In Section 5, by using Mellin transformation formula of character generating function of generalized Barnes' type Changhee q-Bernoulli numbers, we will define the Dirichlet's type Changhee q-L-series. We give relations between these functions, q-zeta functions and Dirichlet's type Changhee q-Bernoulli numbers, as well.

In Section 6, we will construct new generating function of multiple Changhee q-Bernoulli polynomials. Under the Mellin transformation , we give relation between this function and multiple q-zeta function, the multiple Changhee q-Bernoulli numbers.

In Section 7, we give relation between q-gamma functions and zeta functions. We also find some new results related to these functions.

In Section 8, we give analytic properties of q-L-function and q-Hurwitz zeta function. We prove relations between these functions and q-gamma function.

In Section 9, we define generalized multiple Changhee q-Bernoulli numbers attached to the Drichlet character χ. We also construct Dirichlet's type multiple Changhee q-L-functions. These will lead to relations between Dirichlet's type multiple Changhee q-L-functions and generalized multiple Changhee q-Bernoulli numbers as well.

In Section 10, the main purpose is to prove analytic continuation of the Euler-Barnes' type multiple q-Daehee zeta functions depending on the parameters $a_1, ..., a_r$ which are taken positive parts in the Complex Field. Thus, we construct generating function of q-Euler-Barnes' type multiple Frobenius-Euler polynomials. We define Euler-Barnes' type multiple q-Daehee zeta Functions. Euler-Barnes' type multiple q-Daehee zeta functions have a certain connection with Topology and Physics, together with the algebraic relations among them. We give the values of these functions at negative integers as well.

In Section 11, we give analytic continuation of Euler-Barnes' type Daehee q-zeta functions. We also give some remarks related to these functions.

2. A Family of p-Adic q-Zeta Functions

In this section, we need the following definitions and notations.

Every continuous function $f : \mathbb{Z}_p \to \mathbb{K}$, $\mathbb{K}$ is an non-Archemedian valued field, has unique expansion as

$$f(x) = \sum_{n \geq 0} (\Delta_q^n f)(0) \begin{pmatrix} x \\ n \end{pmatrix}_q , \tag{2.1}$$

where

$$(\Delta_q^n f)(0) \in \mathbb{K}$$

and

$$(\Delta_q^n f)(0) \to 0 \text{ as } n \to \infty$$

and Δ is the q-difference operator. We also have

$$\begin{pmatrix} m \\ n \end{pmatrix}_q = \frac{[m]!}{[n]![m-n]!} \quad (\ m \geq n \ )$$

(see, for example [31]).

Definition 2. *A function f is called uniform differentiable function if it satisfies the following conditions:*

1) $F_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$,
$F_f(x, y) = \frac{f(x)-f(y)}{x-y}$.
2) $\lim_{x \to y} F_f(x, y) = f^{'}(y)$.

From (2.1), let $f:\mathbb{Z}_p\to\mathbb{Z}_p$ be a function. Then f can be written as

$$f(x)=\sum_{n=0}^{\infty}\left(\begin{array}{c}x\\n\end{array}\right)_q\Delta^n f(0), \tag{2.2}$$

where

$$\Delta^n f(0)=\sum_{i=0}^{n}\left(\begin{array}{c}n\\i\end{array}\right)_q(-1)^{n-i}f(i).$$

By using (2.2), we have

1) f is continuous function $\Leftrightarrow \mid \Delta^n f(0)\mid_p\to 0$ as $n\to\infty$.
2) f is differentiable function $\Leftrightarrow \mid \frac{\Delta^n f(0)}{n}\mid_p\to 0$ as $n\to\infty$.
3) f is analytic function $\Leftrightarrow \mid \frac{\Delta^n f(0)}{n!}\mid_p\to 0$ as $n\to\infty$.
4) f is uniform differentiable function, $f\in UD\Leftrightarrow n\mid \Delta^n f(0)\mid_p\to 0$ as $n\to\infty$.
5) $f\in C^{(n)}$ function $\Leftrightarrow \lim_{m\to\infty} m^n \mid \Delta^m f(0)\mid_p\to 0$,

where

$$C^{(m)}=C^{(m)}(\mathbb{Z}_p,\mathbb{K})=\{f:f:\mathbb{Z}_p\to\mathbb{K},\ m \text{ times strictly differentiable}\}.$$

If $f\in C^{(m)}(\mathbb{Z}_p,\mathbb{K})$, then $f^{(m-1)}\in C^{(1)}$. We also note that $(\Delta_q^n f)(0)$ is the nth Mahler coefficient of f at $q=1$.

The function f is differentiable at

$$x\in\mathbb{Z}_p\Leftrightarrow \lim_{m\to\infty}\frac{(\Delta_q^m f)(0)}{[m]}=0,$$

in which case

$$f^{(1)}(x)=\frac{\log q}{q-1}\sum_{m=1}^{\infty}\frac{(\Delta_q^m f)(x)}{[m]}(-1)^{m-1}q^{-\left(\begin{array}{c}m\\2\end{array}\right)},$$

where $f^{(1)}$ denotes first derivative. We say that f is strictly differentiable at a point $a\in\mathbb{Z}_p$, and denote this property by $f\in C^{(1)}$, if the difference quotients

$$F_f(x,y)=\frac{f(x)-f(y)}{x-y},$$

have a limit $l=f^{'}(a)$ as $(x,y)\to(a,a)$. Recall that if $f\in C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$ function, then there exist a unique continuos function

$$(sf)(x)=\sum_{k \bmod x}f(k).$$

The function sf satisfies the following properties:

1) $(sf)(x+1)-(sf)(x)=f(x)$,
2) If $f\in C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$, then

$$(sf)\in C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p),$$

and

$$\parallel sf\parallel_1\leq p\parallel f\parallel_1,$$

where

$$\parallel f\parallel_1=\parallel f\parallel_p\vee\parallel \Delta_n f\parallel_\infty$$

and (cf. [31])

$$\parallel f\parallel_m=\mid f^{(m-1)}\mid_{\sup}\vee\mid F_{f^{(m-1)}}(x,y)\mid_{\sup},$$

$$\| f \|_{\infty}= \sup_{x\in\mathbb{Z}_p} \mid f(x) \mid_p .$$

We note that

$$\Delta_n f(m_1,...,m_n;x) = \Delta_1(\Delta_{n-1} f(m_1,...,m_{n-1};x)),$$

and

$$\Delta_1 f(m,x) = \frac{f(x+m)-f(x)}{n}.$$

Therefore, by (1.9), it is easy to see that

$$\int_{\mathbb{Z}_p} q^{-x} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{sf(p^N)-sf(0)}{[p^N]} = \frac{q-1}{\log q}(sf)^{'}(0),$$

(see [31]).

Proposition 1. *1) If $f\in C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$, then*

$$\int_{\mathbb{Z}_p} q^{-x} f(x) d\mu_q(x) \leq p \| f \|_1 .$$

that is, the q-Volkenborn integral a linear, continuos function on $C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p)$.

2) Let

$$f(x) = \sum_{n=0}^{\infty} \left(\begin{array}{c} x \\ n \end{array} \right)_q \Delta^n f(0).$$

Then the q-Mahler representation of the $C^{(1)}$-function f satisfies the following equation:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \sum_{n=0}^{\infty} (\Delta^n f)(0) \frac{(-1)^n}{[n+1]} q^{n+1-\left(\begin{array}{c} n+1 \\ 2 \end{array} \right)}.$$

3) In particular, the Fourier transformation of the q-Volkenborn integral is given by

$$\int_{\mathbb{Z}_p} q^{-x} T^{[x]} d\mu_q(x) = \frac{q-1}{\log q} T^{\frac{1}{1-q}} - \log T \sum_{n=0}^{\infty} q^n T^{[n]}. \tag{2.3}$$

4)

$$\int_{\mathbb{Z}_p} q^{-x} [x]^n d\mu_q(x) = \beta_n(q), \tag{2.4}$$

where $\beta_n(q)$ the nth q-Bernoulli numbers (see, for detail [19]*).*

5) If

$$f\in C^{(1)}(\mathbb{Z}_p,\mathbb{C}_p),$$

$n\geq 0$, $j\in\{0,1,2,...,p-1\}$ and

$$\mathbb{T}_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p,$$

then

$$\begin{aligned}\int_{j+p^n\mathbb{Z}_p} q^{-(j+xp^n)} f(x)d\mu_q(x) &= \lim_{N\to\infty}\sum_{x=0}^{p^N-1} f(j+p^n x)\frac{1}{[p^{n+N}]}\\ &= \frac{1}{[p^n]}\lim_{N\to\infty}\frac{1}{[p^N:q^{p^n}]}\sum_{x=0}^{p^N-1} f(j+p^n x)\\ &= \frac{1}{[p^n]}\int_{\mathbb{Z}_p} q^{-xp^n} f(j+p^n x)d\mu_{q^{p^n}}(x),\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{T}_p} f(x)d\mu_q(x) &= \int_{\mathbb{Z}_p\setminus p\mathbb{Z}_p} f(x)d\mu_q(x)\\ &= \int_{\mathbb{Z}_p} f(x)d\mu_q(x) - \frac{1}{[p]}\int_{\mathbb{Z}_p} f(px)d\mu_{q^p}(x)\\ &= \frac{1}{[p]}\left([p]\int_{\mathbb{Z}_p} f(x)d\mu_q(x) - \int_{\mathbb{Z}_p} f(px)d\mu_{q^p}(x)\right).\end{aligned}$$

We note that, substituting $T=e^t$ into (2.3), than we obtain (2.4).

We now consider p-adic q-zeta function

$$\zeta_{p,q,1}, \zeta_{p,q,2}, ..., \zeta_{p,q,p-1},$$

which are given by

$$\zeta_{p,q,j}(s) = \frac{1}{j+(p-1)s}\int_{\mathbb{T}_p} q^{-x}[x]^j[x]^{(p-1)s}d\mu_q(x),\ \ (j=0,1,2,...,p-1),$$

where

$$\mathbb{T}_p = \mathbb{Z}_p\setminus p\mathbb{Z}_p, \text{ and } \mid s\mid_p < p^{\frac{p-2}{p-1}}\ \ (\ s\neq -\frac{j}{p-1}).$$

We note that

$$\begin{aligned}\int_{\mathbb{Z}_p\setminus p\mathbb{Z}_p} q^{-x}[x]^j[x]^{(p-1)s}d\mu_q(x) &= \int_{\mathbb{T}_p} q^{-x}[x]^j[x]^{(p-1)s}d\mu_q(x)\\ &= \int_{\mathbb{T}_p} q^{-x}[x]^j\sum_{n=0}^{\infty} s^n(p-1)^n\frac{(\log[x])^n}{n!}d\mu_q(x)\\ &= \sum_{n=0}^{\infty} a_{n,q}s^n,\end{aligned}$$

where, as before,

$$\mid s\mid_p < p^{\frac{p-2}{p-1}}\ \ (\ s\neq -\frac{j}{p-1}),$$

and

$$a_{n,q} = \frac{(p-1)^n}{n!}\int_{\mathbb{T}_p} q^{-x}[x]^j\log^n[x]d\mu_q(x).$$

If $n \to \infty$, then $\mid a_{n,q} \mid_p \to 0$. By using the above definition, we obtain

$$\begin{aligned}\zeta_{p,q,0}(s) &= \frac{1}{(p-1)s} \int_{\mathbb{T}_p} q^{-x}[x]^{(p-1)s} d\mu_q(x) \\ &= \frac{1}{(p-1)s} \sum_{n=0}^{\infty} a_{n,q} s^n,\end{aligned}$$

where

$$\begin{aligned}a_{0,q} &= \int_{\mathbb{T}_p} q^{-x} d\mu_q(x) = \int_{\mathbb{Z}_p} q^{-x} d\mu_q(x) - \int_{p\mathbb{Z}_p} q^{-x} d\mu_q(x) \\ &= \frac{q-1}{\log q} - \frac{q-1}{\log q}\frac{1}{p} = \frac{(q-1)(p-1)}{p\log q}.\end{aligned}$$

Therefore, we arrive at the following result.

Theorem 1. *By means of the following transformation:*

$$s \to \zeta_{p,q,0} - \frac{1}{ps}\left(\frac{q-1}{\log q}\right),$$

each of the functions

$$\zeta_{p,q,1},\ \zeta_{p,q,2}, ..., \zeta_{p,q,p-1}$$

can be extend to the corresponding analytic function on the following set:

$$\mathbb{B} := \left\{ s : s \in \mathbb{C}_p \text{ and } \mid s \mid_p < p^{\frac{p-2}{p-1}} \right\}.$$

Remark 1. *It easily follows from the above observations that*

$$\zeta_{p,q,0}(s) = \frac{q-1}{\log q}\frac{1}{ps} + \Theta(s),$$

where $\Theta(s)$ is analytic function. Thus, before giving the connection between the p-adic q-zeta functions and the classical q-zeta functions, we determine the values of

$$\zeta_{p,q,1}(s),\ \zeta_{p,q,2}(s), ..., \zeta_{p,q,p-1}(s).$$

Then, by using these values as well as the p-adic interpolation of sequences of values of ζ_q at certain negative integers , we will construct the p-adic q-zeta functions.

The following theorems provides us with the relationship between $\zeta_{p,q,j}(s)$ and $\beta_n(q)$ are given by

Proposition 2. *Let n and p be positive integers with p prime. Then*

$$\zeta_{p,q,0}(n) = \frac{1}{(p-1)n}(\beta_{(p-1)n}(q) - [p]^{(p-1)n-1}\beta_{(p-1)n-1}(q^p)).$$

Proof.

$$\begin{aligned}
\zeta_{p,q,0}(s) &= \frac{1}{(p-1)s}\int_{T_p} q^{-x}[x]^{(p-1)s}d\mu_q(x)\\
&= \frac{1}{(p-1)s}\int_{\mathbb{Z}_p\setminus p\mathbb{Z}_p} q^{-x}[x]^{(p-1)s}d\mu_q(x)\\
&= \frac{1}{(p-1)s}\left(\int_{\mathbb{Z}_p} q^{-x}[x]^{(p-1)s}d\mu_q(x)-\int_{p\mathbb{Z}_p} q^{-x}[x]^{(p-1)s}d\mu_q(x)\right)\\
&= \frac{1}{(p-1)s}\left(\int_{\mathbb{Z}_p} q^{-x}[x]^{(p-1)s}d_q x-\int_{\mathbb{Z}_p} q^{-px}[px]^{(p-1)s}d\mu_{q^p}(x)\right)
\end{aligned}$$

By setting $s=n$, ($n\in\mathbb{Z}^+$) and using (2.4) in the above, we easily obtain

$$\zeta_{p,q,0}(n)=\frac{1}{(p-1)n}\left(\beta_{(p-1)n}(q)-[p]^{(p-1)n-1}\beta_{(p-1)n-1}(q^p)\right),$$

which completes our proof of Proposition 2. ∎

By applying a similar method for $\zeta_{p,q,j}$, we arrive at the desired result asserted by Theorem 2 below.

Theorem 2. *Let n,p,j be positive integers with p prime. Then*

$$\zeta_{p,q,j}(n)=\frac{1}{j+(p-1)n}(\beta_{j+(p-1)n}(q)-[p]^{j+(p-1)n-1}\beta_{j+(p-1)n-1}(q^p)).$$

The proof of Theorem 2 is simillar to that of Proposition 2. So we omit it.

The classical q-zeta function was defined by Kim[29] as follows:

$$\zeta_q(s)=\sum_{n=0}^{\infty}\frac{q^n}{[n]^s}-\frac{1}{s-1}\frac{(1-q)^s}{\log q},$$

for $s\in\mathbb{C}$.

For any positive integer n, we have

$$\zeta_q(1-n)=-\frac{\beta_n(q)}{n}.$$

Furthermore, if we define

$$\zeta_q^*(s)=\zeta_q(s)-[p]^{-s}\zeta_{q^p}(s),$$

then, for $j\in\{0,1,2,...,p-1\}$, and $n\geq 0$, we find that

$$\zeta_{p,q,j}(n)=-\zeta_q^*(1-(j+(p-1)n).$$

3. The q-Bernoulli Numbers and the q-Bernoulli Polynomials

Our primary aim in this section is to give generating functions of q-Bernoulli numbers and q-Bernoulli polynomials. These numbers will be used to prove analytic continuation of q-L-series.

We first define q-version of each of the functions $F(t,x)$ and $F(t)$ occurring in (1.1) and (1.2), respectively. The generating function $F_q(t)$ of q-Bernoulli numbers $\beta_n(q)$ ($n\geq 0$) is given by [20]:

$$F_q(t)=\frac{q-1}{\log q}\exp(\frac{t}{1-q})-t\sum_{n=0}^{\infty}q^n e^{[n]t}=\sum_{n=0}^{\infty}\frac{\beta_n(q)t^n}{n!}, \tag{3.1}$$

We note that the definition (1.2) and (3.1) that

$$\lim_{q\to 1}\beta_n(q) = B_n,$$

and

$$\lim_{q\to 1}F_q(t) = F(t) = \frac{t}{e^t-1}.$$

The generating function $F_q(x,t)$ of the q-Bernoulli polynomials $\beta_n(x:q)$ ($n \geq 0$) is defined analogously as follows:

$$F_q(x,t) = \frac{q-1}{\log q}\exp(\frac{t}{1-q}) - t\sum_{n=0}^{\infty} q^{n+x}e^{[n+x]t} = \sum_{n=0}^{\infty}\frac{\beta_n(x:q)t^n}{n!}. \tag{3.2}$$

We note from (1.1) and (3.2) that

$$\lim_{q\to 1}\beta_n(x:q) = B_n(x),$$

and

$$\lim_{q\to 1}F_q(x,t) = F(x,t) = \frac{te^{xt}}{e^t-1}.$$

The remarkable point here is that the series on the righet-hand side of (3.1) and (3.2) are uniformly convergent in the wider sense. Therefore, we shall explicitly determine the q-Bernoulli numbers as follows:

$$\beta_{0,q} = \frac{q-1}{\log q}, \quad q(q\beta_q+1)^n - \beta_n(q) = \begin{cases} 1, \text{ if } n = 1 \\ 0, \text{ if } n > 1, \end{cases}$$

with the usual convention about replacing β^n by β_n.

For the q-Bernoulli polynomials are defined by (3.2), we first derive an explicit representation given by Theorem 3 below.

Theorem 3.

$$\beta_n(x:q) = \sum_{l=0}^{n}\binom{n}{l} q^{lx}\beta_l(q)[x]^{n-l}.$$

Proof. By using Cauchy product in (3.1) and (3.2), we have

$$\sum_{n=0}^{\infty}\frac{\beta_n(x:q)t^n}{n!} = \sum_{n=0}^{\infty}(\sum_{l=0}^{n}\binom{n}{l} q^{lx}\beta_l(q)[x]^{n-l})\frac{t^n}{n!}.$$

After some elementary calculations in the above, we easily arrive at the desired result. ∎

We note that

$$\begin{aligned}\beta_n(x \quad : \quad q) &= \sum_{l=0}^{n}\binom{n}{l} q^{lx}\beta_l(q)[x]^{n-l} \\ &= (q^x\beta(q)+[x])^n.\end{aligned}$$

Now we construct generalized q-Bernoulli numbers associated with a Dirichlet characater.

Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. We define the generating function of generalized q-Bernoulli numbers attached to χ as follows:

$$F_{q,\chi}(t) = -t\sum_{a=1}^{f}\chi(a)\sum_{n=0}^{\infty}q^{fn+a}e^{[fn+a]t} = -t\sum_{n=0}^{\infty}\chi(n)q^n e^{[n]t} = \sum_{n=0}^{\infty}\beta_{n,\chi}(q)\frac{t^n}{n!}. \tag{3.3}$$

where the coefficients, $\beta_{n,\chi}(q)$ ($n \geq 0$) are called generalized q-Bernoulli numberswith a Dirichlet characater. We note from the definitions in (1.3) and (3.3) that

$$\lim_{q\to 1}\beta_{n,\chi}(q) = B_{n,\chi},$$

and

$$\lim_{q\to 1}F_{q,\chi}(t) = F_\chi(t) = \sum_{a=1}^{f}\chi(a)\frac{te^{at}}{e^{tf}-1} = \sum_{n=0}^{\infty}B_{n,\chi}\frac{t^n}{n!}.$$

By using (3.3), we also have

$$F_{q,\chi}(t) = \frac{1}{[f]}\sum_{a=1}^{f}\chi(a)F_{q^f}(\frac{a}{f},[f]t) \tag{3.4}$$

By applying Cauchy product in (3.2), (3.3) and (3.4), we easily obtain the following theorems:

Theorem 4. *Let χ be a Dirichlet character with conductor $f \in \mathbb{Z}^+$. Then we have*

$$\beta_{n,\chi}(q) = [f]^{n-1}\sum_{a=1}^{f}\chi(a)\beta_n(\frac{a}{f}:q^f).$$

We now construct generating function of generalized q-Bernoulli polynomials associated with a Dirichlet characater as follows:

$$F_{q,\chi}(x,t) = q^x te^{-[x]t}\sum_{n=0}^{\infty}\chi(n)q^n e^{[n]q^x t} = -t\sum_{n=0}^{\infty}\chi(n)q^{n+x}e^{[n+x]t} = \sum_{n=0}^{\infty}\beta_{n,\chi}(x:q)\frac{t^n}{n!}. \tag{3.5}$$

By using (3.2), (3.3) and (3.5), we obtain

$$F_{q,\chi}(x,t) = -t\sum_{n=0}^{\infty}\chi(n)q^{n+x}e^{[n+x]t} = -e^{[x]t}q^x t\sum_{a=1}^{f}\chi(a)\sum_{n=0}^{\infty}q^{fn+a}e^{[fn+a]tq^x},$$

which, in view of the following well-known identity

$$[x+a]=[x]+q^x[a]$$

yields

$$F_{q,\chi}(x,t)=-t\sum_{a=1}^{f}\chi(a)\sum_{n=0}^{\infty}q^{fn+a+x}e^{[fn+a+x]t}.$$

After some elementary calculation we arrive at the following theorem:

Theorem 5. *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$. Then we have*

$$F_{q,\chi}(x,t)=\frac{1}{[f]}\sum_{a=1}^{f}\chi(a)F_{q^f}(\frac{a+x}{f},[f]t) \tag{3.6}$$

Note that substituting $x=0$ into (3.6), then we obtain (3.4).

By comparing the coefficients on both sides of (3.5) and (3.6), we easily see that

$$\begin{aligned}\beta_{n,\chi}(x\quad:\quad q)&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]^{n-l}\sum_{a=1}^{f}\chi(a)\beta_l(\frac{a}{f}:q^f)\\&=\sum_{l=0}^{n}\binom{n}{l}q^{lx}[x]^{n-l}\beta_{l,\chi}(q).\end{aligned}$$

By using definition of $\beta_{l,\chi}(q)$ into the above, we have obtain the following result.

Theorem 6. *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$. Then*

$$\beta_{n,\chi}(x:q)=\frac{1}{[f]^{1-n}}\sum_{a=0}^{f-1}\chi(a)\beta_n(\frac{a+x}{f}:q^f).$$

4. A Class of q-Multiple Zeta Functions

In this section, we give new generating functions which produce new definitions of Barnes' type of Changhee q-Bernoulli polynomials and the generalized Barnes' type Changhee q-Bernoulli numbers with attached to χ, Dirichlet character with conductor with conductor $f\in\mathbb{Z}^+$. These generating functions are very important in case of multiple zeta function. Therefore, by using these generating functions, we will give relation between Barnes' type Changhee q-zeta function and Barnes' type Changhee q-Bernoulli numbers.

Let $w,w_1,w_2,\dots,w_r$ be complex numbers such that $w_i\neq 0$ for $i=1,2,\dots,r$. We define Barnes' type of Changhee q-Bernoulli polynomials of w with parameters w_1 as follows:

$$\begin{aligned}F_q(w,t\quad|\quad w_1)&=\frac{q-1}{\log q}e^{\frac{t}{1-q}}-w_1t\sum_{n=0}^{\infty}q^{w_1n+w}e^{[w_1n+w]t}\\&=\sum_{n=0}^{\infty}\frac{\beta_n(w:q\mid w_1)t^n}{n!}\quad(\mid t\mid<2\pi\ ),\end{aligned}\tag{4.1}$$

where the coefficients, $\beta_n(w:q\mid w_1)$ ($n\geq 0$) are called Barnes' type of Changhee q-Bernoulli polynomials in w with parameters w_1.

We note that

$$\lim_{q\to 1}\beta_n(w:q\mid w_1)=w_1^n\beta_n(w),$$

and

$$\lim_{q\to 1}F_q(w,t\mid w_1)=\frac{w_1t}{e^{w_1t}-1}e^{wt},$$

where $\beta_n(w)$ are the ordinary Barnes Bernoulli polynomials.

By using (4.1), we easily obtain[29], [23]

$$\beta_n(w:q\mid w_1)=\frac{1}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}q^{lw}(-1)^l\frac{lw_1}{[lw_1]}.$$

If $w=0$ in the above, then

$$\beta_n(0:q\mid w_1)=\beta_n(q\mid w_1),$$

where $\beta_n(q\mid w_1)$ are called Barnes' type Changhee q-Bernoulli numbers with parameter w_1.

Let χ be the Dirichlet character with conductor f. Then the generalized Barnes' type Changhee q-Bernoulli numbers with attached to χ are defined as follows:

$$\begin{aligned}F_{q,\chi}(t\mid w_1)&=-w_1t\sum_{n=1}^{\infty}\chi(n)q^{w_1n}e^{[w_1n]t}\\&=\sum_{n=0}^{\infty}\frac{\beta_{n,\chi}(q\mid w_1)t^n}{n!}\quad(\mid t\mid<2\pi).\end{aligned}\tag{4.2}$$

We easily see from (4.2) that

$$F_{q,\chi}(t\mid w_1)=\frac{1}{[f]}\sum_{a=1}^{f}\chi(a)F_{q^f}(\frac{w_1a}{f},[f]t\mid w_1).\tag{4.3}$$

Now by using (4.1), (4.2) and (4.3), and after some elementary calculations, we arrive at the following theorem.

Theorem 7. *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$. Then*

$$\beta_{n,\chi}(q\mid w_1)=\frac{1}{[f]^{1-n}}\sum_{a=0}^{f-1}\chi(a)\beta_n(\frac{aw_1}{f}:q^f\mid w_1).$$

By applying Mellin transformation in (4.1), we obtain

$$\frac{1}{\Gamma(s)}\int_0^{\infty}t^{s-2}F_q(w,-t\mid w_1)dt=-\frac{(1-q)^s}{s-1}\frac{1}{\log q}+w_1\sum_{n=0}^{\infty}\frac{q^{w_1n+w}}{[w_1n+w]^s},\tag{4.4}$$

where $\Gamma(s)$ is denoted Euler gamma function.

Note that, by substituting $w=w_1=q=1$ into (4.4), then we obtain Hurwitz zeta function.

We define Barnes' type Changhee q-zeta function as follows:

Definition 3. *For $s\in\mathbb{C}$, we have*

$$\zeta_q(s,w\mid w_1)=-\frac{(1-q)^s}{s-1}\frac{1}{\log q}+w_1\sum_{n=0}^{\infty}\frac{q^{w_1n+w}}{[w_1n+w]^s}.\tag{4.5}$$

Theorem 8. *If $n \in \mathbb{Z}^+$, then*

$$\zeta_q(1-n, w \mid w_1) = -\frac{\beta_n(w : q \mid w_1)}{n}.$$

Proof. In view of (4.4), we define $Y(s)$ by means of the following contour integral:

$$Y(s) = \int_C z^{s-2} F_q(w, -z \mid w_1) dz, \tag{4.6}$$

where C is Hankel's contour along the cut joining the points $z = 0$ and $z = \infty$ on the real axis, which starts from the point at ∞, encircles the origin ($z = 0$) once in the positive (counter-clockwise) direction, and returns to the point at ∞ (see for details, [60] p. 245). Here, as usual,we interpret z^s to mean $\exp(s \log z)$, where we asume log to defined by $\log t$ on the top part of the real axis and by $\log t + 2\pi i$ on the bottom part of the real axis. We thus find from the definition (4.6) that

$$\begin{aligned} Y(s) = (e^{2\pi i s} - 1) \int_\varepsilon^\infty t^{s-2} F_q(w, -t \mid w_1) dt \\ + \int_{C_\varepsilon} z^{s-2} F_q(w, -z \mid w_1) dz, \end{aligned}$$

where C_ε denotes a circle of radius $\varepsilon > 0$ (and centred at the origin), which is described in the positive (counter-clockwise) direction. Assume first that $\mathrm{Re}(s) > 1$. Then

$$\int_{C_\varepsilon} \to 0 \text{ as } \varepsilon \to 0,$$

so we have

$$Y(s) = (e^{2\pi i s} - 1) \int_0^\infty t^{s-2} F_q(w, -t \mid w_1) dt,$$

which, upon substituting from (4.1) into it, yields

$$\begin{aligned} Y(s) &= (e^{2\pi i s} - 1)(-\frac{1-q}{\log q} \int_0^\infty t^{s-2} \exp(-\frac{t}{1-q}) dt \\ &\quad + w_1 \sum_{n=0}^\infty q^{w_1 n + w} \int_0^\infty t^{s-1} e^{-[w_1 n + w] t} dt). \end{aligned}$$

After some elementary calculations, we thus find that

$$Y(s) = (e^{2\pi i s} - 1)\Gamma(s) \left(-\frac{(1-q)^s}{(s-1)\log q} + w_1 \sum_{n=0}^\infty \frac{q^{w_1 n + w}}{[w_1 n + w]^s} \right).$$

By using (4.5) in the above, we get

$$Y(s) = (e^{2\pi i s} - 1)\Gamma(s) \zeta_q(s, w \mid w_1).$$

Therefore

$$\zeta_q(s, w \mid w_1) = \frac{Y(s)}{(e^{2\pi i s} - 1)\Gamma(s)}, \tag{4.7}$$

which, by analytic continuation, holds true for all $s \neq 1$. This evidently provides us with an analytic continuation of $\zeta_q(s, w \mid w_1)$.

We now consider the situation when we let $s \to 1 - n$ in (4.7), where n is a positive integer. Then since

$$e^{2\pi i s} = e^{2\pi i (1-n)} = 1 \quad (n \in \mathbb{Z}^+),$$

we have the following limit relationship:

$$\lim_{s\to 1-n}\left\{(e^{2\pi is}-1)\Gamma(s)\right\} = \lim_{s\to 1-n}\left\{\frac{(e^{2\pi is}-1)}{\sin(\pi s)}\frac{\pi}{\Gamma(1-s)}\right\} = \frac{2\pi i(-1)^{n-1}}{(n-1)!} \quad (\ n\in\mathbb{Z}^+\ ) \tag{4.8}$$

by means of the familiar reflection formula for $\Gamma(s)$. Furthermore, since the integrand in (4.6) has simple pole order $n+1$ at $z=0$, where also find from the definition (4.6) with $s=1-n$ that

$$\begin{aligned} Y(1-n) &= \int_C z^{-n-1}F_q(w,-z\mid w_1)dz \\ &= 2\pi i\mathrm{Res}_{z=0}\left\{\ z^{-n-1}F_q(w,-z\mid w_1)\right\} \\ &= (2\pi i)\frac{(-1)^n}{n!}\beta_n(w:q\mid w_1), \end{aligned} \tag{4.9}$$

where we have made of the power-series representation in (4.1). Thus by Cauchy Residue Theorem, we easily complete the proof of Theorem 8 upon suitably combiningfin (4.8) and (4.9) with (4.7). ∎

Remark 2. *The representation in (4.4) can be used to show that $\zeta_q(s,w\mid w_1)$ admits itself of an analytical continuation to whole complex s- plane except for simple pole at $s=1$.*

5. The Dirichlet Type Changhee q-L-Function

We consider the following contour integral:

$$\begin{aligned} \frac{1}{\Gamma(s)}\oint_C t^{s-2}F_{\chi,q}(-t\mid w_1)dt &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}F_{\chi,q}(-t\mid w_1)dt \\ &= \frac{w_1}{\Gamma(s)}\sum_{n=1}^\infty \chi(n)q^{w_1n}\int_0^\infty t^{s-1}e^{-[w_1n]t}dt \\ &= w_1\sum_{n=0}^\infty \frac{\chi(n)q^{w_1n}}{[w_1n]^s}, \end{aligned} \tag{5.1}$$

where C denote a positively oriented circle of radius R, centered at origin. The function

$$B(t)=\frac{1}{\Gamma(s)}t^{s-2}F_{\chi,q}(-t\mid w_1)$$

has pole $t=0$ inside the contour C. Therefore, if want to integrate $B(t)$ function, than we have modify the contour by indentation at this point. We take as indentation identical small semicircle, which has radius r, leaving $t=0$.

We now define the Dirichlet's type Changhee q-L-function as follows:

Definition 4. *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$.*

$$L_q(s,\chi\mid w_1)=w_1\sum_{n=0}^\infty \frac{\chi(n)q^{w_1n}}{[w_1n]^s}. \tag{5.2}$$

We now give a relationship between $L_q(s,\chi\mid w_1)$ and generalized Changhee q-Bernoulli numbers. Thus we give the numbers $L_q(1-n,\chi\mid w_1), n\in\mathbb{Z}^+$, explicitly.

Theorem 9. *Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$ and let $n \in \mathbb{Z}^+$. Then*

$$L_q(1-n, \chi \mid w_1) = -\frac{\beta_{n,\chi}(q \mid w_1)}{n}.$$

Proof. Proof of Theorem 9 runs parallel to that of Theorem 8 above, so we choose to omit the details involved. ∎

The Dirichlet's Type Changhee q-L-function and Hurwitz type Changhee q-zeta function are closely related, too. We gave this relation as follows:

Theorem 10. *Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Then*

$$L_q(s, \chi \mid w_1) = [f]^{-s} \sum_{a=1}^{f} \chi(a) \zeta_{q^f}(s, \frac{aw_1}{f} \mid w_1). \tag{5.3}$$

Proof. By setting $n = a + kf$, where ($k = 0, 1, 2, ..., \infty$; $a = 1, 2, ..., f$) in (5.2), we have

$$\begin{aligned} L_q(s, \chi \quad \mid \quad w_1) &= w_1 \sum_{a=1}^{f} \chi(a) \sum_{k=0}^{\infty} \frac{q^{(aw_1 + kfw_1)}}{[aw_1 + kfw_1]^s} \\ &= w_1 \sum_{a=1}^{f} \chi(a) \sum_{k=0}^{\infty} \frac{q^{f(\frac{aw_1}{f} + kw_1)}}{[f]^s [\frac{aw_1}{f} + kw_1 : q^f]^s} \\ &= [f]^{-s} \sum_{a=1}^{f} \chi(a) \left\{ -\frac{(1-q^f)^s}{s-1} \frac{1}{\log q^f} + w_1 \sum_{n=0}^{\infty} \frac{q^{f(\frac{aw_1}{f} + kw_1)}}{[\frac{aw_1}{f} + kw_1 : q^f]^s} \right\}. \end{aligned}$$

By using (4.5) in the above, we easily arrive at the desired result (5.3). ∎

6. Barnes' Type Changhee q-Bernoulli Numbers

We now define new generating functions as follows:

$$\begin{aligned} G_q(t \quad \mid \quad w_1) &= F_q(t \mid w_1) - \frac{q-1}{\log q} \exp(\frac{t}{1-q}) \\ &= \sum_{k=0}^{\infty} B_k(q \mid w_1) \frac{t^k}{k!} \quad (\mid t \mid < 2\pi), \end{aligned} \tag{6.1}$$

where the coefficients $B_k(q \mid w_1)$ are called Barnes' type Changhee q-Bernoulli numbers. By (6.1), we easily see that

$$B_k(q \mid w_1) = \frac{q-1}{\log q} (\frac{1}{q-1})^k + \beta_k(q \mid w_1),$$

where $\beta_k(q \mid w_1)$ is given by (4.1).

Analogous to (3.1), we can also consider the *modified* Changhee q-Bernoulli polynomials as follows:

$$\begin{aligned} G_q(w, t \quad \mid \quad w_1) &= F_q(w, t \mid w_1) - \frac{q-1}{\log q} \exp(\frac{t}{1-q}) \\ &= \sum_{n=0}^{\infty} \frac{B_n(w : q \mid w_1) t^n}{n!} \quad (\mid t \mid < 2\pi). \end{aligned} \tag{6.2}$$

By applying (6.2), we easily see that

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-2}G_q(w,-t \quad | \quad w_1)dt = w_1\sum_{n=0}^{\infty}\frac{q^{w_1n+w}}{[w_1n+w]^s} = \zeta_{q,1}(s,w\mid w_1) \tag{6.3}$$

By using (6.2), we give the following relationship between $\zeta_{q,1}(s,w\mid w_1)$ and $B_n(w:q\mid w_1)$.

Theorem 11. *For positive integer n,*

$$\zeta_{q,1}(1-n,w\mid w_1) = -\frac{B_n(w:q\mid w_1)}{n}.$$

We next define Barnes' type multiple Changhee q-Bernoulli polynomials as follows:

$$\begin{aligned} G_q^{(r)}(w,t \quad | \quad w_1,w_2,...,w_r) &= (-t)^r(\prod_{i=1}^{r}w_i)\sum_{n_1,n_2,...,n_r=0}^{\infty} q^{w+n_1w_1+n_2w_2+...+n_rw_r}e^{[w+n_1w_1+n_2w_2+...+n_rw_r]t} \\ &= \sum_{n=0}^{\infty}\frac{B_n^{(r)}(w:q\mid w_1,w_2,...,w_r)t^n}{n!} \quad (\mid t\mid < 2\pi\ ), \end{aligned} \tag{6.4}$$

whith, as usual,

$$\sum_{n_1,n_2,...,n_r=0}^{\infty} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_r=0}^{\infty}.$$

It follows from (6.4) that

$$\lim_{q\to1}G_q^{(r)}(w,t\mid w_1,w_2,...,w_r) = \frac{e^{tw}(tw_1)(tw_2)...(tw_r)}{(e^{tw_1}-1)(e^{tw_2}-1)...(e^{tw_r}-1)}.$$

This gives generating function of Barnes' type multiple Bernoulli numbers. Thus we get the following limit relationship:

$$\lim_{q\to1}B_n^{(r)}(w:q\mid w_1,w_2,...,w_r) = B_n^{(r)}(w\mid w_1,w_2,...,w_r).$$

This gives Barnes' type multiple Bernoulli numbers as a limit when q approaches.

By using (6.4), we give Barnes' type Changhee multiple q-zeta functions. For $s\in\mathbb{C}$, we consider the below integral which is known Mellin transformation of $G_q^{(r)}(w,t\mid w_1,w_2,...,w_r)$.

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1-r}G_q^{(r)}(w,-t \quad | \quad w_1,w_2,...,w_r)dt = (\prod_{i=1}^{r}w_i)\sum_{n_1,n_2,...,n_r=0}^{\infty}\frac{q^{w+n_1w_1+n_2w_2+...+n_rw_r}}{[w+n_1w_1+n_2w_2+...+n_rw_r]^s}. \tag{6.5}$$

By using (6.5), we can define Barnes' type Changhee multiple q-zeta functions as follows:

Definition 5. *Let $s,w,w_1,w_2,...,w_r\in\mathbb{C}$ with $\mathrm{Re}(w)>0$ and $r\in\mathbb{Z}^+$.*

$$\zeta_{q,r}(s,w\mid w_1,w_2,...,w_r) = (\prod_{i=1}^{r}w_i)\sum_{n_1,n_2,...,n_r=0}^{\infty}\frac{q^{w+n_1w_1+n_2w_2+...+n_rw_r}}{[w+n_1w_1+n_2w_2+...+n_rw_r]^s}. \tag{6.6}$$

We note that $\zeta_{q,r}(s,w \mid w_1,w_2,...,w_r)$ is analytic continuation for $\mathrm{Re}(s) > r$. By using (6.4) and (6.5), we arrive at the following theorem.

Theorem 12. *Let* $r \in \mathbb{Z}^+$. *Then*

$$\zeta_{q,r}(1-n,w \mid w_1,w_2,...,w_r) = (-1)^r \frac{(n-1)!}{(n+r-1)!} B^{(r)}_{n+r-1}(w : q \mid w_1,w_2,...,w_r).$$

We record the following limit relationship:

$$\begin{aligned}\lim_{q\to 1} \zeta_{q,r}(1-n,w \quad \mid \quad & w_1,w_2,...,w_r) = \zeta_{1,r}(1-n,w \mid w_1,w_2,...,w_r) \\ &= \quad (-1)^r \frac{(n-1)!}{(n+r-1)!} B^{(r)}_{n+r-1}(w \mid w_1,w_2,...,w_r)\end{aligned}$$

between the ordinary Barnes' type multiple zeta functions and Barnes' type Bernoulli numbers.

7. Relations Between Γ_q, $\zeta_{q,r}(s,w \mid 1,1,...,1)$ and $L_{q,r}(s,\chi)$

Γ_q-function is defined by (see, for example, [21], [26], and [27])

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q:q)_\infty}{(q^z:q)_\infty}, \tag{7.1}$$

where

$$(q^z : q)_\infty = \prod_{k=0}^{\infty} (1-q^{k+z}).$$

Thus the function $\zeta_q(s,x)$ is defined by

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^{n+x}}{[n+x]^s}. \tag{7.2}$$

If, for convenience, we denote

$$\frac{d}{ds}\zeta_q(s,x) \mid_{s=0} = \zeta_q^{'}(0,x), \tag{7.3}$$

we readily observe that

$$\zeta_q(0,a+1) = \zeta_{q,1}^{'}(0,1) + (q-1)\sum_{m=1}^{a} \log[m]^{-s-1} - \sum_{n=0}^{\infty}[n]^{-s} \tag{7.4}$$

By using (7.1), (7.2) and (7.3), we have the following theorem.

Theorem 13. *([27])*

$$\Gamma_q(a) = \frac{e^{\zeta_q^{'}(0,a+1)}}{e^{\zeta_q^{'}(0,1)}} \prod_{m=1}^{a} [m]^{(1-q)[m]}. \tag{7.5}$$

By means of (7.5), we obtain

$$\lim_{q\to 1} \Gamma_q(a) = \frac{e^{\zeta^{'}(0,a+1)}}{e^{\zeta^{'}(0,1)}} = \Gamma^{'}(a).$$

We now define $\zeta_{q,2}(s,x)$ as follows

$$\zeta_{q,2}(s,x) = \sum_{n_1,n_2=0}^{\infty} \frac{q^{n_1+n_2+x}}{[n_1+n_2+x]^s}. \tag{7.6}$$

We give some properties of the zeta function defined by (7.6) as follows:

1)

$$[n]^{-s}\sum_{m=0}^{n-1}\sum_{k=0}^{n-1}\zeta_{q^n,2}(s,x+\frac{k+m}{n})=\zeta_{q,2}(s,nx). \tag{7.7}$$

2) By (7.7), we have

$$\sum_{m=0}^{n-1}\sum_{k=0}^{n-1}\zeta^{'}_{q^n,2}(s,x+\frac{k+m}{n})=[n]^s\zeta_{q,2}(s,nx)\log[n]+[n]^s\zeta^{'}_{q,2}(s,nx). \tag{7.8}$$

3) In (7.8), if we take $s=0$, so that we have

$$\prod_{m=0}^{n-1}\prod_{k=0}^{n-1}e^{\zeta^{'}_{q^n,2}(0,x+\frac{k+m}{n})}=[n]^{\zeta_{q,2}(0,nx)}e^{\zeta^{'}_{q,2}(0,nx)}.$$

This function is called di-gamma function.

For any integer k with $k\geq 0$, we define the function $W_{m,\chi,q}(k)$ functions as follows:

$$W_{m,\chi,q}(k)=\sum_{a=1}^{k}\chi(a)q^a[a]^m,\ m\geq 0.$$

If $\chi\equiv 1$ in the above, we have

$$W_{m,q}(k)=W_{m,1,q}(k)=\sum_{a=1}^{k}q^a[a]^m.$$

By using (3.1) and (3.5), we easily obtain

$$\frac{1}{t}(F_{q,\chi}(x,t)-F_{q,\chi}(t))=\sum_{k=0}^{\infty}(\frac{B_{k+1,\chi}(x:q)-B_{k+1,\chi}(q)}{k+1})\frac{t^k}{k!}.$$

Thus we arrive at the following results:

$$W_{k,\chi,q}(n)=\frac{B_{k+1,\chi}(n:q)-B_{k+1,\chi}(q)}{k+1}$$

and

$$W_{k,q}(n)=\frac{B_{k+1}(n:q)-B_{k+1}(q)}{k+1}.$$

8. Analytic Properties of q-L-Function and the q-Hurwitz zeta function

Let $s\in\mathbb{C}$. q-zeta function is defined by[30]

$$\zeta_q(s)=-\frac{(1-q)^s}{s-1}\frac{1}{\log q}+\sum_{n=0}^{\infty}\frac{q^n}{[n]^s}. \tag{8.1}$$

We note that $\zeta_q(s)$ are analytically continued for $\mathrm{Re}(s)>1$.

q-Hurwitz zeta function is defined by

$$\zeta_q(s,x)=-\frac{(1-q)^s}{s-1}\frac{1}{\log q}+\sum_{n=0}^{\infty}\frac{q^{n+x}}{[n+x]^s}. \tag{8.2}$$

From the definition (8.2), one can easily get

$$\zeta_q(0,v) = -(\frac{q-1}{\log q})[v] + \frac{q^v}{\log q} - \frac{q^v}{q-1}.$$

If $q \to 1$, then we have

$$\zeta_q(0,v) \to \zeta(0,v) = \frac{1}{2} - v. \tag{8.3}$$

More generally, we have

$$\zeta_q(-m,v) = -\frac{\beta_{m+1}(v:q)}{m+1},\ m \geq 0. \tag{8.4}$$

For $s \in \mathbb{C}$, we define q-L-function as follows:
Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$.

$$L_q(s,\chi) = \sum_{n=0}^{\infty} \frac{\chi(n)q^n}{[n]^s}. \tag{8.5}$$

By using (8.1) and (8.5), we easily obtain the following relation:

$$L_q(s,\chi) = [f]^{-s} \sum_{a=1}^{f} \chi(a)\zeta_{q^f}(s, \frac{a}{f}). \tag{8.6}$$

By using (8.6) and (8.4), we easily have the following theorem.

Theorem 14. *Let $k \in \mathbb{Z}^+$. We have*

$$L_q(1-k,\chi) = -\frac{\beta_{k,\chi}(q)}{k}.$$

In particular, if we define

$$H_q(s,a,F) = \sum_{\substack{m > 0 \\ m \equiv a (\mathrm{mod}\, F)}} \frac{q^m}{[m]^s}.$$

then we have

$$H_q(s,a,F) = \sum_{n=0}^{\infty} \frac{q^{nF+a}}{[nF+a]^s} = \frac{1}{[F]^s}\zeta_{q^F}(s, \frac{a}{F}).$$

It is well-known for the Hurwitz zeta function that (cf., e.g., [54], p. 91, Equation (15)), we have

$$\lim_{s \to \infty}(\zeta(s,a) - \frac{1}{s-1}) = -\frac{\Gamma'(a)}{\Gamma(a)} = -\psi(a)$$

in terms of the familiar digamma (or ψ) function. Hence, $H_q(s,a,F)$ has a simple pole at $s = 1$ with residue

$$\frac{1}{[F]}\frac{1}{F}\frac{q^F - 1}{\log q}.$$

Remark 3. *Barnes-Changhee multiple q-zeta functions are defined by (see [20], [28])*

$$\zeta_{q,r}(s, w \mid a_1, a_2, ..., a_r) = \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{q^{w+n_1+n_2+\ldots+n_r}}{[w+n_1a_1+n_2a_2+\ldots+n_ra_r]^s},\ \Re(w) > 0,\ q \in C \quad \textit{with} \quad |\, q \,| < 1,$$

which, for $a_1 = a_2 = ... = a_r = 1$, yields

$$\zeta_{q,r}(s, w \mid 1, 1, ..., 1) = \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{q^{w+n_1+n_2+...+n_r}}{[w+n_1+n_2+...+n_r]^s}.$$

Moreover, if $w = r$ and $s = 1 - n$ ($n \in \mathbb{Z}^+$), we have

$$\zeta_{q,r}(1-n, r \mid 1, 1, ..., 1) = (-1)^r \frac{(n-1)!}{(n+r-1)!} B_{n+r-1}^{(r)}(r : q).$$

We also note that

$$\begin{aligned} \lim_{q \to 1} \zeta_{q,r}(1-n, r \quad &| \quad 1, 1, ..., 1) = \zeta_r(1-n, r \mid 1, 1, ..., 1) \\ &= \quad (-1)^r \frac{(n-1)!}{(n+r-1)!} B_{n+r-1}^{(r)}(r). \end{aligned}$$

(see [20], [28]*).*

Similarly, by using the analogous approaches to the multiple L-functions, we have

$$L_{q,r}(s, \chi) = \sum_{n_1, n_2, ..., n_r = 1}^{\infty} \frac{\chi(n_1)\chi(n_2)...\chi(n_r) q^{n_1+n_2+...+n_r}}{[n_1+n_2+...+n_r]^s}. \tag{8.7}$$

For $s = -n$ ($n \in \mathbb{Z}^+$), we find from (4.3) that

$$L_{q,r}(-n, \chi) = (-1)^r \frac{n!}{(n+r)!} B_{n+r,\chi}^{(r)}(q)$$

(see [20], [28]*).*

The following theorem provides a relationship between $L_{q,r}(s, \chi)$ and $\zeta_{q,r}(s, w \mid a_1, a_2, ..., a_r)$.

Theorem 15. *Let χ be a Dirichlet character of conductor $f \in \mathbb{Z}^+$. Also let $a_1, a_2, ..., a_r$ and $n_1, ..., n_r$ be in $\mathbb{Z}^+$. Then*

$$L_{q,r}(s, \chi) = [f]^{-s} \sum_{a_1, ..., a_r = 1}^{f} \chi(a_1)\chi(a_2)...\chi(a_r) \zeta_{q^f, r}(s, \frac{a_1 + ... + a_r}{f} \mid 1, ..., 1).$$

Proof. By setting $n_j = a_j + n_j f$, ($j \in \{1, 2, ..., r\}$, $n_j = 0, 1, ..., \infty$, and $a_j = 1, 2, ..., f$) in (8.7), we have

$$\begin{aligned} L_{q,r}(s, \chi) &= \sum_{a_1, ..., a_r = 1}^{f} \chi(n_1)\chi(n_2)...\chi(n_r) \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{q^{a_1+...+a_r+f(n_1+n_2+...+n_r)}}{[a_1+...+a_r+f(n_1+n_2+...+n_r)]^s} \\ &= [f]^{-s} \sum_{a_1, ..., a_r = 1}^{f} \chi(n_1)\chi(n_2)...\chi(n_r) \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{q^{f(\frac{a_1+...+a_r}{f}+n_1+n_2+...+n_r)}}{[\frac{a_1+...+a_r}{f}+n_1+n_2+...+n_r]^s} \\ &= [f]^{-s} \sum_{a_1, ..., a_r = 1}^{f} \chi(n_1)\chi(n_2)...\chi(n_r) \zeta_{q^f, r}(s, \frac{a_1 + ... + a_r}{f} \mid 1, 1, ..., 1). \end{aligned}$$

Thus we obtain the desired result asserted by Theorem 15. ∎

By putting $w_1 = w_2 = ... = w_r = 1$ and $w = x$ in (6.6), we obtain

$$\zeta_{q,r}(s, x \mid 1, 1, ..., 1) = \sum_{l=0}^{\infty} \begin{pmatrix} l+r-1 \\ r-1 \end{pmatrix} \frac{q^{x+l}}{[x+l]^s}, \tag{8.8}$$

which, for $r = 1$, reduces immediately to the q-Hurwitz zeta function:

$$\zeta_{q,r}(s, x \mid 1, 1, ..., 1) = \sum_{l=0}^{\infty} \frac{q^{x+l}}{[x+l]^s}.$$

Furthermore, by setting $s = -n$, with ($n \in \mathbb{Z}^+$) in (8.8), we have

$$\begin{aligned} \zeta_{q,r}(-n, x \quad | \quad 1, 1, ..., 1) &= \sum_{l=0}^{\infty} \begin{pmatrix} l+r-1 \\ r-1 \end{pmatrix} \frac{q^{x+l}}{[x+l]^{-n}} \\ &= (-1)^r \frac{(n-1)!}{(n+r-1)!} \beta_{n+r-1}^{(r)}(x : q \mid 1, 1, ..., 1), \end{aligned}$$

where $B_n^{(r)}(x : q \mid 1, 1, ..., 1)$ numbers are defined as follows ([23], [24]):

For $n, k \in \mathbb{Z}^+$ $(k > 1)$, if $S_{n,q}$ denotes the sums of the nth powers of positive q-integers up to $k-1$ (see[24]):

$$S_{n,q}(k) = \sum_{m=0}^{k-1} q^m [m]^n,$$

then we have

$$B_{n,q}^{(r)}(x : q \mid 1, 1, ..., 1) = (-1)^r \frac{(r+1)!}{r!} \sum_{k=0}^{\infty} \begin{pmatrix} k+r-1 \\ r-1 \end{pmatrix} q^{x+k} \sum_{m=0}^{r-1} S_{m,q^{r-m}}(x+k).$$

Remark 4. *By using (6.6), we have*

$$\lim_{q \to \infty} \zeta_{q,r}(s, w \mid a_1, a_2, ..., a_r) = \zeta_r(s, w \mid a_1, a_2, ..., a_r)$$

in termes of Barnes multiple zeta function in (1.12).

9. Generalized Multiple Changhee q-Bernoulli Numbers and the Dirichlet Type Multiple Changhee q-L-functions

In this section, we define generalized multiple Changhee q-Bernoulli numbers attached to the Drichlet character χ. We also construct Dirichlet's type multiple Changhee q-L-functions. We then give relation between Dirichlet's type multiple Changhee q-L-functions and generalized multiple Changhee q-Bernoulli numbers as well.

The generalized multiple Changhee q-Bernoulli numbers attached to the Drichlet character χ are defined by means of the following generating function:

$$\begin{aligned} F_{q,\chi}^{(r)}(t \quad | \quad w_1, ..., w_r) &= (-t)^r \left(\prod_{j=1}^{r} w_j \right) \sum_{n_1,...,n_r=1}^{\infty} \left(\prod_{k=1}^{r} \chi(n_k) \right) q^{\left(\sum_{m=1}^{r} w_m n_m\right)} e^{[\sum_{m=1}^{r} w_m n_m] t} \\ &= \sum_{n=0}^{\infty} B_{n,\chi}^{(r)}(q \mid w_1, ..., w_r) \frac{t^n}{n!} \quad (\mid t \mid < 2\pi), \end{aligned} \tag{9.1}$$

where $w_1, ..., w_r \in \mathbb{R}^+$, $r \in \mathbb{Z}^+$.

By simple calculations in (9.1), we have

$$\begin{aligned}
&(-t)^r\left(\prod_{j=1}^r w_j\right)\sum_{n_1,\ldots,n_r=1}^{\infty}\left(\prod_{k=1}^r\chi(n_k)\right)q^{\left(\sum_{m=1}^r w_m n_m\right)}e^{[\sum_{m=1}^r w_m n_m]t}\\
= &(-t)^r(\prod_{j=1}^r w_j)\sum_{a_1,\ldots,a_r=1}^{f}\left(\prod_{k=1}^r\chi(n_k)q^{a_k w_k}\right)\exp(t[\sum_{m=1}^r w_m a_m])\\
&.\sum_{n_1,\ldots,n_r=1}^{\infty}q^{\left(\sum_{m=1}^r w_m n_m f\right)}\exp\left(t[\sum_{m=1}^r w_m n_m f]q^{\sum_{m=1}^r w_m n_m f}\right)\\
(9.2)\quad = &F_{q,\chi}^{(r)}(t\mid w_1,\ldots,w_r)
\end{aligned}$$

Now by applying (6.4), we obtain

(9.3)

$$F_{q,\chi}^{(r)}(t\mid w_1,\ldots,w_r)=[f]^{-r}\sum_{a_1,\ldots,a_r=1}^{f}\prod_{i=1}^r\chi(a_i)G_{q^f}^{(r)}(\frac{w_1a_1+\ldots+w_ra_r}{f},[f]t\mid w_1,w_2,\ldots,w_r).$$

By using (6.4) and (9.3), we readily arrive at the following theorem:

Theorem 16. *Let χ be a Dirichlet character of conductor $f\in\mathbb{Z}^+$. Then*

(9.4)

$$B_{n,\chi}^{(r)}(q\mid w_1,\ldots,w_r)=[f]^{n-r}\sum_{a_1,\ldots,a_r=1}^{f}\prod_{i=1}^r\chi(a_i)B_n^{(r)}(\frac{w_1a_1+\ldots+w_ra_r}{f}:q^f\mid w_1,w_2,\ldots,w_r).$$

Here, we can now construct Dirichlet's type multiple Changhee q-L-function. By using Mellin transformation and Residue Theorem in (9.1), then we obtain

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1-r}F_{q,\chi}^{(r)}(-t\quad\mid\quad w_1,\ldots,w_r)dt$$

$$(9.5)\quad =\left(\prod_{j=1}^r w_j\right)\sum_{n_1,\ldots,n_r=1}^{\infty}\left(\prod_{k=1}^r\chi(n_k)\right)q^{\left(\sum_{m=1}^r w_m n_m\right)}\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-[\sum_{m=1}^r w_m n_m]t}dt$$

By using (9.5), we can define Dirichlet's type multiple Changhee q-L-functions as follows.

Definition 6. *For a Dirichlet character $\chi$with conductor $f\in\mathbb{Z}^+$, we define*

$$(9.6)\quad L_{q,r}(s,\chi\mid w_1,\ldots,w_r)=\left(\prod_{j=1}^r w_j\right)\sum_{n_1,n_2,\ldots,n_r=1}^{\infty}\frac{\left(\prod_{k=1}^r\chi(n_k)\right)q^{\left(\sum_{m=1}^r w_m n_m\right)}}{[\sum_{m=1}^r w_m n_m]^s}.$$

By (9.5) and (9.6), we can easily obtain relationship the following relationship between $L_{q,r}(s,\chi\mid w_1,\ldots,w_r)$ and $\zeta_{r,q}(s,w_1a_1+\ldots+w_ra_r\mid w_1,\ldots,w_r)$.

Theorem 17.

(9.7)

$$L_{q,r}(s,\chi\mid w_1,\ldots,w_r)=[f]^{r-s}\sum_{a_1,\ldots,a_r=1}^{f}\left(\prod_{k=1}^r\chi(a_k)\right)\zeta_{q^f,r}(s,\frac{w_1a_1+\ldots+w_ra_r}{f}\mid w_1,\ldots,w_r).$$

By using (9.1) to (9.7), the numbers $L_{q,r}(-n,\chi\mid w_1,\ldots,w_r)$, ($n>0$) are given explicitly aby Theorem 18 bellow.

Theorem 18.

$$L_{q,r}(-n,\chi \mid w_1,...,w_r) = (-1)^r \frac{n!}{(n+r)!} B^{(r)}_{n,\chi}(q \mid w_1,...,w_r).$$

10. Euler-Barnes' type Multiple q-Daehee Zeta Functions

The main purpose of this section is to prove analytic continuation of the Euler-Barnes' type multiple q-Daehee zeta functions depending on the parameters $a_1,...,a_r$ which are taken positive parts in the Complex Field. Therefore, we construct generating function of q-Euler-Barnes'type multiple Frobenius-Euler polynomials. We define Euler-Barnes' type multiple q-Daehee zeta Functions. As we remarked earlier, the Euler-Barnes' type multiple q-Daehee zeta functions have a potentially useful connection with Topology and Physics, together with the algebraic relations among them. We give the values of these functions at negative integers as well.

Recently, by using an invariant p-adic integral, Kim[17] showed that the Daehee numbers are related to the q-Bernoulli and Eulerian numbers. Here we construct our generating function in complex case.

We now define the generating function of Euler-Barnes' type q-Daehee numbers as follows.

If $w_1 \in \mathbb{C}$ with positive real part and $u \in \mathbb{C}$ with $\mid u \mid< 1$, then

$$\begin{aligned} F_{u^{-1},q}(t \quad \mid \quad w_1) &= (1-u)\exp\left(\frac{t}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{1}{1-q}\right)^j\left(\frac{1}{1-q^{w_1 j}u}\right)\frac{(-t)^j}{j!} \\ &= \sum_{n=0}^{\infty} H_n(u^{-1}:q \mid w_1)\frac{t^n}{n!} \quad (\mid t \mid< 2\pi\ ). \end{aligned} \tag{10.1}$$

By letting $q \to 1$ in (10.1), we arrive at (1.5), that is,

$$\lim_{q\to 1} F_{u^{-1},q}(t \mid w_1) = \frac{1-u^{-1}}{e^{wt}-u^{-1}} = \sum_{n=0}^{\infty} H_n(u^{-1} \mid w_1)\frac{t^n}{n!},$$

which implies that

$$\lim_{q\to 1} H_n(u^{-1}:q \mid w_1) = H_n(u^{-1} \mid w_1).$$

By using (10.1), we also get

$$F_{u^{-1},q}(t \mid w_1) = (1-u)\sum_{k=0}^{\infty}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n}{1-q^{w_1 n}u}\frac{1}{(1-q)^k}\frac{t^k}{k!}. \tag{10.2}$$

By applying (10.1) and (10.2), we easily obtain the following result:

For $w_1 \in \mathbb{C}$ with positive real part, $u \in \mathbb{C}$ with $\mid u \mid< 1$ and $\mathrm{Re}(w_1) > 0$,

$$H_k(u^{-1}:q \mid w_1) = \frac{(1-u)}{(1-q)^k}\sum_{n=0}^{k}\binom{k}{n}\frac{(-1)^n}{1-q^{w_1 n}u}. \tag{10.3}$$

We note that the numbers in (10.3) are called Euler-Barnes' type Daehee q-Euler numbers.

By (10.1), we have

$$F_{u^{-1},q}(t \quad | \quad w_1) = (1-u)e^{\frac{t}{1-q}}\sum_{j=0}^{\infty}(-\frac{1}{1-q})^j(\sum_{n=0}^{\infty}q^{w_1jn}u^n)\frac{t^j}{j!}$$

$$= \quad (1-u)\sum_{n=0}^{\infty}u^n e^{[w_1n]t} \quad (\mid t \mid < 2\pi \ ). \tag{10.4}$$

Using (10.1) and (10.4), we define generating function of Euler-Barnes' type Daehee q-Euler polynomials as follows:

$$\begin{aligned} F_{u^{-1},q}(t, w \quad | \quad w_1) &= e^{[w]t}F_{u^{-1},q}(q^w t \mid w_1) \\ &= \sum_{n=0}^{\infty}H_n(u^{-1}, w : q \mid w_1)\frac{t^n}{n!} \quad (\mid t \mid < 2\pi \ ), \end{aligned} \tag{10.5}$$

which implies that

$$F_{u^{-1},q}(t, w \mid w_1) = (1-u)\sum_{n=0}^{\infty}u^n e^{[w+w_1n]t}. \tag{10.6}$$

Next we note that

$$\begin{aligned} \lim_{q\to\infty}F_{u^{-1},q}(t, w \quad | \quad w_1) &= (1-u)\sum_{n=0}^{\infty}u^n e^{(w+w_1n)t} \\ &= \frac{(1-u^{-1})}{e^{w_1t}-u^{-1}}e^{wt} \\ &= \sum_{n=0}^{\infty}H_n(u^{-1}, w \mid w_1)\frac{t^n}{n!} \quad (\mid t \mid < 2\pi \ ), \end{aligned}$$

which gives (1.6). Hence

$$\lim_{q\to\infty}H_n(u^{-1}, w : q \mid w_1) = H_n(u^{-1}, w \mid w_1).$$

Now by applying (10.1), (10.3) , (10.5) and (10.6), we easily arrive at the following theorem.

Theorem 19.

$$H_n(u^{-1}, w : q \mid w_1) = \sum_{l=0}^{n}\binom{n}{l}[w]^{n-l}q^{wl}H_l(u^{-1} : q \mid w_1). \tag{10.7}$$

We remark that the numbers $H_n(u^{-1}, w : q \mid w_1)$ are called Euler-Barnes' type Daehee q-Euler polynomials.

Let us define the Euler-Barnes' type multiple Daehee q-Euler polynomials. The generating function of this polynomials are defined as follows:

$$\begin{aligned} F^{(r)}_{u^{-1},q}(t, w \quad | \quad w_1, ..., w_r) &= (1-u)^r\sum_{n_1,\ldots,n_r=0}^{\infty}u^{n_1+\ldots+n_r}e^{[w+w_1n_1+\ldots+w_rn_r]t} \\ &= \sum_{n=0}^{\infty}H^{(r)}_{u^{-1},q}(u^{-1}, w \mid w_1, ..., w_r)\frac{t^n}{n!} \quad (\mid t \mid < 2\pi \ ). \end{aligned} \tag{10.8}$$

where $r \in \mathbb{Z}^+$, $w_1, ..., w_r \in \mathbb{C}$ with positive real part, $u \in \mathbb{C}$ with $\mid u^{-1} \mid < 1$.

Note that that cf [22]

$$\begin{aligned}\lim_{q\to 1} F^{(r)}_{u^{-1},q}(t,w \mid w_1,...,w_r) &= (1-u)^r \sum_{n_1,...,n_r=0}^{\infty} u^{n_1+...+n_r} e^{(w+w_1n_1+...+w_rn_r)t}\\ &= \frac{(1-u^{-1})...(1-u^{-1})}{(e^{w_1t}-u^{-1})...(e^{w_rt}-u^{-1})} e^{wt} \quad (10.9)\\ &= \sum_{n=0}^{\infty} H^{(r)}_n(u^{-1},w \mid w_1,...,w_r)\frac{t^n}{n!} \quad (\mid t \mid < 2\pi).\end{aligned}$$

By using (10.7) to (10.9), we have [22]

$$\lim_{q\to\infty} H^{(r)}_n(u^{-1},w:q \mid w_1,...,w_r) = H^{(r)}_n(u^{-1},w \mid w_1,...,w_r).$$

11. Euler-Barnes' Type Daehee q-Zeta Functions

By applying Mellin transformation and Residue Theorem in (10.1) and (10.6), we have

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1}\frac{1}{1-u}F_{u^{-1},q}(-t,w \mid w_1)dt\\ &= \sum_{n=0}^{\infty} u^n \frac{1}{\Gamma(s)}\int_0^{\infty} e^{-[w+w_1n]t}t^{s-1}dt \quad (11.1)\\ &= \sum_{n=0}^{\infty}\frac{u^n}{[w+w_1n]^s},\end{aligned}$$

where $\Gamma(s)$ is the Euler gamma function.

Thus by virtue of (11.1), we consider Euler-Barnes' type Daehee q-zeta functions as follows.

For $s \in \mathbb{C}$,

$$\zeta_q(s,w,u \mid w_1) = \sum_{n=0}^{\infty}\frac{u^n}{[w+w_1n]^s}. \quad (11.2)$$

We note that $\zeta_q(s,w,u \mid w_1)$ is analytic for $\mathrm{Re}(s) > 1$, and that $\zeta_q(s,u \mid w_1)$ is called the Euler-Barnes' type Daehee q-zeta functions which are defined as follows:

$$\zeta_q(s,u \mid w_1) = \sum_{n=0}^{\infty}\frac{u^n}{[w_1n]^s}.$$

By using (11.1) and (11.2), we easily see that

Theorem 20. *Let $n \in \mathbb{Z}^+$. Then*

$$\zeta_q(-n,w,u \mid w_1) = \frac{1}{1-u}H_n(u^{-1},w:q \mid w_1).$$

By using the same method as in (11.1), we shall construct the analytic Euler-Barnes' type multiple Daehee q-zeta functions as follows:

$$\begin{aligned}&\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1-r}\frac{1}{(1-u)^r}F^{(r)}_{u^{-1},q}(-t,w \mid w_1,...,w_r)dt\\ &= \sum_{n_1,n_2,...,n_r=0}^{\infty}\frac{u^{n_1+n_2+...+n_r}}{[w+n_1w_1+n_2w_2+...+n_rw_r]^s}, \quad (11.3)\end{aligned}$$

where Re$(w) > 0$ and Re$(s) > r$.

By means of (11.3), we define Euler-Barnes' type multiple Daehee q-zeta functions.

Definition 7.

$$\zeta_q(s, w, u \mid w_1, w_2, ..., w_r) = \sum_{n_1, n_2, ..., n_r = 0}^{\infty} \frac{u^{n_1+n_2+...+n_r}}{[w+n_1w_1+n_2w_2+...+n_rw_r]^s},$$

where $\Re(w) > 0$ *and* $\Re(s) > r$*.*

We note that for Re$(s) > 1$, $\zeta_q(s, w, u \mid w_1, w_2, ..., w_r)$ provides an analytic continuation in the Complex Field $\mathbb{C}$ and that [22]

$$\lim_{q \to 1} \zeta_q(s, w, u \mid w_1, w_2, ..., w_r) = \zeta(s, w, u \mid w_1, w_2, ..., w_r).$$

Analytic continuation and special values of Euler-Barnes' type multiple Daehee q-zeta functions are given by integral representation of $\zeta_q(s, w, u \mid w_1, w_2, ..., w_r)$ as follows:

$$\zeta_q(s, w, u \mid w_1, w_2, ..., w_r) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1-r} \frac{1}{(1-u)^r} F^{(r)}_{u^{-1},q}(-t, w \mid w_1, ..., w_r) dt.$$

Now, by using Cauchy Residue Theorem for $s = n$ ($n \in \mathbb{Z}^+$), we obtain the following theorem.

Theorem 21. *Let* $n \in \mathbb{Z}^+$*. Then*

$$\zeta_q(-n, w, u \mid w_1, w_2, ..., w_r) = \frac{1}{(1-u)^r} H^{(r)}_n(u^{-1}, w : q \mid w_1, ..., w_r).$$

12. Further Remarks and Observations

Shiratani[50] defined the following zeta functions:

$$\zeta(s \mid u) = \sum_{n=0}^{\infty} \frac{u^{-n}}{n^s}, \ (\ \Re(s) > 1 \ ).$$

By using (1.5),the values of this function at negative integers are obtained explicitly as follows:

$$\zeta(-k \mid u) = -\frac{H_k(u)}{k}, \ (\ k \in \mathbb{Z}^+ \ ).$$

This functions generalized by Kim [22] to the form which is given already in (1.17). He also gave the analytic continuation of multiple zeta functions (the Euler-Barnes multiple zeta functions) depending on parameters $a_1, ..., a_r$ taking positive values in the complex number field. If $q \to 1$ in (10.9), we get the rth Frobenius -Euler polynomials with parameters $w, w_1, ..., w_r$ taking positive values in the complex number field

$$\frac{(1-u^{-1})...(1-u^{-1})}{(e^{w_1t}-u^{-1})...(e^{w_rt}-u^{-1})} e^{wt} = \sum_{n=0}^{\infty} H^{(r)}_n(u^{-1}, w \mid w_1, ..., w_r) \frac{t^n}{n!} \quad (\mid t \mid < 2\pi \ ),$$

If we set $w = 0$ in the above generating function, then we readily obtain multiple Euler-Barnes numbers given by

$$H^{(r)}_n(u^{-1}, 0 \mid w_1, ..., w_r) = H^{(r)}_n(u^{-1} \mid w_1, ..., w_r).$$

We now define

$$F_{u,q}^{(r)}(t,x\mid w_1,...,w_r) = \frac{(1-u)^r e^{-xt}}{(e^{-w_1 t}-u)...(e^{-w_r t}-u)}(\frac{u}{u-1})^r$$

$$= (\frac{u^f}{u^f-1})^r \sum_{a_1,...,a_r=1}^{f-1} u^{-\sum_{j=1}^r a_j} \frac{(1-u^f)^r \exp(-ft\frac{x+\sum_{j=1}^r a_j w_j}{f})}{(e^{-w_1 ft}-u^f)...(e^{-w_r ft}-u^f)} \tag{12.1}$$

By using (12.1), we have

$$\frac{u^r}{(u-1)^r}H_n^{(r)}(u,x \mid w_1,...,w_r)$$
$$= f^n \sum_{a_1,...,a_r=1}^{f} \frac{u^{fr-\sum_{j=1}^r a_j}}{(u^f-1)^r} H_n^{(r)}(u^f, \frac{x+a_1w_1+...+a_rw_r}{f} \mid w_1,...,w_r).$$

This is known as distribution function. Now, by using this function, Frobenius-Barnes' type measure is defined as follows.

Let χ be a Dirichlet character with conductor $f \in \mathbb{Z}^+$. We define

$$\sum_{a_1,...,a_r=1}^{f} \prod_{j=1}^{r} \chi(a_j) u^{rf-\sum_{j=1}^r a_j} \frac{e^{t\sum_{j=1}^r a_j w_j}(1-u^f)^r}{(e^{w_1 ft}-u^f)...(e^{w_r ft}-u^f)}$$
$$= \sum_{n=0}^{\infty} H_{n,\chi}^{(r)}(u \mid w_1,...,w_r)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}(f^n \sum_{a_1,...,a_r=1}^{f} \prod_{j=1}^{r} \chi(a_j) u^{rf-\sum_{j=1}^r a_j} H_n^{(r)}(u^f, \frac{a_1w_1+...+a_rw_r}{f} \mid w_1,...,w_r))\frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$, we easily see that

$$H_{n,\chi}^{(r)}(u \mid w_1,...,w_r)$$
$$= f^n \sum_{a_1,...,a_r=1}^{f} \prod_{j=1}^{r} \chi(a_j) u^{rf-\sum_{j=1}^r a_j} H_n^{(r)}(u^f, \frac{a_1w_1+...+a_rw_r}{f} \mid w_1,...,w_r).$$

By using this generating function, we can obtain an analytic continuation of $\zeta(s,w,u \mid w_1,w_2,...,w_r)$.

By using Mellin transformation in (12.1), we easily see that

$$\zeta_r(s,w,u \mid w_1,w_2,...,w_r) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1-r}\frac{1}{(1-u)^r}F_{u,q}^{(r)}(-t,w \mid w_1,...,w_r)dt.$$

Putting $s=-n$ ($n>0$) in the above, we have

$$\zeta_r(-n,w,u \mid w_1,w_2,...,w_r) = \frac{u^r}{(u-1)^r}H_n^{(r)}(u,w \mid w_1,...,w_r).$$

In p-adic case, we similarly obtain the following results.

For $u \in \mathbb{C}_p$, with $\mid 1-u \mid_p \geq 1$ we consider the integral

$$\lim_{N\to\infty}\frac{1}{1-u^{p^N}}\sum_{j=0}^{p^N-1} u^{p^N-j}g(j) = \sum_{j=0}^{p^N-1} g(j)E_u(j+p^N\mathbb{Z}_p).$$

For $g \in UD(\mathbb{Z}_p, \mathbb{C}_p)$, the above limit exist. Thus Euler integral defined as follows:

$$\int_{\mathbb{Z}_p} g(x) dE_u(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} g(x) \frac{u^{p^N-x}}{1-u^{p^N}}.$$

We now define p-adic Barnes' type Frobenius-Euler measure as follows:
Let $w_1, w_2, ..., w_r$ be the nonzero p-adic integers. Thus we have

$$E^{(k)}_{u,w_1}(x + p^N\mathbb{Z}_p) = \frac{u^{p^N-x}}{1-u^{p^N}} H^{(1)}_k(u^{p^N}, \frac{w_1 x}{p^N} \mid w_1).$$

This is a measure, because it is easily observe that

$$\sum_{i=0}^{p^N-1} E^{(k)}_{u,w_1}(x + ip^N + p^{N+1}\mathbb{Z}_p) = E^{(k)}_{u,w_1}(x + p^N\mathbb{Z}_p).$$

Thus $E^{(k)}_{u,w_1}$ is a distribution. Now we give bounded property of

$$E^{(k)}_{u,w_1}(x + p^N\mathbb{Z}_p), \text{ when } \mid 1-u\mid_p \geq 1.$$

Hence $E^{(k)}_{u,w_1}$ is a measure on $\mathbb{Z}_p$. By using this measure and the above relations, we have

$$\int_{\mathbb{X}} \chi(x) dE^{(k)}_{u,w_1}(x) = \frac{1}{1-u^f} H^{(1)}_{k,\chi}(u \mid w_1). \tag{12.2}$$

By using simple calculation, we see that

$$\int_{\mathbb{X}} dE^{(k)}_{u,w_1}(x) = w_1^k \int_{\mathbb{X}} dE_u(x).$$

Hence, substituting $\chi \equiv 1$ into (12.2), we have

$$\int_{\mathbb{X}} dE^{(k)}_{u,w_1}(x) = \frac{u}{1-u} H^{(r)}_k(u \mid w_1),$$

which finally yields

$$\int_{\mathbb{X}}\int_{\mathbb{X}} ... \int_{\mathbb{X}} e^{(x_1w_1+...+x_rw_r+w)t} dE_u(x_1)dE_u(x_2)...dE_u(x_r)$$
$$= \frac{u^r}{(e^{w_1t}-u)(e^{w_2t}-u)...(e^{w_rt}-u)} e^{wt}.$$

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