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# The infinite sum of reciprocal Pell numbers ☆

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ABSTRACT

In this paper, we consider infinite sums derived from the reciprocals of the Pell numbers. Then applying the floor function to the reciprocals of this sums, we obtain a new and interesting identity involving the Pell numbers.



## 1. Introduction

For any integer $n \geqslant 0$, the famous Pell numbers $P_n$ are defined by the second-order linear recurrence sequence $P_{n+2} = 2P_{n+1} + P_n$, where $P_0 = 0$ and $P_1 = 1$. From the characteristic equations $x^2 - 2x - 1 = 0$ we also have the computational formula

$$P_n = \frac{1}{2\sqrt{2}}\left[\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n\right].$$

About the properties of this sequence, some authors had studied it, and obtained many interesting results, see [1–6]. For example, Santos and Sills [3] had studied the arithmetic properties of the $q$-Pell sequence, and obtained two identities. Kilic [4] had studied the generalized order-$k$ Fibonacci–Pell sequences, and given several congruences.

On the other hand, Ohtsuka and Nakamura [7] studied the properties of the Fibonacci numbers, and proved the following conclusions:

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{F_k}\right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geqslant 2; \\ F_{n-2}-1, & \text{if } n \text{ is odd and } n \geqslant 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{F_k^2}\right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n-1, & \text{if } n \text{ is even and } n \geqslant 2; \\ F_{n-1}F_n, & \text{if } n \text{ is od and } n \geqslant 1. \end{cases}$$

These two identities are interesting, at least they showed us some new properties of the Fibonacci numbers. It is naturally that one can ask whether there exist some corresponding formulae for the other second-order linear recurrence sequences, such as the Pell sequence? Unfortunately, we have not found any related results in [7]. The main purpose of this paper is using a new method to give a similar identity for the Pell numbers. That is, we shall prove the following conclusion:

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**Theorem 1.** *For any positive integer n, we have the identity*

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_{n-1} + P_{n-2}, & \text{if } n \text{ is even and } n \geqslant 2; \\ P_{n-1} + P_{n-2} - 1, & \text{if } n \text{ is odd and } n \geqslant 1, \end{cases}$$

*where providing* $P_{-1} = P_1 = 1$.

For any integer $m \geqslant 2$, whether there exists a computational formula for

$$\left\lfloor \left( \sum_{k=n}^{\infty} \frac{1}{P_k^m} \right)^{-1} \right\rfloor$$

is an open problem, suggest interested readers study it with us.

## 2. Proof of the theorem

In this section, we shall prove our theorem directly. First we consider the case that $n = 2m$ is an even number. At this time, our theorem become into

$$\left\lfloor \left( \sum_{k=2m}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = P_{2m-1} + P_{2m-2}.$$

It is equivalent to

$$P_{2m-1} + P_{2m-2} \leqslant \left( \sum_{k=2m}^{\infty} \frac{1}{P_k} \right)^{-1} < P_{2m-1} + P_{2m-2} + 1,$$

or

$$\frac{1}{P_{2m-1} + P_{2m-2} + 1} < \sum_{k=2m}^{\infty} \frac{1}{P_k} \leqslant \frac{1}{P_{2m-1} + P_{2m-2}}. \tag{1}$$

Clearly, $P_1 + P_0 = 1$ and

$$\frac{1}{2} < \sum_{k=2}^{\infty} \frac{1}{P_k} = \frac{1}{2} + \frac{1}{5} + \cdots < \frac{1}{2} + \frac{1}{5} + \frac{1}{2 \times 5} + \frac{1}{2^2 \times 5} + \frac{1}{2^3 \times 5} + \cdots = \frac{1}{2} + \frac{2}{5} < 1,$$

so our theorem is true for $m = 1$. Now without loss of generality we can assume $m \geqslant 2$. We prove that for any integer $k \geqslant 2$,

$$\frac{1}{P_{2k}} + \frac{1}{P_{2k+1}} < \frac{1}{P_{2k-1} + P_{2k-2}} - \frac{1}{P_{2k+1} + P_{2k}}. \tag{2}$$

This inequality equivalent to

$$\frac{P_{2k+2}}{P_{2k+1}(P_{2k+1} + P_{2k})} < \frac{P_{2k-1}}{P_{2k}(P_{2k-1} + P_{2k-2})},$$

or

$$P_{6k+1} + P_{2k+3} + P_{6k} - P_{2k+4} - 6P_{2k-1} - 6P_{2k-2} < P_{6k+1} + P_{2k-1} + P_{6k} - P_{2k} + 6P_{2k+1} + 6P_{2k},$$

or

$$P_{2k+3} < P_{2k+4} + 5P_{2k} + 7P_{2k-1} + 6(P_{2k+1} + P_{2k-2}). \tag{3}$$

It is clear that inequality (3) is correct. So inequality (2) is true.

Now applying (2) repeatedly we have

$$\sum_{k=2m}^{\infty} \frac{1}{P_k} = \sum_{k=m}^{\infty} \left( \frac{1}{P_{2k}} + \frac{1}{P_{2k+1}} \right) < \sum_{k=m}^{\infty} \left( \frac{1}{P_{2k-1} + P_{2k-2}} - \frac{1}{P_{2k+1} + P_{2k}} \right) = \frac{1}{P_{2m-1} + P_{2m-2}}. \tag{4}$$

On the other hand, we prove the inequality

$$\frac{1}{P_{2k}} + \frac{1}{P_{2k+1}} > \frac{1}{P_{2k-1} + P_{2k-2} + 1} - \frac{1}{P_{2k+1} + P_{2k} + 1}. \tag{5}$$

This inequality equivalent to

$$P_{2k+2}P_{2k}(P_{2k-1}+P_{2k-2})+P_{2k}(P_{2k-1}+P_{2k-2})+P_{2k+2}P_{2k}+P_{2k} \\ > P_{2k+1}P_{2k-1}(P_{2k+1}+P_{2k})+P_{2k+1}P_{2k-1}-P_{2k+1}(P_{2k+1}+P_{2k})-P_{2k+1}. \tag{6}$$

From the process of proving (3) we know that inequality (6) equivalent to

$$P_{2k}(P_{2k-1}+P_{2k-2})+P_{2k+2}P_{2k}+P_{2k}+P_{2k+1}(P_{2k+1}+P_{2k})+P_{2k+1} \\ > P_{2k+4}-P_{2k+3}+5P_{2k}+7P_{2k-1}+6(P_{2k+1}+P_{2k-2})+P_{2k+1}P_{2k-1}. \tag{7}$$

It is easy to check that inequality (7) holds for all integers $k \geqslant 2$. So inequality (5) is correct.

Applying (5) repeatedly we have

$$\sum_{k=2m}^{\infty}\frac{1}{P_k}=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k}}+\frac{1}{P_{2k+1}}\right)>\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k-1}+P_{2k-2}+1}-\frac{1}{P_{2k+1}+P_{2k}+1}\right)=\frac{1}{P_{2m-1}+P_{2m-2}+1}. \tag{8}$$

Now the inequality (1) follows from (4) and (8).

Similarly, we can consider the case that $n = 2m + 1$ is an odd number. Note that $P_0 + P_{-1} - 1 = 0 + 1 - 1 = 0$ and

$$\sum_{k=1}^{\infty}\frac{1}{P_k}=1+\frac{1}{2}+\frac{1}{5}+\frac{1}{12}+\cdots>1.$$

So our theorem is true if $m = 0$.

If $n = 2m + 1$ with $m \geqslant 1$, then our theorem equivalent to the inequality

$$P_{2m}+P_{2m-1}-1\leqslant\left(\sum_{k=2m+1}^{\infty}\frac{1}{P_k}\right)^{-1}<P_{2m}+P_{2m-1},$$

or

$$\frac{1}{P_{2m}+P_{2m-1}}<\sum_{k=2m+1}^{\infty}\frac{1}{P_k}\leqslant\frac{1}{P_{2m}+P_{2m-1}-1}. \tag{9}$$

First we can prove that the inequality

$$\frac{1}{P_{2k+1}}+\frac{1}{P_{2k+2}}>\frac{1}{P_{2k}+P_{2k-1}}-\frac{1}{P_{2k+2}+P_{2k+1}}, \tag{10}$$

holds for all integers $k \geqslant m$.

The inequality (10) equivalent to

$$\frac{P_{2k+3}}{P_{2k+2}(P_{2k+2}+P_{2k+1})}>\frac{P_{2k}}{P_{2k+1}(P_{2k}+P_{2k-1})},$$

or

$$P_{2k+4}<5P_{2k+1}+6P_{2k+2}+7P_{2k}+P_{2k+5}+6P_{2k-1}. \tag{11}$$

It is clear that the inequality (11) is true for all integers $k \geqslant 1$. So (10) is correct.

Applying (10) repeatedly we have

$$\sum_{k=2m+1}^{\infty}\frac{1}{P_k}=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k+1}}+\frac{1}{P_{2k+2}}\right)>\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k}+P_{2k-1}}-\frac{1}{P_{2k+2}+P_{2k+1}}\right)=\frac{1}{P_{2m}+P_{2m-1}}. \tag{12}$$

On the other hand, we can also prove the inequality

$$\frac{1}{P_{2k+1}}+\frac{1}{P_{2k+2}}<\frac{1}{P_{2k}+P_{2k-1}-1}-\frac{1}{P_{2k+2}+P_{2k+1}-1}. \tag{13}$$

This inequality equivalent to

$$\frac{P_{2k+3}-1}{P_{2k+2}(P_{2k+2}+P_{2k+1}-1)}<\frac{P_{2k}+1}{P_{2k+1}(P_{2k}+P_{2k-1}-1)},$$

or

$$6(P_{2k+1}+P_{2k-1})+7(P_{2k+2}+P_{2k})+P_{2k+2}P_{2k}<P_{2k+3}P_{2k+1}+P_{2k+1}(P_{2k}+P_{2k-1})+P_{2k+2}(P_{2k+2}+P_{2k+1})+P_{2k+4}. \tag{14}$$

It is clear that inequality (14) is correct. So inequality (12) is true.

Applying inequality (12) repeatedly we have

$$\sum_{k=2m+1}^{\infty}\frac{1}{P_k}=\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k+1}}+\frac{1}{P_{2k+2}}\right)<\sum_{k=m}^{\infty}\left(\frac{1}{P_{2k}+P_{2k-1}-1}-\frac{1}{P_{2k+2}+P_{2k+1}-1}\right)=\frac{1}{P_{2m}+P_{2m-1}-1}. \tag{15}$$

Combining (12) and (15) we deduce the inequality (9).

This completes the proof of our theorem.

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