Now let $y = \left(\frac{f'}{f}\right)_{\theta=\theta_0}$, and we find from (1) and (2) that

$$N = \left[\frac{U_\alpha \sqrt{E_0(y^2)} + U_\beta \sqrt{E_1(y^2) - [E_1(y)]^2}}{E_1(y)}\right]^2.$$

Now

$$E_1(y) = \int_{-\infty}^{\infty} \frac{f'(x, \theta_0)}{f(x, \theta_0)} f(x, \theta_1)\, dx$$

$$= \Delta \int_{-\infty}^{\infty} \frac{f'(x, \theta_0)}{f(x, \theta_0)} f'(x, \theta_0)\, dx + \Delta \int_{-\infty}^{\infty} \frac{f'(x, \theta_0)}{f(x, \theta_0)} [f'(x, \theta)]_{\theta=\theta_0}^{\theta=\theta^*} dx$$

$= \Delta E_0 y^2[1 + o(1)]$ from assumption $B$.

Proceeding in a similar manner, we find

$$[U_\alpha \sqrt{E_0(y)^2} + U_\beta \sqrt{E_1(y)^2 - [E_1(y)]^2}]^2 = E_0(y^2)[U_\alpha + U_\beta(1 + o(1))]^2.$$

We now have

$$\frac{E_0(n)}{N} = \frac{\Delta^2[E_0(y^2)]^2(1 + o(1))^2}{E_0(y^2)[U_\alpha + U_\beta(1 + o(1))]^2} \times \frac{\alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)}{-\frac{\Delta^2}{2}[E_0(y^2) + o(1)]}$$

therefore

$$\lim_{\Delta\to 0}\left\{\frac{E_0(n)}{N}\right\} = -2\frac{\left[\alpha \log\left(\frac{1-\beta}{\alpha}\right) + (1-\alpha)\log\left(\frac{\beta}{1-\alpha}\right)\right]}{(U_\alpha + U_\beta)^2}.$$

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# A NOTE ON THE POISSON-CHARLIER[1] FUNCTIONS

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The polynomials $p_n(m, z)$ given by the definition

$$(1) \qquad p_n(m, z) \equiv (-)^m e^z z^{-m} \frac{d^n}{dz^n}[e^{-z} z^m],$$

[1] This note was written while the author was employed by the Radiation Laboratory, M.I.T.

called the Poisson-Charlier polynomials, and the associated function $\psi_n(m, z)$ given by the definition

$$\psi_n(m, z) \equiv p_n(m, z)\psi_0(m, z), \tag{2}$$

$$\psi_0(m, z) \equiv \frac{e^{-z} z^m}{m!}, \tag{3}$$

occur in statistics. Doetsch [1] has devoted a memoir to them, and they are noticed in Szegö's *Orthogonal Polynomials* (pp. 33–34).

I suggest that they are most directly and easily studied in connection with the "F-equation"

$$\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1), \tag{4}$$

whose properties and application to various special functions I have summarized in a recent note [2]. Using the theorems of that note, which I shall cite by number, I shall now generalize the Poisson-Charlier polynomials and sketch the speediest derivation of their most interesting formal properties.

Greek letters shall represent unrestricted real numbers, while Latin letters shall represent integers.

From the existence theorem for the F-equation (Theorem 4) we know that there exists an integral function of $z$, $F_\beta(z, \alpha)$, which satisfies the F-equation and the condition

$$F_\beta(0, \alpha) = \cos(\alpha + \beta)\pi \binom{\beta}{-\alpha}. \tag{5}$$

From the uniqueness theorem for the F-equation (Theorem 4) it follows that

$$F_\beta(z, n - \beta + \tfrac{1}{2}) = 0, \tag{6}$$

$$F_\beta(z, n) = 0, \qquad n > 0. \tag{7}$$

From the general power series solution for the F-equation (Theorem 4) we have the formula

$$F_\beta(z, \alpha) = \cos(\alpha + \beta)\pi \binom{\beta}{-\alpha} {}_1F_1(\alpha; \beta + \alpha + 1; z). \tag{8}$$

We now define the Poisson-Charlier functions in general by the formulas

$$p_\beta(\alpha, z) \equiv \Gamma(\alpha + 1) z^{-\alpha} F_\beta(z, -\alpha), \tag{9}$$

$$\psi_\beta(\alpha, z) \equiv \frac{e^{-z} z^\alpha}{\Gamma(\alpha + 1)} p_\beta(\alpha, z). \tag{10}$$

From the formulas (6) and (7) we see that [1, p. 263]

$$\psi_\beta(-n, z) = 0, \qquad n > 0; \tag{11}$$

(12) $$\psi_\beta(-n+\beta-\tfrac{1}{2}, z) = 0, \qquad p_\beta(-n+\beta-\tfrac{1}{2}, z) = 0.$$

From the formula (8) we see that

(13) $$p_\beta(\alpha, z) = \cos(\beta-\alpha)\pi \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} z^{-\alpha} {}_1F_1(-\alpha; \beta-\alpha+1; z),$$

whence it follows at once that

(14) $$p_\beta(m, z) = \cos\beta\pi \sum_{k=0}^{m} \binom{m}{k}\binom{\beta}{k} k!(-z)^{-k}.$$

This is the usual explicit expression for the Charlier polynomials [1, p. 257]. From formula (13) we see that

(15) $$p_0(-\alpha, z) = \frac{\sin 2\alpha\pi}{2\pi} \Gamma(1-\alpha) z^{\alpha} \gamma(\alpha, z).$$

In the indeterminate case when $\alpha$ is a negative integer we see from the formula (14) that

(16) $$p_0(m, z) = 1, \qquad m \geqq 0.$$

Hence

(17) $$\psi_0(-\alpha, z) = \frac{\sin 2\alpha\pi}{2\pi} e^{-z} \gamma(\alpha, z),$$

(18) $$\psi_0(m, z) = \frac{e^{-z} z^m}{m!}.$$

From the definition (10) we now see that

(19) $$\psi_\beta(m, z) = p_\beta(m, z)\psi_0(m, z),$$

a generalization of the formula (2). From the formula (13) and the definition (10) we see that

(20) $$\psi_\alpha(\beta, z) = \cos(\beta-\alpha)\pi \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} e^{-z} {}_1F_1(-\beta; \alpha-\beta+1; z).$$

Then by Kummer's first transformation,

(21) $$\psi_\alpha(\beta, z) = \cos(\beta-\alpha)\pi \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)} {}_1F_1(\alpha+1; \alpha-\beta+1; -z),$$

from which it follows from the power series formula for solutions of the F-equation (Theorem 4) that $\psi_\alpha(\beta, z)$ is a solution of the F-equation (4).

We now have two different solutions of the F-equation based on the Poisson-Charlier functions:

(A) $$F(z, \alpha) = e^{z}\psi_\beta(-\alpha, z).$$

(B) $$F(z, \alpha) = \psi_\alpha(\beta, z).$$

From the F-equation it is evident that

$$\psi_n(\beta, z) = \frac{\partial^n}{\partial z^n} \psi_0(\beta, z), \tag{22}$$

whence we at once deduce the formula (1). Applying Taylor's theorem for the F-equation (Theorem 8) to the solution (B) we see that [1, p. 259]

$$\psi_\alpha(\beta, z + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \psi_{\alpha+n}(\beta, z); \tag{23}$$

putting $\alpha$ equal to zero we find that

$$-\frac{\sin 2\beta\pi}{2\pi} e^{-z-h} \gamma(-\beta, z + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \psi_n(\beta, z), \tag{24}$$

and, more specially [1, p. 260]

$$\left(1 + \frac{h}{z}\right)^m e^{-h} = \sum_{n=0}^{\infty} \frac{h^n}{n!} p_n(m, z). \tag{25}$$

Applying the same theorem to the solution (A) we obtain the formula

$$e^{h} \psi_\beta(\alpha, z + h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \psi_\beta(\alpha - n, z), \tag{26}$$

whence we recover the formula (11) by putting $\alpha$ equal to zero.

Applying Theorem 9 to the solution (B) yields the result

$$\sum_{n=0}^{\infty} t^n \psi_{\alpha+n}(\beta, z) = \int_0^\infty e^{-\theta} \psi_\alpha(\beta, z + \theta t) \, d\theta, \tag{27}$$

which contains as a special case the formula

$$\sum_{n=0}^{\infty} t^n p_n(m, z) = (1 + t)^{-m-1} \left(\frac{t}{z}\right)^m e^{z(1+1/t)} \left[m! - \gamma\left(m + 1, z\left(1 + \frac{1}{t}\right)\right)\right]. \tag{28}$$

Appell's generating expansion (see Theorem 10, part C or [3, p. 120]) applied to the solution (A) yields the result

$$\sum_{n=0}^{\infty} \psi_\beta(n, z + y) t^n = e^{z(t-1)} \sum_{n=0}^{\infty} \psi_\beta(n, y) t^n; \tag{29}$$

hence

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} p_\beta(n, z + y) = e^{(zt)/(z+y)} \sum_{n=0}^{\infty} \left(\frac{yt}{z + y}\right)^n \frac{p_\beta(n, y)}{n!}. \tag{30}$$

Putting $y$ equal to zero and using the formula (13) we see that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} p^\beta(n, z) = e^t \left(1 - \frac{t}{z}\right)^\beta \cos \beta\pi. \tag{31}$$

Comparing this result with the formula (25) we see that

$$(-)^n p_n(m, z) = (-)^m p_m(n, z). \tag{32}$$

It would be possible to proceed in this same fashion and discover many other formal properties of the Poisson-Charlier functions, but it is perhaps easier to notice from the formula (13) that

$$p_\beta(\alpha, z) = \cos (\beta - \alpha)\pi\Gamma(\alpha + 1)z^{-\alpha}L_\alpha^{(\beta-\alpha)}(z). \tag{33}$$

$L_\delta^\gamma(x)$ being Laguerre's function suitably generalized for complex lower index [4, p. 53]. By means of this formula every relationship involving Laguerre functions may be translated into one involving Poisson-Charlier functions.

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