BARNES' TYPE MULTIPLE CHANGHEE q-ZETA FUNCTIONS

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ABSTRACT. In this paper, we give new generating functions which produce Barnes' type multiple generalized Changhee q-Bernoulli polynomials and polynomials. These functions are very important to construct multiple zeta functions. By using Mellin transform's formula and Cauchy Theorem , we prove the analytic continuation of Barnes' type multiple Changhee q-zeta function. Finally we give some relations between Barnes' type multiple Changhee q-zeta function and Barnes' type multiple generalized Changhee q-Bernoulli numbers.

1. INTRODUCTION, DEFINITION AND NOTATIONS

Multiple zeta functions have studied by many mathematicians. These functions are of interest and importance in many areas. These functions and numbers are in used not only in Complex Analysis and Mathematical Physics, but also in used in p-adic Analysis and other areas. In particular, multiple zeta functions occur within the context of knot theory, Quantum Field Theory, Applied Analysis and Number TheoryIn particular, multiple zeta functions occur within the context of knot theory and quantum field theory. The Barnes multiple zeta functions and gamma functions were also encountered by Shintani within the context of analytic number theory. They showed up in the form of the factor program for integrable field theories and in the studies of XXZ model correlation function. The computation of Feynman diagrams has confronted physicists with classes of integrals that are usually hard to be evaluated, both analytically and numerically. The newer techniques applied in the more popular computer algebra packages do not offer much relief. Therefore, it seems reasonable to occasionally study some alternative methods to come to a result. In the case of the computation of structure functions in deep inelastic scattering, the Mellin moments of these functions are often of interest. Each individual moment can be computed directly in a much simples way than that needed to compute the entire structure function and take its moments afterwards. Meanwhile, the special values of multiple zeta functions at positive integers have come to the foreground in the recent years, both in connection with theorical physics (Feynman diagrams) and the theory of mixed Tate motives. Historically, Euler already investigated the double zeta values in the $XVIII$th century (see for detail [1], [2], [3], [4], [28], [32], [24], [30], [38], [37], [8], [23], [7], [25], [33], [9], [10], [11], [12], [13], [16], [19], [22]).

In 1904, Barnes defined multiple gamma functions and multiple zeta functions, which are given as follows:

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In the complex plane, Barnes' multiple zeta function $\zeta_r(s, w|a_1, \ldots, a_r)$ depends on the parameters $a_1, \ldots, a_r$ (which will be taken nonzero). This function can be defined by the series

$$\zeta_r(s, w|a_1, \ldots, a_r) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{1}{(w + m_1a_1 + \cdots + m_ra_r)^s},$$

for $Re(w) > 0$ and $Re(s) > r$ [3].

In [31], Ruijsenaars showed how various known results concerning the Barnes multiple zeta and gamma functions can be obtained as specializations of simple features shared by a quite extensive class of functions. The pertinent functions involve Laplace transforms, and their asymptotic was obtained by exploiting this. He demonstrated how Barnes' multiple zeta and gamma functions fit into a recently developed theory of minimal solutions to first order analytic difference equations. Both of these approaches to the Barnes functions gave rise to novel integral representations. In [15], T Kim studied on the multiple L-series and functional equation of this functions. He found the value of this function at negative integers in terms of generalized Bernoulli numbers. In [29], Russias and Srivastava presented a systematic investigation of several families of infinite series which are associated with the Riemann zeta functions, the Digamma functions,the harmonic numbers, and the stirling numbers of the first kind. In [26], Matsumoto considered general multiple zeta functions of multi-variables, involving both Barnes multiple zeta functions and Euler-Zagier sums as special cases. He proved the meromorphic continuation to the whole space , asymptotic expansions, and upper bounded estimates. These results were expected to have applications to some arithmetical L-functions. His method was based on the classical Mellin-Barnes integral formula. Ota[27] studied on Kummer-type congruences for derivatives of Barnes' multiple Bernoulli polynomials. Ota generalized these congruences to derivatives of Barnes' multiple Bernoulli polynomials by an elementary method and gave a p-adic interpolation of them. The Barnes' multiple Bernoulli polynomials $B_n(x, r|a_1, \ldots, a_r)$ are defined by [3]

$$\frac{t^r e^{xt}}{\prod_{j=1}^{r} (e^{a_jt} - 1)} = \sum_{n=0}^{\infty} B_n(x, r|a_1, \ldots, a_r) \frac{t^n}{n!}, \tag{1.1}$$

for $|t| < 1$. In [14], T. Kim constructed the Barnes-type multiple Frobenius-Euler polynomials. He gave a Witt-type formula for these polynomials. To give the Witt-type formula for Barnes-type multiple Frobenius-Euler polynomials, he employed the p-adic Euler integrals on $\mathbb{Z}_p$ [12]. He also investigated the properties of the p-adic Stieltjes transform and p-adic Mellin transform. He defined multiple zeta functions (the Euler-Barnes multiple zeta functions) depending on the parameters $a_1, a_2, \ldots, a_r$ that are taken positive in the complex number field ([15], [16], [17], [18], [19], [20]):

$$\zeta_r(s, w, u|a_1, \ldots, a_r) = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{u^{-(m_1+m_2+\ldots+m_r)}}{(w + m_1a_1 + \cdots + m_ra_r)^s}, \tag{1.2}$$

where $Re(w) > 0, u \in \mathbb{C}$ with $|u| > 1$. These values have a certain connection with topology and physics, together with the algebraic relations among them. He also showed that multiple zeta functions can be continued analytically to $\mathbb{C}^k$.

The Euler numbers E_n are defined by [36],

$$\frac{2}{e^t-1}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}.$$

They are classical and important in number theory. Frobenius extended E_n to Euler numbers $H^n(u)$ belonging to an algebraic number u, with $\mid u \mid> 1$, and many authors investigate their properties [11], [14], . Shiratani and Yamamoto [35] constructed a p-adic interpolation $G_p(s,u)$ of the Euler numbers $H^n(u)$ and as its application, they obtained an explicit formula for $L_p'(0,\chi)$ with any Dirichlet character χ. In [36], Tsumura defined the generalized Euler numbers $H^n_\chi(u)$ for any Dirichlet character χ, which are analogous to the generalized Bernoulli numbers. He constructed their p-adic interpolation, which is an extension of Shiratani and Yamamoto's p-adic interpolation $G_p(s,u)$ of $H^n(u)$

For $u\in\mathbb{C}$ with $|u|>1$, the Frobenius-Euler polynomial were also defined by

$$\frac{1-u}{e^t-u}e^{xt}=e^{H(x,u)}=\sum_{n=0}^{\infty}H_n(x,u)\frac{t^n}{n!},\tag{1.3}$$

where we use the notation by symbolically replacing $H^m(x,u)$ by $H_m(x,u)$. In the case of $x=0$, the Frobenius-Euler polynomials are called Frobenius-Euler numbers. We write $H_m(u)=H_m(0,u)$ ([11], [14]). Note that $H_m(-1)=E_m$.

The Frobenius-Euler polynomials of order r, denoted by $H_n^{(r)}(u,x)$, were defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt}=\sum_{n=0}^{\infty}H_n^{(r)}(u,x)\frac{t^n}{n!}\tag{1.4}$$

([11], [14]). The values at $x=0$ are called Frobenius-Euler numbers of order r, when $r=1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers. When $x=0$ or $r=1$, we often suppress that part of the notation; e.g., $H_n^{(r)}(u)$ denotes $H_n^{(r)}(u,0)$, $H_n(u)$ denotes $H_n^{(1)}(u,0)$ [11], [14]. Let $w,a_1,a_2,\ldots,a_r$ be complex numbers such that $a_i\neq 0$ for each i, $i=1,2,\ldots,r$. Then the Euler-Barnes' polynomials of w with parameters $a_1,\ldots,a_r$ are defined as

$$\frac{(1-u)^r}{\prod_{j=1}^{r}(e^{a_jt}-u)}=\sum_{n=0}^{\infty}H_n^{(r)}(w,u|a_1,\ldots,a_r)\frac{t^n}{n!},\tag{1.5}$$

for $u\in\mathbb{C}$ with $|u|>1$ [11], [14].

In the special case $w=0$, the above polynomials are called the r-th Euler Barnes' numbers. We write

$$H_n^{(r)}(u|a_1,\ldots,a_r)=H_n^{(r)}(0,u|a_1,\ldots,a_r).$$

In [11], [14], by using p-adic (Euler) integrals on $\mathbb{Z}_p$, Kim and Rim constructed Changhee-Barnes' q-Euler numbers and polynomials which are related to the q-analogue of Euler-Barnes' polynomials and numbers. The q-analogue of Frobenius-Euler numbers, by using p-adic Euler integral, are given as follows [11], [14]:

$$H_{n,q}(u)=\int_{\mathbb{Z}_p}[x]_q^n d\mu_u(x)=\frac{1-u}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l\frac{1}{1-uq^l},$$

where p-adic q-integral is given by: Let f be uniformly differentiable function at a point of $\mathbb{Z}_p$. Then we have [20]

$$\int_{\mathbb{Z}_p} f(x)d\mu_u(x) = \lim_{N\to\infty}[p^N : u]^{-1} \sum_{0\le x<p^N} f(x)u^x,$$

with $u \in \mathbb{C}_p$, where $\mathbb{C}_p$ is the completion of algebraic closure of $\mathbb{Q}_p$. In the above we assume that $u \in \mathbb{C}_p$ with $|1-u|_p \ge 1$. If $q \in \mathbb{C}_p$, one normally assumes that $|q| < 1$ and $|1-q|_p \le p^{-1/(p-1)}$, so that $q^x = exp(x \log q)$. We use the notation

$$[x] = [x]_q = \frac{1-q^x}{1-q}, \text{ and } [x:z]_q = \frac{1-z^x}{1-z}$$

([13], [14], [20]). The q-analogue of Euler-Barnes' multiple numbers, which reduce to Euler-Barnes' multiple numbers at $q = 1$, are given as follows [20]:

Let $a_1, \dots, a_r$; $b_1, \dots, b_r$ will be taken in the nonzero integers and let $q \in \mathbb{C}_p$ with $|1-q|_p < 1, w \in \mathbb{Z}_p, u \in \mathbb{C}_p$ with $|1-u|_p \ge 1$. Then [20]

$$\begin{aligned} & H^{(r)}_{n,q}(u, w|a_1, \dots, a_r : b_1, \dots, b_r) \\ (1.6)\qquad = & \int_{\mathbb{Z}_p^r} [w + \sum_{j=1}^r a_j x^j]^n u^{\sum_{j=1}^r (b_j-1)x_j} d\mu_u(x_1)\cdots\mu_u(x_r) \end{aligned}$$

Theorem 1. *For $n \ge 0$, we have*

$$\begin{aligned} & H^{(r)}_{n,q}(u, w|a_1, \dots, a_r : b_1, \dots, b_r) \\ (1.7)\qquad = & \frac{(1-u)^r}{(1-q)^n} \sum_{l=0}^n \binom{n}{l} (-1)^l q^{lw} \frac{1}{\prod_{j=1}^r (1-q^{la_j}u^{b_j})}, \ \textit{cf.}\ [20]. \end{aligned}$$

Note that

$$\lim_{q\to 1} H^{(r)}_{n,q}(u, w|a_1, \dots, a_r : 1, \dots, 1) = H^{(r)}_n(u^{-1}, w|a_1, \dots, a_r).$$

In this paper, we define generating functions of Changhee-Barnes' q-Euler numbers and polynomials on $\mathbb{C}$-plane. These functions are very important to construct multiple zeta functions. By using Mellin transform's formula and Cauchy Theorem , we prove the analytic continuation of Barnes' type multiple Changhee q-zeta function. Finally we give some relations between Barnes' type multiple Changhee q-zeta function and Barnes' type multiple generalized Changhee q-Bernoulli numbers. Moreover, we give the value of these functions at negative integers.

2. Changhee-Barnes' Type q-Euler Numbers and Polynomials

In this chapter, we define generating function of the Changhee-Barnes' type q-Euler numbers and polynomials as follows:

Let u be the algebraic element of the complex number field $\mathbb{C}$, with $|u| < 1$. For $w_1, v_1, s \in \mathbb{C}$ with $Re(w_1) > 0, Re(v_1) > 0$, we then define

$$\begin{aligned} F_{u^{-1},q}(t|w_1; v_1) &= (1-u)e^{t/(1-q)} \sum_{j=0}^\infty \frac{1}{(1-q)^j}(-1)^j \frac{1}{1-q^{w_1 j}u^{v_1}} \frac{t^j}{j!} \\ (2.1)\qquad &= \sum_{k=0}^\infty H_{k,q}(u^{-1}|w_1; v_1)\frac{t^k}{k!}, \ |t| < 2\pi. \end{aligned}$$

Theorem 2. *Let u be algebraic integer with $|u| < 1, u \in \mathbb{C}$ and $w_1, v_1 \in \mathbb{C}$ with $Re(w_1) > 0, Re(v_1) > 0$. Then we have*

$$H_{k,q}(u^{-1}|w_1; v_1) = \frac{1-u}{(1-q)^k} \sum_{j=0}^{\infty} \binom{k}{j} (-1)^j \frac{1}{1-q^{w_1 j} u^{v_1}}.$$

Proof. By using (2.1) we have

$$\begin{aligned} &\sum_{k=0}^{\infty} H_{k,q}(u^{-1}|w_1; v_1) \frac{t^k}{k!} \\ = \quad & (1-u) \left(\sum_{n=0}^{\infty} \frac{1}{(1-q)^n} \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} \frac{1}{(1-q)^j} (-1)^j \frac{1}{1-q^{w_1 j} u^{v_1}} \frac{t^j}{j!} \right). \end{aligned}$$

By applying Cauchy product in the above, we easily obtain

$$\sum_{k=0}^{\infty} H_{k,q}(u^{-1}|w_1; v_1) \frac{t^k}{k!} = (1-u) \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{1}{1-q^{w_1 j} u^{v_1}} \right) \frac{1}{(1-q)^k} \frac{t^k}{k!}.$$

Now, by comparing coefficients of $\frac{t^k}{k!}$ in the above series, we arrive at the desired result. ∎

Remark 1. *We note that if $v_1 = 1$ in (2.1), then (2.1) reduces to (8.1) in* [22]. *By substituting $r = 1$ and $v_1 = v_2 = \cdots = v_r = 1$ in (1.6) and (1.7), then Changhee q-Euler numbers reduce to Daehee numbers. Indeed, by using (2.1), we obtain*

$$\begin{aligned} F_{u^{-1},q}(t|w_1; v_1) \quad &= \quad (1-u) \sum_{l=0}^{\infty} u^{v_1 l} e^{t/(1-q)} \sum_{j=0}^{\infty} \frac{1}{(1-q)^j} (-1)^j q^{w_1 j l} \frac{t^j}{j!} \\ &= \quad (1-u) \sum_{l=0}^{\infty} u^{v_1 l} e^{t[w_1 l]_q}. \end{aligned}$$

Hence

$$\begin{aligned} F_{u^{-1},q}(t|w_1; v_1) \quad &= \quad (1-u) \sum_{l=0}^{\infty} u^{v_1 l} e^{t[w_1 l]_q} \\ &= \quad \sum_{n=0}^{\infty} H_{n,q}(u^{-1}|w_1; v_1) \frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

In the above for $v_1 = 1$ the function $F_{u^{-1},q}(t|w_1; 1)$ is the generating function of q-Daehee polynomials.

We now define the generating function of Changhee q-Euler polynomials as follows:

$$\begin{aligned} F_{u^{-1},q}(t, w|w_1; v_1) \quad &= \quad e^{[w]_q t} F_{u^{-1},q}(q^w t|w_1; v_1) \\ &= \quad \sum_{n=0}^{\infty} H_{n,q}(u^{-1}, w|w_1; v_1) \frac{t^n}{n!}, \ |t| < 2\pi. \end{aligned} \tag{2.3}$$

By using (2.2) and (2.3), we easily see that

$$\begin{aligned} F_{u^{-1},q}(t,w|w_1;v_1) &= e^{[w]_q t}(1-u)\sum_{l=0}^{\infty} u^{v_1 l} e^{[w_1 l]_q q^w t} \\ &= (1-u)\sum_{l=0}^{\infty} u^{v_1 l} e^{[w+w_1 l]_q t}. \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we obtain

$$\begin{aligned} F_{u^{-1},q}(t,w|w_1;v_1) &= (1-u)\sum_{l=0}^{\infty} u^{v_1 l} e^{[w+w_1 l]_q t} \\ &= \sum_{n=0}^{\infty} H_{n,q}(u^{-1},w|w_1;v_1)\frac{t^n}{n!}. \end{aligned}$$

Now, we will generalize (2.1). Let $w_1, w_2, \ldots, w_r; v_1, v_2, \ldots, v_r \in \mathbb{C}$. Then generating function of multiple Changhee q-Euler polynomials are given as follows:

$$\begin{aligned} & F_{u^{-1},q}(t,w|w_1,w_2,\ldots,w_r;v_1,v_2,\ldots,v_r) \\ &= (1-u)\sum_{n_1,n_2,\ldots,n_r=0}^{\infty} u^{\sum_{i=1}^r n_i v_i} e^{\left[\sum_{i=1}^r n_i w_i\right]_q t} \\ &= \sum_{n=0}^{\infty} H_{n,q}^{(r)}(u^{-1},w|w_1,w_2,\ldots,w_r;v_1,v_2,\ldots,v_r)\frac{t^n}{n!},\ |t|<2\pi. \end{aligned} \tag{2.5}$$

By using (2.5), we arrive at the following theorem easily:

Theorem 3. *For $n \geq 0$, we have*

$$\begin{aligned} & H_{n,q}^{(r)}(u^{-1},w|w_1,w_2,\ldots,w_r;v_1,v_2,\ldots,v_r) \\ &= \frac{(1-u)^r}{(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lw}\frac{1}{\prod_{j=1}^{r}(1-q^{la_j}u^{b_j})}. \end{aligned}$$

The proof of this theorem is also given by Kim and Rim [20]. Their proof is related to p-adic Euler integration.

Note that

$$\lim_{q\to 1} H_{n,q}^{(r)}(u^{-1},w|w_1,w_2,\ldots,w_r:1,\ldots,1) = H_n^{(r)}(u^{-1},w|w_1,w_2,\ldots,w_r).$$

3. Analytic Continuation of Changhee q-Euler-Zeta Function

By using (2.1) and (2.5), we consider Changhee q-zeta functions $\zeta_q^{(r)}(u,w|w_1,\ldots,w_r;v_1,\ldots,v_r)$. We also give analytic continuation of $\zeta_q^{(r)}(u,w|w_1,\ldots,w_r;v_1,\ldots,v_r)$ by using Mellin transform's formula of (2.1) and (2.5). Let w, w_1, v_1 be complex numbers with positive real parts. For $s \in \mathbb{C}$, we consider Changhee q-zeta functions as follows:

$$\frac{1}{1-u}\frac{1}{\Gamma(s)}\int_0^{\infty} F_{u^{-1},q}(-t,w|w_1;v_1)t^{s-1}dt = \sum_{n=1}^{\infty}\frac{u^{v_1 n}}{[w+w_1 n]_q^s}. \tag{3.1}$$

(3.1) is the Mellin transform of (2.1). By (3.1), we define Changhee q-zeta function as follows:

For $s \in \mathbb{C}$,

$$\zeta_q(s,w,u|w_1;v_1) = \sum_{n=0}^{\infty} \frac{u^{v_1 n}}{[w+w_1 n]_q^s}.$$

Note that, for $v_1 = 1$, $\zeta_q(s,w,u|w_1;1)$ reduces to [22]. Thus, $\zeta_q(s,w,u|w_1;v_1)$ is the analytic continuation on $\mathbb{C}$, with simple pole at $s = 1$.

By using Cauchy Theorem and Residue Theorem in (3.1), for positive integer n, we easily arrive at the following theorem:

Theorem 4. *For $n \in \mathbb{Z}^+$,*

$$\zeta_q(-n,w,u|w_1;v_1) = \frac{1}{1-u} H_{n,q}(u^{-1},w|w_1;v_1).$$

Similarly we define multiple Changhee q-zeta function as follows:

Let $w, w_1, w_2, \ldots, w_r, v_1, v_2, \ldots, v_r$ be complex numbers with positive real parts and $r \in \mathbb{Z}^+$. Then we construct multiple Changhee q-zeta functions as follows: For $s \in \mathbb{C}$,

$$\frac{1}{(1-u)^r}\frac{1}{\Gamma(s)}\int_0^{\infty} F^{(r)}_{u^{-1},q}(t,w|w_1,w_2,\ldots,w_r;v_1,v_2,\ldots,v_r)t^{s-1}dt$$

$$= \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{u^{\sum_{i=1}^r v_i n_i}}{[w+\sum_{i=1}^r w_i n_i]_q^s}. \tag{3.2}$$

The above equation is the Mellin transform of (2.5). By using (3.2), we define multiple Changhee q-zeta functions as follows:

$$\zeta_q^{(r)}(s,u,w|w_1,\ldots,w_r;v_1,\ldots,v_r) = \sum_{n_1,n_2,\ldots,n_r=0}^{\infty} \frac{u^{\sum_{i=1}^r v_i n_i}}{[w+n_1 w_1+\cdots+n_r w_r]_q^s}.$$

Note that, for $v_1 = v_2 = \cdots = v_r = 1$, $\zeta_q^{(r)}(s,u,w|w_1,\ldots,w_r;1,\ldots,1)$ is reduced to Definition 6 in [22]. By using Cauchy Theorem and Residue Theorem in (3.2), we easily obtain the following theorem:

Theorem 5. *For $n \in \mathbb{Z}^+$, we have*

$$\zeta_q^{(r)}(-n,w,u|w_1,\ldots,w_r;v_1,\ldots,v_r) = \frac{1}{(1-u)^r} H_{n,q}(u^{-1},w|w_1,\ldots,w_r;v_1,\ldots,v_r).$$

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