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APPROXIMATION OF THE JACOBI POLYNOMIALS AND THE RACAH COEFFICIENTS

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ABSTRACT. This is the second part of a project which provides asymptotic approximations to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and to the Racah coefficients $P_n^{(an+c,bn+d)}(x)$, as $n \to \infty$, where a, b, c, d are constants. The approximations to $P_n^{(\alpha,\beta)}(x)$ are generated by the construction of certain fundamental sets of solutions to a hypergeometric differential equation. In a first step we construct approximations to the Jacobi polynomials and the Racah coefficients on a closed interval $[z_1, 1]$ where the solutions are free of zeros. This poses a special challenge since the two endpoints of the interval are a regular–singular point and a turning point of the corresponding differential equation. In the second step we "connect" the approximations of the Jacobi polynomials on $[1, \infty)$ through the singular regular point $x = 1$ to yield a global approximation on $[z_1, \infty)$. Our global approximation of the Jacobi polynomials on $[z_1, \infty)$ is obtained without the intervention of "special functions."

1. Introduction. This work is a continuation of the studies in [**5**]. In this recent paper [**5**], a technique was developed and employed that rendered new asymptotic approximations to solutions of the differential equation

$$y'' = p(x)y, \quad a \le x \le b \tag{1.1}$$

and their derivatives. This technique was applied to the hypergeometric equation

$$y'' = \left[\frac{\alpha^2-1}{4(x-1)^2} + \frac{\beta^2-1}{4(x+1)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{x^2-1}\right]y \tag{1.2}$$

on the interval $(1, \infty)$. One solution of equation (1.2) is the function $(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x)$, so an approximation of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is obtained on $[1, \infty)$. The same method was

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applied also to some specialized Jacobi polynomials where α, β are replaced by linear functions of n. These polynomials, $P_n^{(an+\alpha, bn+\beta)}(x)$, are employed in the representation of the Racah coefficients. The Racah coefficients play an important role in quantum theory, see e.g., [**1, 2**]. See [**3, 14, 17**] for other numerous applications.

The asymptotic approximations that we utilize are

(1.3)
$$y_1 = \left[(1+q_{11})\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2} + q_{21}\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{-1/4} \times \exp\left(\int_{x_0}^{x} \lambda\, ds\right),$$

(1.4)
$$y_2 = \left[q_{12}\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2} + (1+q_{22})\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{-1/4} \times \exp\left(-\int_{x_0}^{x} \lambda\, dx\right),$$

$$y_1' = \left[(1+q_{11})\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2} - q_{21}\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{1/4} \times \exp\left(-\int_{x_0}^{x} \lambda\, dx\right), \tag{1.5}$$

$$y_2' = \left[q_{12}\left(1-\frac{l}{\sqrt{1+l^2}}\right)^{1/2} - (1+q_{22})\left(1+\frac{l}{\sqrt{1+l^2}}\right)^{1/2}\right]p^{1/4} \times \exp\left(-\int_{x_0}^{x} \lambda\, dx\right), \tag{1.6}$$

where
$$\lambda = \sqrt{p+(p'/4p)^2}, \qquad l(x) = p'/4p^{3/2}.$$

The four perturbation terms $q_{i,j}(x)$ are solutions of four separate Volterra integral equations of the second type

(1.7)
$$q_{11}(x) = -\int_a^x r(t_1)\left[\int_a^{t_1} r(t_2)\exp\left(-2\int_{t_2}^{t_1}\lambda\, ds\right)dt_2\right]dt_1 - \int_a^x r(t_1)\left[\int_a^{t_1} r(t_2)\exp\left(-2\int_{t_2}^{t_1}\lambda\, ds\right)q_{11}(t_2)\,dt_2\right]dt_1,$$

$$
\begin{aligned}
q_{22}(x) = &- \int_x^b r(t_1) \bigg[\int_{t_1}^b r(t_2) \exp\left(2 \int_{t_2}^{t_1} \lambda\, ds \right) dt_2 \bigg] dt_1 \\
&- \int_x^b r(t_1) \bigg[\int_{t_1}^b r(t_2) \exp\left(2 \int_{t_2}^{t_1} \lambda\, ds \right) q_{22}(t_2)\, dt_2 \bigg] dt_1,
\end{aligned}
\tag{1.8}
$$

$$
\begin{aligned}
q_{12}(x) = &- \int_x^b r(t_1) \exp\left(-2 \int_x^{t_1} \lambda\, ds \right) dt_1 \\
&- \int_x^b r(t_1) \exp\left(-2 \int_x^{t_1} \lambda\, ds \right) \\
&\qquad \times \bigg[\int_{t_1}^b r(t_2) q_{12}(t_2)\, dt_2 \bigg] dt_1,
\end{aligned}
\tag{1.9}
$$

$$
\begin{aligned}
q_{21}(x) = &- \int_a^x r(t_1) \exp\left(2 \int_x^{t_1} \lambda\, ds \right) dt_1 \\
&- \int_a^x r(t_1) \exp\left(2 \int_x^{t_1} \lambda\, ds \right) \\
&\qquad \times \bigg[\int_a^{t_1} r(t_2) q_{21}(t_2)\, dt_2 \bigg] dt_1
\end{aligned}
\tag{1.10}
$$

with $r = l'/2(1 + l^2)$. They are calculated by a rapidly convergent resolvent series. It follows that

$$q_{11}(a) = q_{21}(a) = q_{22}(b) = q_{12}(b) = 0.$$

If, in addition, $\int_{a^+} \lambda(t)\, dt = +\infty$, then also $q_{12}(a) = 0$ and if $\int^{b^-} \lambda(t)\, dt = +\infty$, then also $q_{21}(b) = 0$ [**5**, Theorem 3.1]. Moreover, $q_{11}(b) = q_{22}(a)$. Note that all other terms in the asymptotic approximations (1.3)–(1.6) are explicitly given by algebraic expressions and by certain integrals of algebraic expressions.

The asymptotic techniques that are utilized in [**5**] and here are influenced by [**7, 8, 9**]. The reader who is interested in more details in how our technique differs from the WKB technique and from the asymptotic techniques utilized in [**4, 10–13, 15, 16**] may want to consult the comparisons in [**5**].

The asymptotic approximations (1.3)–(1.6) may be applied to a wide variety of problems. They hold on bounded or unbounded intervals, at regular points and at regular-singular points as well. In [**5**] we derived new representations and approximations of the Jacobi polynomials as $x \to 1^+$, $x \to +\infty$, $n \to +\infty$ or $\alpha \to \infty$.

Our plan is to provide global approximations to the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and to the Racah coefficients on the entire interval $(-\infty, \infty)$. In this quest the interval $(-\infty, \infty)$ is decomposed into a union of a finite number of intervals. On each subinterval an asymptotic approximation is provided, and the different approximations are "connected" on neighboring intervals. (This type of connection was called in [**16**] "central connection" in contrast to "lateral connection" that requires analytic continuation in the complex plane.)

In the present work we extend the results of [**5**] to a subinterval of the interval of orthogonality $(-1, 1)$. Let $z_0, z_1, -1 < z_0 < z_1 < 1$ denote the turning points, i.e., the two points where $p(x)$ vanishes. We shall obtain asymptotic approximations of the solutions of (1.2) on the intervals $(-1, z_0]$ and $[z_1, 1)$ where $p(x) > 0$. The analysis of solutions of (1.2) on the intervals $(-1, z_0]$ and $[z_1, 1)$ should not be underestimated, since each such interval poses a challenge to the task of determining an asymptotic approximation to the solutions of (1.2). This is so because of several reasons:

(1) One endpoint, say z_1, is a turning point while the other end point $x = 1$ is a regular-singular point of equation (1.2). It is noteworthy that the approximations (1.3)–(1.6) are valid at turning points as well. This property turns out to be of prime importance in the current work.

(2) Our technique extracts the values of solutions of differential equations at the turning points z_0, z_1 from the asymptotic formulas without the intervention of special functions. All these benefits follow from one and the same set of asymptotic approximations, (1.3)–(1.6) that are written in several variations. At the turning points, some of the perturbation terms $q_{jk}(x)$ are shown to play an important role in the approximations. These are uncommon practices in the literature.

(3) For Jacobi polynomials with large values of n, the intervals $(-1, z_0]$ and $[z_1, 1)$ shrink to zero, having length $\mathcal{O}(1/n)$. On the other hand, for $P_n^{(an+\alpha,bn+\beta)}(x)$, the lengths of these intervals tend, as $n \to \infty$, to positive numbers. We note that the limiting values of

z_0 and z_1 as functions of n coincide with the limiting values obtained in [**6**]. In either case $(z_1, 1)$ separates the interval $[1, \infty)$, where Jacobi polynomials have no zeros, from the interval (z_0, z_1), where $p(x) < 0$ where the behavior of Jacobi polynomials is dominated by their zeros.

Among other results, we prove

Theorem 1.1. *$(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x)$ is given on $(1, \infty)$ by $y_1(x)/C_R$, while on $[z_1, 1)$ it is given by $y_2(x)/C_L$, where C_R, C_L are explicitly calculated constants.*

In a future work we plan to derive approximations on (z_0, z_1) where $p(x) < 0$ and thus complete a global description on $(-\infty, \infty)$. The upcoming sections build towards a primary goal of this article. Namely, the approximation of the Racah coefficients on the interval $[z_1, \infty)$ as $n \to \infty$, to be given in Section 6. It is shown that from one formula we get the asymptotic approximation of the Racah coefficients as $n \to \infty$ away from the turning point z_1 as well as an approximation of the Racah coefficients in the vicinity of the turning point z_1. The special nature of our formulas also provide us with the value of the Racah coefficients at $x = z_1$.

2. Fundamental systems. Let us write equation (1.2) as $y'' = p(x)y$ with

$$p(x) = \frac{A}{(x-1)^2} + \frac{B}{(x+1)^2} + \frac{C}{x^2-1}. \tag{2.1}$$

We shall primarily be concerned with

$$\begin{aligned} A &= \frac{\alpha^2-1}{4} > 0, \\ B &= \frac{\beta^2-1}{4} > 0, \\ C &= n(n+\alpha+\beta+1) + \frac{1}{2}(\alpha+1)(\beta+1) > 0, \end{aligned} \tag{2.2}$$

and we will give special consideration to the cases where α, β are replaced by $an + \alpha$, $bn + \beta$.

In [**5**] we calculated that, on $(1,\infty)$,

(2.3)
$$p(x) = \frac{A(x+1)^2 + B(x-1)^2 + C(x^2-1)}{(x^2-1)^2} > 0,$$

(2.4)
$$p'(x) = \frac{-2[A(x+1)^3 + B(x-1)^3 + Cx(x^2-1)]}{(x^2-1)^3} < 0,$$

$$l(x) = \frac{p'}{4p^{3/2}} = \frac{p'}{4p^{3/2}}$$

(2.5)
$$= -\frac{A(x+1)^3 + B(x-1)^3 + Cx(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{3/2}} < 0,$$

and

(2.6)
$$l'(x) = \frac{4AC(x+1)^2 + 4BC(x-1)^2 + (12AB + C^2)(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{5/2}} > 0.$$

Consider (1.2) on $(-1,1)$. For $A, B, C > 0$, $p(x)$ may have two zeros, say $z_0 < z_1$, in $(-1,1)$ and $p(x) > 0$ on $(z_1, 1)$ and $(-1, z_0)$. Since we are interested in A, B and C that are possibly more general than those which are specified in (2.2), the change of sign of $p(x)$ in $(-1,1)$ is not always guaranteed. The following discussion applies to $[z_1, 1)$, regardless of its length.

As we turn from $(1,\infty)$ to $(z_1, 1)$, (2.5)–(2.6) change their external appearance. $p(x) > 0$, but for $x^2 < 1$, we have $[(x^2-1)^2]^{3/2} = |x^2-1|^3 = -(x^2-1)^3$, hence

(2.7)
$$l(x) = \frac{p'}{4p^{3/2}} = \frac{A(x+1)^3 + B(x-1)^3 + Cx(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{3/2}} \text{ on } (z_1, 1),$$

which differs by a minus sign from the expression in (2.5) that holds on $(1,\infty)$. Equation (2.7) implies that $l(1^-) = (2A)^{-1/2} = (\alpha^2-1)^{-1/2} > 0$. Thus $l(x)$ has a jump discontinuity at $x = 1$. From (2.7) it follows that

(2.8)
$$l'(x) = -\frac{4AC(x+1)^2 + 4BC(x-1)^2 + (12AB + C^2)(x^2-1)}{2[A(x+1)^2 + B(x-1)^2 + C(x^2-1)]^{5/2}} \text{ on } (z_1, 1).$$

We need the following lemma for the sequel.

Lemma 2.1. *Let $p(x)$ be given in* (2.1) *with arbitrary $A, B, C > 0$. If $p(x)$ has real zeros, say $-1 < z_0 < z_1 < 1$, then on $(z_1, 1)$, $l(x)$ is positive and decreasing.*

Proof. Consider $l(x) = p'/4p^{3/2}$ on $(z_1, 1)$, where $p(x) > 0$. Since $p(1^-) = +\infty$ and $p(x)$ is positive on $(z_1, 1)$, we have $p'(z_1^+) > 0$ and $p'(1^-) > 0$. Thus, if $p'(x)$ changes its sign in $(z_1, 1)$, this must happen twice. In this situation the numerator of (2.7), $m(x) = A(x+1)^3 + B(x-1)^3 + C(x^3 - x)$, must have two zeros in $(z_1, 1)$. But this is impossible, since $m'(x) = 3[A(x+1)^2 + B(x-1)^2 + C(x^2-1)] + 2C > 0$ on $(z_1, 1)$ by the positivity of C and $p(x)$ there. Hence, $p'(x)$ must have a fixed positive sign on $(z_1, 1)$. Consequently, $l(x) > 0$.

By (2.8), the numerator of $l'(x)$ can be written as

$$4C[A(x+1)^2 + B(x-1)^2 + C(x^2-1)] + 3(4AB - C^2)(x^2-1). \tag{2.9}$$

The first group of terms is positive on $(z_1, 1)$ since $p(x) > 0$ there. If $p(x)$ has two real zeros, they are, by (2.1), $z_{0,1} = (B - A \pm \sqrt{C^2 - 4AB}/(A + B + C)$ and $C^2 - 4AB > 0$. Consequently, the last term of (2.9) is positive on $(z_1, 1)$ as well. Thus, $l'(x) < 0$ on $(z_1, 1)$. $\square$

Theorem 2.2. *Let $p(x)$ be given in* (2.1) *with $A, B, C > 0$. If $p(x)$ has real zeros, say $-1 < z_0 < z_1 < 1$, then $y_1(x), y_2(x)$ of* (1.3)–(1.4) *are a fundamental system of solutions of* (1.1) *on $(z_1, 1)$ with $q_{jk}(x)$, $j, k = 1, 2$ continuous functions of x on the closed interval $[z_1, 1]$. Moreover, each of the functions $q_{jk}(x)$, $j, k = 1, 2$ is bounded by the same bound for all $A, B, C > 0$.*

Proof. First let us outline the formal transformations of [**5**] that lead to (1.3)–(1.10). Equation (1.1) is written as

$$Z' = AZ, \qquad A = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}.$$

A sequence of linear transformations leads to $Z = TV(I+Q)\Phi$, where

$$T = \begin{pmatrix} p^{1/4} & p^{-1/4} \\ p^{1/4} & -p^{1/4} \end{pmatrix},$$
$$V = \frac{1}{\sqrt{2}(1+l^2+\sqrt{1+l^2})^{1/2}} \begin{pmatrix} 1+\sqrt{1+l^2} & -l \\ l & 1+\sqrt{1+l^2}, \end{pmatrix}$$
$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \qquad \Phi(x) = \begin{pmatrix} \exp\int_{x_0}^x \lambda & 0 \\ 0 & \exp\int_{x_0}^x -\lambda \end{pmatrix}.$$

As $\operatorname{trace}(A) = 0$, we have $\det(Z) = \text{const}$. Also $\det(T) \equiv -2$, $\det(V) \equiv 1$, $\det(\Phi) \equiv 1$. Therefore, Z is a fundamental system if and only if the constant $\det(I+Q(x))$ is nonzero. For $x = a$, we have $q_{11}(a) = q_{21}(a) = 0$; therefore,

$$\det(I+Q(x)) = \det \begin{pmatrix} 1+q_{11}(a) & q_{12}(a) \\ q_{21}(a) & 1+q_{22}(a) \end{pmatrix} = 1+q_{22}(a).$$

So we have to show that $q_{22}(a) \neq -1$ or analogously that $q_{11}(b) \neq -1$. (In particular, $\det(I+Q(a)) = \det(I+Q)(b)$ implies that $q_{22}(a) = q_{11}(b)$.) $|q_{22}(a)| < 1$ is implied, for example, by the global estimate for $q_{jj}(x)$ that was obtained in [**5**, (3.20)]. From the Voltera equations for $q_{jj}(x)$ it follows that

$$(2.10)\qquad \begin{aligned} \|q_{jj}\| &\le \frac{1}{2}\left[\int_a^b |r(t)|\,dt\right]^2 (1+\|q_{jj}\|), \\ \|q_{jj}\| &= \sup_x |q_{jj}(x)|, \quad j = 1, 2. \end{aligned}$$

This implies

$$(2.11)\qquad \|q_{jj}\| \le \frac{[\int_a^b |r(t)|\,dt]^2}{2 - \int_a^b |r(t)|\,dt]^2} < 1$$

provided that $[\int_a^b |r(t)|\,dt]^2 < 1$. But this indeed happens for $[a,b] = [z_1, 1]$ (and for $[a,b] = [1,\infty)$ as well). Recall that $r = l'/2(1+l^2)$ and by Lemma 3.1, $l(x) > 0$, $l'(x) < 0$ on $[z_1, 1]$. Consequently,

$$\begin{aligned} \int_{z_1}^1 |r(t)|\,dt &= \int_{z_1}^1 \frac{|l'|}{2(1+l^2)}\,dt \\ &= \int_{z_1}^1 \frac{-l'}{2(1+l^2)}\,dt \\ &= -\frac{1}{2}[\arctan l(1) - \arctan l(z_1)]. \end{aligned}$$

Since $l(x)$ is positive, decreasing and continuous on $(z_1, 1)$, the variation of $\arctan l(x)$ does not exceed $\pi/2$ and $\int_{z_1}^1 |r(t)|\,dt < \pi/4$. Thus, $(\int_{z_1}^1 |r(t)|\,dt)^2 < (\pi/4)^2 < 1$. Consequently, $|q_{22}(a)| \le \|q_{22}\| < 1$. In a similar manner, we have

$$\tag{2.12} \begin{aligned} &\|q_{jk}\| \le \int_a^b |r(t)|\,dt + \frac{1}{2}\left[\int_a^b |r(t)|\,dt\right]^2 \|q_{jk}\|, \\ &\|q_{jk}\| = \sup_x |q_{jk}(x)|, \quad j,k = 1,2,\ k \ne j. \end{aligned}$$

Hence,

$$\tag{2.13} \|q_{jk}\| \le \frac{2\int_a^b |r(t)|\,dt}{2 - [\int_a^b |r(t)|\,dt]^2}.$$

Similar considerations had been applied to the interval $(1, \infty)$ in [**5**, (3.20), (3.21), (5.7)]. □

3. The fundamental system near $x = 1^+$. In order to connect the solutions of (1.2) on $(1, \infty)$ with those on $[z_1, 1)$, we compute the asymptotic values of a fundamental solution y_1, y_2 on both sides of $x = 1$. Since $p(x) = ((\alpha^2 - 1)/4)(x-1)^{-2}(1 + \mathcal{O}(x-1)) > 0$, we have

$$\lambda(x) = (p + (p'/4p)^2)^{1/2} = \frac{\alpha}{2(x-1)}(1 + \mathcal{O}(x-1)) > 0.$$

In order to calculate a fundamental system near $x = 1^+$, we choose in equations (1.3)–(1.6) a fixed point $x_0 = x_R$ in the interval $(1, \infty)$. We split the integral $\int_{x_R}^x \lambda\,ds$ as follows.

$$\begin{aligned} \int_{x_R}^x \lambda\,ds = &\int_1^x \left(\lambda - \frac{\alpha}{2(s-1)}\right) ds \\ &- \int_1^{x_R} \left(\lambda - \frac{\alpha}{2(s-1)}\right) ds + \int_{x_R}^x \frac{\alpha}{2(s-1)}\,ds. \end{aligned}$$

Here $\int_1^x (\lambda - \alpha/2(s-1))\,ds \to 0$ as $x \to 1^+$ and

$$\gamma_R := -\int_1^{x_R} \left(\lambda - \frac{\alpha}{2(s-1)}\right) ds$$

is a regular integral, so

$$\exp\left(\int_{x_R}^{x} \lambda\, ds\right) = \left(\frac{x-1}{x_R-1}\right)^{\alpha/2} e^{\gamma_R}(1+\mathcal{O}(x-1))$$

near $x = 1^+$. For practical purposes it is convenient to take $x_R = 2$.

Recall that the asymptotic approximations (1.3)–(1.6) for the interval $(1,\infty)$, are such that

$$q_{11}(1) = q_{21}(1) = q_{22}(\infty) = q_{12}(\infty) = 0.$$

Moreover, since $\int_{1^+} \lambda(t)\, dt = +\infty$ and $\int^{\infty} \lambda(t)\, dt = +\infty$, we also have

$$q_{12}(1) = q_{21}(\infty) = 0.$$

The value $q_{22}(1) = q_{11}(\infty)$ is computed by the absolutely convergent resolvent series of (1.7). Next, $l(1^+) = -(2A)^{-1/2} = -(\alpha^2-1)^{1/2} < 0$,

$$\left(1 + \frac{l}{\sqrt{1+l^2}}\right)^{1/2} = \left(\frac{\alpha-1}{\alpha}\right)^{1/2},$$

and, together with $p^{-1/4}(x) = ((\alpha^2-1)/4)^{-1/4}(x-1)^{1/2}(1+\mathcal{O}(x-1))$, we get from (1.3) that, near $x = 1^+$,

(3.1)

$$y_1(x) = \left(\frac{2}{\alpha}\right)^{1/2}\left(\frac{\alpha-1}{\alpha+1}\right)^{1/4}\frac{e^{\gamma_R}}{(x_R-1)^{\alpha/2}}(x-1)^{(\alpha+1)/2}(1+\mathcal{O}(x-1)).$$

For the calculation of $y_2(x)$ near $x = 1^+$, $\exp\int_{x_R}^{x} \lambda\, ds$ is replaced by $\exp(-\int_{x_R}^{x} \lambda\, ds)$, we utilize $q_{12}(1) = 0$ and

$$\left(1 - \frac{l}{\sqrt{1+l^2}}\right)^{1/2} = \left(\frac{\alpha+1}{\alpha}\right)^{1/2},$$

so by (1.4) we get

$$\begin{aligned} y_2(x) = (1+q_{22}(1))\left(\frac{2}{\alpha}\right)^{1/2}\left(\frac{\alpha+1}{\alpha-1}\right)^{1/4}\frac{(x_R-1)^{\alpha/2}}{e^{\gamma_R}} \\ \times (x-1)^{(-\alpha+1)/2}(1+\mathcal{O}(x-1)). \end{aligned} \tag{3.2}$$

The calculation of $y_1'(x)$ and $y_2'(x)$ requires the already known values of $\exp\int_{x_R}^{x}\lambda\,ds$, $p^{-1/4}(x)$, $q_{11}(1)=q_{21}(1)=0$. Then we obtain

(3.3)
$$y_1'(x)=\frac{(\alpha-1)^{1/4}(\alpha+1)^{3/4}}{(2\alpha)^{1/2}}\frac{e^{\gamma_R}}{(x_R-1)^{\alpha/2}}(x-1)^{(\alpha-1)/2}(1+\mathcal{O}(x-1)),$$

$$y_2'(x)=-(1+q_{22}(1))\frac{(\alpha-1)^{3/4}(\alpha+1)^{1/4}}{(2\alpha)^{1/2}}\frac{(x_R-1)^{\alpha/2}}{e^{\gamma_R}}$$
(3.4)
$$\times(x-1)^{(-\alpha-1)/2}(1+\mathcal{O}(x-1))$$

near $x=1^+$.

It is known [**15**] that the solution $(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x)$ of equation (1.2) is bounded at $x=1$ while any other linearly independent solution is unbounded at $x=1$. On the other hand, it follows from (3.1) and (3.2) that $y_1(x)$ is bounded at $x=1$ and $y_2(x)$ is not. Consequently,

$$y_1(x)=C_R(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x), \tag{3.5}$$

where C_R is a certain number independent of x. A comparison of (3.1) with (3.5) and the well-known normalization $P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$ determines

$$C_R=\left(\frac{2}{\alpha}\right)^{1/2}\left(\frac{\alpha-1}{\alpha+1}\right)^{1/4}\frac{e^{\gamma_R}}{(x_R-1)^{\alpha/2}}/2^{(\beta+1)/2}\binom{n+\alpha}{n}.$$

While $P_n^{(\alpha,\beta)}(x)$ is explicitly expressed in terms of $y_1(x)$, note that for $Q_n^{(\alpha,\beta)}(x)$, Jacobi's function of the second kind [**15**, page 73], $(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2}Q_n^{(\alpha,\beta)}(x)$ is a certain linear combination of $y_1(x)$ and $y_2(x)$.

4. The fundamental system near $x=1^-$. The next step is to produce a fundamental system of solutions y_1,y_2 on $(z_1,1)$. We have

$$q_{11}(z_1)=q_{12}(z_1)=q_{21}(z_1)=q_{12}(1)=q_{21}(1)=q_{22}(1)=0.$$

The values of $q_{11}(1)$, $q_{22}(z_1)$ are obtained from the absolutely converging resolvent series of (1.7) and (1.8).

In this section we calculate the fundamental system near $x = 1^-$. Near $x = 1^-$, we have

$$\lambda(x) = \left| \frac{\alpha}{2(x-1)}(1 + \mathcal{O}(x-1)) \right| = \frac{\alpha}{2(1-x)}(1 + \mathcal{O}(x-1)) > 0.$$

On the interval $(z_1, 1)$, we choose in equations (1.3)–(1.6) a fixed point $x_0 = x_L$ and rewrite the integral as

$$\int_{x_L}^{x} \lambda\, ds = \int_1^x \left(\lambda - \frac{\alpha}{2(1-s)}\right) ds - \int_1^{x_L} \left(\lambda - \frac{\alpha}{2(1-s)}\right) ds \\ + \int_{x_L}^{x} \frac{\alpha}{2(1-s)}\, ds.$$

Here $\int_1^x (\lambda - \alpha/2(1-s))\, ds \to 0$ as $x \to 1^-$. Let us denote

$$\gamma_L := -\int_1^{x_L} \left(\lambda - \frac{\alpha}{2(1-s)}\right) ds.$$

Then

$$\exp\left(\int_{x_L}^{x} \lambda\, ds\right) = \left(\frac{1-x}{1-x_L}\right)^{-\alpha/2} e^{\gamma_L}(1 + \mathcal{O}(x-1))$$

near $x = 1^-$. Finally, $l(1^-) = (\alpha^2 - 1)^{-1/2} > 0$,

$$\left(1 + \frac{l}{\sqrt{1+l^2}}\right)^{1/2} = \left(\frac{\alpha+1}{\alpha}\right)^{1/2}.$$

Together with $p^{-1/4}(x) = ((\alpha^2-1)/4)^{-1/4}(1-x)^{1/2}(1+\mathcal{O}(x-1)) > 0$, we get from (5.6)

(4.1)

$$y_1(x) = (1 + q_{11}(1))\left(\frac{2}{\alpha}\right)^{1/2} \left(\frac{\alpha+1}{\alpha-1}\right)^{1/4} \frac{e^{\gamma_L}}{(1-x_L)^{-\alpha/2}}(1-x)^{(-\alpha+1)/2} \\ \times (1 + \mathcal{O}(x-1))$$

near $x = 1^-$. For $y_1'(x)$, we get

$$y_1'(x) = (1 + q_{11}(1)) \frac{(\alpha-1)^{3/4}(\alpha+1)^{1/4}}{(2\alpha)^{1/2}} \frac{e^{\gamma_L}}{(1-x_L)^{\alpha/2}} \times (1-x)^{(-\alpha-1)/2}(1 + \mathcal{O}(x-1)). \tag{4.2}$$

For the calculation of $y_2(x)$, we replace $\exp(\int_{x_0}^x \lambda\, ds)$ by $\exp(-\int_{x_0}^x \lambda\, ds)$. After some manipulations, we obtain that

$$y_2(x) = \left(\frac{2}{\alpha}\right)^{1/2} \left(\frac{\alpha-1}{\alpha+1}\right)^{1/4} \frac{e^{-\gamma_L}}{(1-x_L)^{\alpha/2}} \times (1-x_L)^{(\alpha+1)/2}(1 + \mathcal{O}(x-1)), \tag{4.3}$$

$$y_2'(x) = \frac{(\alpha-1)^{1/4}(\alpha+1)^{3/4}}{(2\alpha)^{1/2}} \frac{e^{-\gamma_L}}{(1-x_L)^{\alpha/2}} \times (1-x)^{(\alpha-1)/2}(1 + \mathcal{O}(x-1)). \tag{4.4}$$

Now, near $x = 1^-$, $y_2(x)$ is the bounded solution and consequently

$$y_2(x) = C_L(1-x)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2} P_n^{(\alpha,\beta)}(x) \tag{4.5}$$

with a numerical factor C_L. This is in contrast to the situation near $x = 1^+$, where the solution $y_1(x)$ is related to $(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2} \times P_n^{(\alpha,\beta)}(x)$. The normalization $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$ determines

$$C_L = \left(\frac{2}{\alpha}\right)^{1/2} \left(\frac{\alpha-1}{\alpha+1}\right)^{1/4} \frac{e^{-\gamma_L}}{(1-x_L)^{\alpha/2} 2^{(\beta+1)/2}} \binom{n+\alpha}{n}. \tag{4.6}$$

The usefulness of relations (4.5) and (4.6) will become apparent in the sequel.

5. The fundamental system near $x = z_1^+$. In this section we calculate the fundamental system y_1, y_2 in the interval $(z_1, 1)$ near $x = z_1^+$. Since $p(x)$ has a simple zero at z_1, the dominant part of

$\lambda(x) = (p + (p'/4p)^2)^{1/2}$ near z_1^+ is $p'/4p > 0$. Therefore,

$$\begin{aligned}\exp\left(\int_{x_L}^{x} \lambda\, ds\right) &= \exp \int_{x_L}^{x} (\sqrt{p + (p'/4p)^2} - p'/4p)\, ds \\ &\quad \times \exp \int_{x_L}^{x} p'/4p\, ds \\ &= \exp\left(\int_{x_L}^{x} \frac{p\, ds}{\sqrt{p + (p'/4p)^2} + p'/4p}\right)\left(\frac{p(x)}{p(x_L)}\right)^{1/4}.\end{aligned}$$

Denote

$$E(x, x_L) := \exp \int_{x_L}^{x} \frac{p\, ds}{\sqrt{p + (p'/4p)^2} + p'/4p}.$$

Here $E(x, x_L)$ is regular at $x = z_1^+$. In equations (1.3) and (1.4) we need the expressions

$$p^{-1/4}(x) \exp\left(\int_{x_L}^{x} \lambda\, ds\right) = E(x, x_L)/p^{1/4}(x_L), \tag{5.1}$$

$$p^{-1/4}(x) \exp\left(\int_{x_L}^{x} -\lambda\, ds\right) = \frac{p^{1/4}(x_L)}{E(x, x_L)} p^{-1/2}(x). \tag{5.2}$$

Since z_1 is a simple zero of $p(x)$ and $p(x) > 0$ on $(z_1, 1)$, we have $l(z_1^+) = \infty$ and

$$\begin{aligned}\left(1 + \frac{l}{\sqrt{1 + l^2}}\right)^{1/2} &= \sqrt{2}, \\ \left(1 - \frac{l}{\sqrt{1 + l^2}}\right)^{1/2} &= 0 \text{ at } z_1.\end{aligned}$$

With $q_{11}(z_1) = q_{21}(z_1) = 0$, we get from (1.3) and (1.5) that

(5.3)

$$y_1(x) = \sqrt{2} \frac{E(x, x_L)}{p^{1/4}(x_L)} (1 + \mathcal{O}(x - z_1)),$$

(5.4)

$$\begin{aligned}y_1'(x) = &\left[(1 + q_{11})\left(1 - \frac{l}{\sqrt{1 + l^2}}\right)^{1/2} - q_{21}\left(1 + \frac{l}{\sqrt{1 + l^2}}\right)^{1/2}\right] \\ &\times \frac{E(x, x_L)}{p^{1/4}(x_L)} p^{1/2}(x) \longrightarrow 0,\end{aligned}$$

as z_1^+. Thus, $y_1(x)$ is a solution which is characterized by the boundary value $y'(z_1) = 0$. Notice that the estimates in relations (5.3) and (5.4) hold for n fixed.

The calculation of $y_2(x)$ near z_1 is slightly different. Since $l(z_1^+) = \infty$, we get for a fixed value of n

$$(5.5) \qquad \left(1 - \frac{l}{\sqrt{1+l^2}}\right)^{1/2} \approx \frac{1}{\sqrt{2}l} = \frac{4p^{3/2}}{\sqrt{2}p'}, \text{ as } x \to z_1^+.$$

Together with (5.2) this yields near $x = z_1^+$,

$$(1+q_{22})\left(1 - \frac{l}{\sqrt{1+l^2}}\right)^{1/2} p(x)^{1/4} \exp\left(\int_{x_L}^{x} -\lambda\, ds\right) \approx \frac{4p(x_L)^{1/4}}{\sqrt{2}E(x,x_L)p'(x)} p(x),$$

which tends to 0 as $x \to z_1^+$. The other term of $y_2(x)$ in (1.4) is

$$q_{12}\left(1 + \frac{l}{\sqrt{1+l^2}}\right)^{1/2} p^{1/4}(x) \exp\left(\int_{x_L}^{x} -\lambda\, ds\right) \approx \frac{\sqrt{2}p^{1/4}(x_L)}{E(x,x_L)} p^{-1/2}(x) q_{12}(x),$$

so

$$(5.6) \qquad y_2(z_1^+) = \frac{\sqrt{2}p^{1/4}(x_L)}{E(z_1,x_L)} \lim_{x \to z_1^+} \left(p^{-1/2}(x) q_{12}(x)\right).$$

The limit at z_1^+ is of the form $0 \cdot \infty$. Due to the importance of $y_2(x)$ in the representation of $P_n^{(\alpha,\beta)}(x)$, we proceed to show that the function $p^{-1/2}(x)q_{12}(x)$ can be obtained nevertheless by a regular integral representation.

Proposition 5.1. *The function $p^{-1/2}(x)q_{12}(x)$ is continuous on the closed interval $[z_1, 1]$. Moreover, it can be written as a sum of an absolutely convergent series which is uniformly convergent and bounded for all $x \in [z_1, 1]$, $A, B, C > 0$, and $A/C + B/C < \rho$, where ρ is a constant.*

Proof. To this end we write the Volterra equation (1.9) as

$$q_{12}(x) = w(x) + L[q_{12}],$$

where
(5.7)

$$\begin{aligned} w(x) &:= -\int_x^1 r(t_1)\exp\Big(-2\int_x^{t_1}\lambda\,ds\Big)\,dt_1, \\ L[q_{12}] &:= -\int_x^1 r(t_1)\exp\Big(-2\int_x^{t_1}\lambda\,ds\Big)\Big[\int_{t_1}^1 r(t_2)q_{12}(t_2)dt_2\Big]\,dt_1, \end{aligned}$$

and express its solution as the absolutely converging resolvent series

$$q_{12}(x) = w(x) + \sum_{\nu=1}^{\infty} L^{\nu}[w]. \tag{5.8}$$

Let us multiply equation (1.9) by $p^{1/2}(x)$. It follows that the function $p^{-1/2}(x)q_{12}(x)$ satisfies

$$\begin{aligned} p^{-1/2}(x)q_{12}(x) = &-\int_x^1 r(t_1)\exp\Big(-2\int_x^{t_1}\lambda\,ds\Big)p^{-1/2}(x)\,dt_1 \\ &-\int_x^1 r(t_1)\exp\Big(-2\int_x^{t_1}\lambda\,ds\Big)p^{-1/2}(x) \\ &\times\Big[\int_{t_1}^1 r(t_2)q_{12}(t_2)\,dt_2\Big]\,dt_1. \end{aligned} \tag{5.9}$$

Notice that

$$\begin{aligned} &\exp\Big(-2\int_x^{t_1}\lambda\,ds\Big)p^{-1/2}(x) \\ &\quad= \exp\Big(2\int_{t_1}^{x}\lambda\,ds\Big)\exp\Big(\int_{t_1}^{x}-\frac{p'(s)}{2p(s)}\,ds\Big)p^{-1/2}(t_1) \\ &\quad= \exp\Big(2\int_{t_1}^{x}\Big(\lambda-\frac{p'(s)}{4p(s)}\Big)\,ds\Big)p^{-1/2}(t_1) \\ &\quad= E^2(x,t_1)p^{-1/2}(t_1), \end{aligned} \tag{5.10}$$

so equation (5.9) becomes
(5.11)

$$p^{-1/2}(x)q_{12}(x) = -\int_x^1 E^2(x,t_1)r(t_1)p^{-1/2}(t_1)\,dt_1 - \int_x^1 E^2(x,t_1)r(t_1)p^{-1/2}(t_1)\left[\int_{t_1}^1 r(t_2)q_{12}(t_2)\,dt_2\right]dt_1.$$

Note that (5.9) and (5.11) are different ways to write

$$p^{-1/2}(x)q_{12}(x) = p^{-1/2}(x)w(x) + p^{-1/2}(x)L[q_{12}](x). \tag{5.12}$$

Since $p, p' > 0$ on $[z_1, 1]$ where z_1 is the zero of $p(x)$, we have $\lambda - p'/4p > 0$, and so $E(x,t_1) \le 1$ for $z_1 \le x \le t_1 \le 1$. Next, the expression $r(t_1)p^{-1/2}(t_1)$ is a continuous function on $[z_1, 1]$. To see this, recall that $r = l'/2(1+l^2)$. Therefore,

$$rp^{-1/2} = \frac{l'}{2(1+l^2)}p^{-1/2} = \frac{-3p'^2 + 2pp''}{p'^2 + 16p^3}. \tag{5.13}$$

Since p has a simple zero at z_1 and a double pole at 1, it is easy to see that $r(t_1)p^{-1/2}(t_1)$ has limits both as $t_1 \to z_1^+$ and as $t_1 \to 1^-$. Hence, $rp^{-1/2}$ is a continuous function on $[z_1, 1]$ and $E^2(x,t_1)r(t_1)p^{-1/2}(t_1)$ in (5.11) is absolutely integrable on $[z_1, 1]$ as a function of t_1. It follows that $p^{1/2}(x)q_{12}(x)$ as well as $p^{-1/2}(x)w(x)$ are continuous functions of x on the closed interval $[z_1, 1]$ and $p^{-1/2}(x)L[.](x)$ on the righthand side of (5.12) is a Volterra operator with a continuous kernel.

The substitution of the absolutely convergent series (5.8) into the righthand side of (5.12) takes this one step further and provides a direct representation of $p^{-1/2}(x)q_{12}(x)$. It leads to the explicit representation

$$p^{-1/2}(x)q_{12}(x) = p^{-1/2}(x)w(x) + p^{-1/2}L[w](x) + \sum_{\nu=1}^{\infty} p^{-1/2}L[L^\nu[w]](x) \tag{5.14}$$

in terms of an absolutely converging series for $x \in [z_1, 1]$.

Our next goal is to show that the series (5.14) is uniformly convergent and bounded for the range of all parameters $x \in [z_1, 1]$, $A, B, C > 0$

and A/C, B/C are restricted to a fixed, bounded interval. To this end, rewrite $p(x)$ as

$$p(x) = C\tilde{p}(x). \tag{5.15}$$

$\tilde{p}(x)$ depends only on the ratios A/C, B/C. A straightforward calculation reveals that

$$r(x)p^{-1/2}(x) = \frac{-3\tilde{p}'^2 + 2\tilde{p}\tilde{p}''}{\tilde{p}'^2 + 16C\tilde{p}^3}, \tag{5.16}$$

is bounded for the entire range of x and the parameters. This, combined with the fact that $E^2(x, t_1)r(t_1)p^{-1/2}(t_1)$ is bounded, leads to the desired conclusion. $\square$

Finally, in (1.6),

$$y_2'(z_1^+) = -(1 + q_{22}(z_1^+))\frac{p(x_L)^{1/4}}{E(z_1, x_L)}, \tag{5.17}$$

and the fundamental solution on the righthand side of the turning point z_1 is

$$\begin{pmatrix} y_1(z_1^+) & y_2(z_1^+) \\ y_1'(z_1^+) & y_2'(z_1^+) \end{pmatrix} \tag{5.18}$$
$$= \begin{pmatrix} \sqrt{2}E(z_1^+, x_L)/p^{1/4}(x_L) & \sqrt{2}p^{(1/4)}(x_L)\lim_{x\to z_1^+}(p^{-1/2}(x)q_{12}(x))/E(z_1^+, x_L) \\ 0 & -(1 + q_{22}(z_1^+))p^{1/4}(x_L)/E(z_1^+, x_L). \end{pmatrix}$$

Note that the terms $q_{22}(z_1^+)$ and $\lim_{x\to z_1^+}(p^{-1/2}(x)q_{12}(x))$ that are evaluated by absolutely convergent resolvent series liberate us from the need to utilize special functions.

This argument completes a global representation of the Jacobi polynomials in $[z_1, \infty)$.

$$P_n^{(\alpha,\beta)}(x) = \begin{cases} y_2(x)/C_L(1-x)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2} & \text{on } [z_1, 1], \\ y_1(x)/C_R(x-1)^{(\alpha+1)/2}(x+1)^{(\beta+1)/2} & \text{on } [1, \infty). \end{cases} \tag{5.19}$$

By the above arguments, the seemingly singular expressions at $x = 1$ converge, in fact, to the same limit from both sides of $x = 1$.

6. The Racah coefficients. Now let us focus on the Racah coefficients $P_n^{(an+\alpha,bn+\beta)}(x)$ which is obtained from equation (1.2) as α, β are replaced, respectively, by $an+\alpha$, $bn+\beta$. An elaboration on the asymptotic behavior of $y_1(x)$ and on the Racah coefficients, on the interval $[1,\infty)$, is given in [**5**]. We now augment it by the asymptotic behavior of the Racah coefficients as $n\to\infty$ on the interval $[z_1,1]$. For fixed numbers a, b, α, β, and $a>0$, $b>0$, let

$$\begin{aligned}\widehat{A} &= \lim_{n\to\infty}\frac{A}{C}\\ &= \lim_{n\to\infty}\frac{((an+\alpha)^2-1)/4}{n(n+an+\alpha+bn+\beta+1)+(an+\alpha+1)(bn+\beta+1)/2}\\ &= \frac{a^2}{1+a+b+ab/2},\\ \widehat{B} &= \lim_{n\to\infty}\frac{B}{C}\\ &= \lim_{n\to\infty}\frac{((bn+\beta)^2-1)/4}{n(n+an+\alpha+bn+\beta+1)+(an+\alpha+1)(bn+\beta+1)/2}\\ &= \frac{b^2}{1+a+b+ab/2}.\end{aligned}$$

Then, there exists

$$w := \lim_{n\to\infty} z_1(n) = \frac{\widehat{B}-\widehat{A}+\sqrt{1-4\widehat{A}\widehat{B}}}{1+\widehat{A}+\widehat{B}}.$$

This is compatible with [**6**, equation (5)]. We are ready now to formulate the next theorem.

Theorem 6.1. *Let $p(x)$ be given in* (2.1) *with $A, B, C > 0$ with α, β replaced, respectively, with $an+\alpha$, $bn+\beta$. Let $p(x)$ possess two real zeros $-1 < z_0 < z_1 < 1$. Then*

(i) *on any closed subinterval I of $(w, 1]$ the asymptotic approximation of the Racah coefficients is given by*

$$P_n^{(an+\alpha,bn+\beta)}(x) \approx \frac{[1+\mathcal{O}(n^{-1})]p^{-1/4}\exp(-\int_{x_L}^{x}\lambda\,ds)}{C_L(1-x)^{(an+\alpha+1)/2}(x+1)^{(bn+\beta+1)/2}} \tag{6.1}$$

as $n\to\infty$.

(ii) *In the vicinity of* z_1 *and on any subinterval of* $[z_1, 1)$ *the Racah coefficients are approximated by* (6.7).

(iii) *At the turning point* z_1 *the Racah coefficients are given precisely by* (6.9).

(iv) *On the interval* $[1, \infty)$ *the Racah coefficients possess an asymptotic approximation*

(6.2)
$$P_n^{(an+\alpha,bn+\beta)}(x) = \frac{[1+\mathcal{O}(n^{-1})]p^{-1/4}(x)\exp(\int_{x_R}^x \lambda\,ds)}{C_R(x-1)^{(an+\alpha+1)/2}(x+1)^{(bn+\beta+1)/2}}, \quad n \to \infty.$$

Proof. (i) Rewrite $l(x)$ in (2.7) as

$$l(x) = \frac{p'}{4p^{3/2}} = C^{-1/2}\frac{[(A/C)(x+1)^3 + (B/C)(x-1)^3 + x(x^2-1)]}{2[(A/C)(x+1)^2 + (B/C)(x-1)^2 + (x^2-1)]^{3/2}}.$$

Notice that on every closed subinterval I of $(w, 1]$ the denominator of $l(x)$ is bounded away from 0 and that A/C, B/C are bounded. Therefore, $l(x) = \mathcal{O}(n^{-1})$. Consequently,

$$\int_{x_1}^{x_2} |r(t)|\,dt = \frac{1}{2}[\arctan l(x_2) - \arctan l(x_1)] = \mathcal{O}(n^{-1}), \tag{6.3}$$

uniformly for $x_1, x_2 \in I$. We are going to use the estimate (6.3), (2.11) and (2.13) in combination with (1.4) and (4.5) in order to obtain the desired formula (6.1).

By virtue of (2.11) the values $\|q_{jj}\|$ are finite. Moreover, from (2.10) together with (6.3) it follows that on every closed subinterval I of $(w, 1]$ we have

$$|q_{jj}(x)| \le \|q_{jj}\| \le \frac{[\int_a^b |r(t)|\,dt]^2}{2 - [\int_a^b |r(t)|\,dt]^2} = \mathcal{O}(n^{-2}). \tag{6.4}$$

In a similar manner we have

$$|q_{jk}(x)| \le \|q_{jk}\| \le \frac{2\int_a^b |r(t)|\,dt}{2 - [\int_a^b |r(t)|\,dt]^2} = \mathcal{O}(n^{-1}) \tag{6.5}$$

on the same subinterval I. By inserting these conclusions into (1.4), we get

$$y_2(x) \approx [1 + \mathcal{O}(n^{-1})]p^{-1/4} \exp\left(- \int_{x_L}^{x} \lambda \, ds \right),$$

and the result (6.1) follows. Note that equation (6.1) is not singular at $x = 1^-$ in spite of its singular apparent form. This follows since, as in (4.3), the numerator has at $x = 1^-$ a zero of the same order as the denominator.

(ii) In the vicinity of $w = \lim_{n\to\infty} z_1(n)$, we write (1.4) as

$$\begin{aligned}(6.6)\quad & P_n^{(an+\alpha,bn+\beta)}(x) \\ &= \frac{[q_{12}(x)p^{-1/2}(x)(1+(l/\sqrt{1+l^2}))^{1/2}+(1+q_{22}(x))(1-(l/\sqrt{1+l^2}))^{1/2}p^{-1/2}(x)]}{C_L(1-x)^{(an+\alpha+1)/2}(x+1)^{(bn+\beta+1)/2}} \\ &\quad \times p^{1/4}(x) \exp\left(- \int_{x_L}^{x} \lambda \, ds \right).\end{aligned}$$

By (5.1)

$$p^{1/4}(x) \exp\left(- \int_{x_L}^{x} \lambda \, ds \right) = E^{-1}(x, x_L)p^{1/4}(x_L);$$

hence, in the vicinity of w we keep the following approximation of the Racah coefficients

$$\begin{aligned}(6.7)\quad & P_n^{(an+\alpha,bn+\beta)}(x) \\ &= \frac{[q_{12}(x)p^{-1/2}(x)(1+(l/\sqrt{1+l^2}))^{1/2}+(1+q_{22}(x))(1-(l/\sqrt{1+l^2}))^{1/2}p^{-1/2}(x)]}{C_L(1-x)^{(an+\alpha+1)/2}(x+1)^{(bn+\beta+1)/2}} \\ &\quad \times E^{-1}(x, x_L)p^{1/4}(x_L).\end{aligned}$$

At this juncture the formula (6.7) is now further replaced by a simpler asymptotic formula. This is so because of the following considerations. Denote

$$\begin{aligned}(6.8)\qquad & T_1(x) := q_{12}(x)p^{-1/2}(x)\left(1 + \frac{l}{\sqrt{1+l^2}}\right)^{1/2}, \\ & T_2(x) := (1 + q_{22}(x))\left(1 - \frac{l}{\sqrt{1+l^2}}\right)^{1/2} p^{-1/2}(x),\end{aligned}$$

that make up a factor in the numerator of (6.7) such that

$$T_1(x)+T_2(x) := \bigg[q_{12}(x)p^{-1/2}(x)\bigg(1+\frac{l}{\sqrt{1+l^2}}\bigg)^{1/2} \\ +(1+q_{22}(x))\bigg(1-\frac{l}{\sqrt{1+l^2}}\bigg)^{1/2} p^{-1/2}(x)\bigg].$$

The identity

$$\bigg(1-\frac{l}{\sqrt{1+l^2}}\bigg)^{1/2} p^{-1/2}(x) = \frac{p^{-1/2}(x)}{(1+l^2)^{1/4}[\sqrt{1+l^2}+l]^{1/2}} \\ = \frac{4\tilde{p}(x)}{(1+l^2)^{1/4}[\sqrt{16C\tilde{p}^3(x)+(\tilde{p}'(x))^2}+\tilde{p}'(x)]^{1/2}}, \\ \tilde{p}=p/C,$$

shows that $T_2(z_1^+)=0$. The term $T_2(x)$ does not converge uniformly as $n\to\infty$. $T_2(x)$ cannot be discarded because the term $T_1(x)$ does not converge uniformly for $x\in[z_1,1]$ as $n\to\infty$ and it is not known if $T_1(x)$ is bounded away from zero on $[z_1,1]$.

(iii) The behavior of $y_2(x)$ at $x=z_1^+$ is given in (5.6), so at the turning point z_1 we have

(6.9)

$$P_n^{(an+\alpha;bn+\beta)}(z_1) = \frac{\sqrt{2}\lim_{x\to z_1^+}(p^{-1/2}(x)q_{12}(x))E^{-1}(z_1,x_L)p^{1/4}(x_L)}{C_L(1-z_1)^{(an+\alpha+1)/2}(z_1+1)^{(bn+\beta+1)/2}}.$$

(iv) Formula (6.2) follows from the uniform estimates in [**5**, page 182] as $\alpha\to\infty$, as follows. The precise formula for the Racah coefficients on $[1,\infty)$ is

$$(6.10)\quad P_n^{(an+\alpha,bn+\beta)}(x) \\ = \frac{[(1+q_{11})(1+(l/\sqrt{1+l^2}))^{1/2}+q_{21}(1-(l/\sqrt{1+l^2}))^{1/2}]p^{-1/4}\exp(\int_{x_R}^{x}\lambda\,ds)}{C_R(x-1)^{(an+\alpha+1)/2}(x+1)^{(bn+\beta+1)/2}}.$$

It is shown in [**5**, page 182] that

$$\int_1^\infty |r(t)|\,dt = \mathcal{O}(\alpha^{-1}) \quad \text{as } \alpha\to\infty.$$

Here we replace α by $an+\alpha$ and let $n \to \infty$. It follows from (6.4) and (6.5) that

$$q_{11}(x) = \mathcal{O}(n^{-2}), \qquad q_{21}(x) = \mathcal{O}(n^{-1}),$$

uniformly on $[1,\infty)$ as $n \to \infty$. The uniform convergence of $l(x)$ to 0 on $[1,\infty)$ as $n \to \infty$ is verified in [**5**]. This implies the desired result (6.2). $\square$

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