

k-Pell, *k*-Pell-Lucas and Modified *k*-Pell Numbers: Some Identities and Norms of Hankel Matrices

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Abstract

In this paper we present some identities involving terms of *k*-Pell, *k*-Pell-Lucas and Modified *k*-Pell sequences. We also give some results on the column and row norms of Hankel matrices which entries are numbers of these sequences.

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1. INTRODUCTION

Hankel matrices play an important role in many science fields namely signal processing in engineering or times series in statistics. An Hankel matrix is a matrix which entries are the same along the anti-diagonals. This type of matrices has been subject of study with respect to its spectrum (collection of eigenvalues) and other interesting results were derived.

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Some norms of Hankel matrices involving Fibonacci and Lucas numbers were presented in [1] and, in what concerns spectral norms of these matrices their work was continued in [9].

In this paper we are interested in the special cases of Hankel matrices involving k -Pell, k -Pell-Lucas and Modified k -Pell numbers, denoted by $P_{k,n}$, $Q_{k,n}$ and $q_{k,n}$, with k a real number and n a non negative integer, respectively.

These sequences are particular cases of the sequences $W_n(a, b; p, q)$ defined by the general recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, n \geq 2 \quad (1)$$

with $W_0 = a$, $W_1 = b$, and a, b, p, q , integers, with $p > 0$, $q \neq 0$, by Horadam (see [5], [6] and [7]).

In fact, in the Horadam notation we have

$$P_{k,n} = W_n(0, 1; 2, -k) \quad (2)$$

$$Q_{k,n} = W_n(2, 2; 2, -k) \quad (3)$$

and

$$q_{k,n} = W_n(1, 1; 2, -k). \quad (4)$$

The corresponding characteristic equation of (1) is

$$x^2 = px - q,$$

and its roots are $r_1 = \frac{p + \sqrt{p^2 - 4q}}{2}$ and $r_2 = \frac{p - \sqrt{p^2 - 4q}}{2}$. For (2), (3) and (4), their roots are

$$r_1 = 1 + \sqrt{1 + k} \quad \text{and} \quad r_2 = 1 - \sqrt{1 + k}, \quad (5)$$

and verify

$$r_1 + r_2 = 2, \quad r_1 - r_2 = 2\sqrt{1 + k} \quad \text{and} \quad r_1 r_2 = -k. \quad (6)$$

Their Binet's formulas are well known and given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (7)$$

$$Q_{k,n} = r_1^n - r_2^n \quad (8)$$

$$q_{k,n} = \frac{r_1^n + r_2^n}{2}, \quad (9)$$

respectively.

The main purpose of this paper is the study of some norms of Hankel matrices involving k -Pell, k -Pell-Lucas and Modified k -Pell numbers. With this purpose, after this introductory section, we present, in section two, a set of identities involving the terms of these sequences. These identities are an attempt to generalize the study started in [4]. In section 3 we present some results for the norms $\|\dots\|_1$ and $\|\dots\|_\infty$ of Hankel matrices involving k -Pell, k -Pell-Lucas and Modified k -Pell numbers.

2. SOME IDENTITIES

Let $P_{k,n}$, $Q_{k,n}$ and $q_{k,n}$, with k a real number and n a nonnegative integer, be defined as above. The following proposition establishes several identities between the terms of the referred sequences.

Proposition 2.1. *For any real number k and a nonnegative integer n (different from zero in items i), iii), vii) and viii)),*

- i) $P_{k,n}^2 + P_{k,n+1}P_{k,n-1} = \frac{1}{2(1+k)} [Q_{k,n}^2 - 2(-k)^{n-1}(1-k)]$;
- ii) $(P_{k,n+1} - P_{k,n})^2 = (1+k)P_{k,n}^2 + (-k)^n$;
- iii) $2P_{k,n}P_{k,n-1} + P_{k,n-1}^2 - P_{k,n}^2 = (1-k)P_{k,n-1}^2 + (-k)^n$;
- iv) $P_{k,n}^2 = \frac{1}{4(1+k)} [Q_{k,2n} - 2(-k)^n]$;
- v) $P_{k,2n+1} = P_{k,n+1}^2 + kP_{k,n}^2$;
- vi) $P_{k,n+1} - P_{k,n} = q_{k,n}$;
- vii) $(2+2k)P_{k,n} = Q_{k,n} + kQ_{k,n-1}$;
- viii) $P_{k,n} + kP_{k,n-1} = q_{k,n}$.

Proof. In order to prove i), ii), iv) and v) we use the Binet's formulas (7), (8), (9), the expressions of the roots (5) and the relations involving the roots (6). To prove iii) we also use ii) in the form

$$(P_{k,n} - P_{k,n-1})^2 = (1+k)P_{k,n-1}^2 + (-k)^{n-1}.$$

The fact that $2q_{k,n} = Q_{k,n}$, $n \geq 0$, is used in the proof of vi) and vii). Proposition 3 of [2] is required to obtain vi) and Proposition 4 of the same article to obtain vii). \square

We note that for $k = 1$ these identities generalize the relationships (9), (10), (11), (12), (14) and (15) established by Halici in [4].

3. NORMS OF HANKEL MATRICES INVOLVING k -PELL, k -PELL-LUCAS AND MODIFIED k -PELL NUMBERS

An Hankel matrix is an $n \times n$ matrix

$$H_n = (h_{ij})$$

where $h_{ij} = h_{i+j-1}$, that is a matrix of the form

$$H_n = \begin{bmatrix} h_1 & h_2 & h_3 & \cdots & h_n \\ h_2 & h_3 & h_4 & \cdots & h_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_n & h_{n+1} & h_{n+2} & \cdots & h_{2n-1} \end{bmatrix}.$$

H_n is symmetric and consequently its column norm $\|H_n\|_1$ is equal to its row norm $\|H_n\|_\infty$,

$$\|H_n\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |h_{ij}| = \max_{1 \leq i \leq n} \sum_{j=1}^n |h_{ij}| = \|H_n\|_\infty.$$

For this reason, we will calculate, from now on, one of them in each case. We start by defining the following Hankel matrices

$$A = (a_{ij}), \quad a_{ij} = P_{k,i+j-1} \quad (10)$$

$$B = (b_{ij}), \quad b_{ij} = Q_{k,i+j-1} \quad (11)$$

and

$$C = (c_{ij}), \quad c_{ij} = q_{k,i+j-1}. \quad (12)$$

We shall give the column and row norms of the Hankel matrices involving k -Pell, k -Pell-Lucas and Modified k -Pell numbers and note that these results generalize the corresponding results for Pell, Pell-Lucas and Modified Pell numbers given in [4].

Theorem 3.1. *If A is an $n \times n$ matrix with $a_{ij} = P_{k,i+j-1}$ then*

$$\|A\|_1 = \|A\|_\infty = \frac{1}{2(k+1)}(Q_{k,2n} - Q_{k,n}) = \frac{1}{k+1}(q_{k,2n} - q_{k,n}).$$

Proof. From the definition of the matrix A , we can write

$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}| \\ &= \max_{1 \leq j \leq n} \{|a_{1j}| + |a_{2j}| + \cdots + |a_{nj}|\} \\ &= P_{k,n} + P_{k,n+1} + \cdots + P_{k,2n-1} \\ &= \sum_{i=0}^{2n-1} P_{k,i} - \sum_{i=0}^{n-1} P_{k,i}. \end{aligned}$$

Now, using Proposition 1 of [3], this last equality can be written as

$$\frac{1}{k+1}(-1 + P_{k,2n} + kP_{k,2n-1}) - \frac{1}{k+1}(-1 + P_{k,n} + kP_{k,n-1})$$

and by Proposition 4 of [2] as

$$\frac{1}{k+1} \left(\frac{1}{2}Q_{k,2n} - \frac{1}{2}Q_{k,n} \right),$$

consequently

$$\|A\|_1 = \|A\|_\infty = \frac{1}{2(k+1)}(Q_{k,2n} - Q_{k,n}),$$

which is equal to

$$\frac{1}{k+1}(q_{k,2n} - q_{k,n}),$$

since $Q_{k,j} = 2q_{k,j}$. □

Theorem 3.2. *If B is an $n \times n$ matrix with $b_{ij} = Q_{k,i+j-1}$ then*

$$\|B\|_1 = \|B\|_\infty = \frac{1}{k+1}(Q_{k,2n+1} + Q_{k,n} - Q_{k,2n} - Q_{k,n+1}) = 2(P_{k,2n} - P_{k,n}).$$

Proof. Similarly to the proof of the previous theorem and using now Proposition 3 of [10], $\sum_{i=0}^n Q_{k,i} = \frac{1}{k+1} (Q_{k,n+1} + kQ_{k,n})$, and Propositions 3 and 4 of [2] we obtain

$$\begin{aligned}
\|B\|_1 &= Q_{k,n} + Q_{k,n+1} + \cdots + Q_{k,2n-1} \\
&= \sum_{i=0}^{2n-1} Q_{k,i} - \sum_{i=0}^{n-1} Q_{k,i} \\
&= \frac{1}{k+1} (Q_{k,2n} + kQ_{k,2n-1} - Q_{k,n} - kQ_{k,n-1}) \\
&= \frac{1}{k+1} [(2Q_{k,2n} + kQ_{k,2n-1}) - Q_{k,2n} - (2Q_{k,n} - kQ_{k,n-1}) + Q_{k,n}] \\
&= \frac{1}{k+1} (Q_{k,2n+1} - Q_{k,2n} - Q_{k,n+1} + Q_{k,n}) \\
&= \frac{1}{k+1} (2(P_{k,2n+1} + kP_{k,2n}) - 2(P_{k,2n+1} - P_{k,2n}) - 2(P_{k,n+1} + kP_{k,n}) + \\
&\quad 2(P_{k,n+1} - P_{k,n})) \\
&= \frac{1}{k+1} [2(k+1)P_{k,2n} - 2(k+1)P_{k,n}] \\
&= 2(P_{k,2n} - P_{k,n}).
\end{aligned}$$

□

Theorem 3.3. *If C is an $n \times n$ matrix with $c_{ij} = q_{k,i+j-1}$ then*

$$\|C\|_1 = \|C\|_\infty = \frac{1}{2(k+1)} (Q_{k,2n+1} + Q_{k,n} - Q_{k,2n} - Q_{k,n+1}) = P_{k,2n} - P_{k,n}.$$

Proof. Since $Q_{k,j} = 2q_{k,j}$ we have $B = 2C$ and $\|B\|_1 = \|2C\|_1 = |2| \|C\|_1$, and the result follows by the previous theorem. □

In addition if we consider the Hankel matrices $D = A + B = (d_{ij})$ where

$$d_{ij} = P_{k,i+j-1} + Q_{k,i+j-1},$$

$E = A + C = (e_{ij})$ where

$$e_{ij} = P_{k,i+j-1} + q_{k,i+j-1}$$

and $F = B + C = (f_{ij})$ where

$$f_{ij} = Q_{k,i+j-1} + q_{k,i+j-1} = \frac{3}{2} Q_{k,i+j-1}$$

and the definitions of the used norms, we get

$$\|D\|_1 = \|D\|_\infty = 2 \|E\|_1 - \|A\|_1 = 2 \|E\|_\infty - \|A\|_\infty \quad (13)$$

and

$$\|F\|_1 = \|F\|_\infty = \frac{3}{2} \|B\|_1 = \frac{3}{2} \|B\|_\infty. \quad (14)$$

To obtain $\|E\|_1$ we can use Theorem 3.1 and Theorem 3.3 since, in this case, $\|E\|_1 = \|A\|_1 + \|C\|_1$. So we have

$$\|E\|_1 = \frac{1}{2(k+1)} (Q_{k,2n+1} - Q_{k,n+1}) = \frac{1}{k+1} (q_{k,2n+1} - q_{k,n+1}). \quad (15)$$

The same result could be obtained if we follow the same steps of Theorem 3.1 and use Proposition 4 item 2 of [10]

$$\sum_{i=0}^n (P_{k,i} + q_{k,i}) = \frac{1}{k+1} (-1 + kP_{k,n} + (k+2)P_{k,n+1}).$$

In this case we obtain

$$\|E\|_1 = \frac{1}{(k+1)} (k(P_{k,2n-1} - P_{k,n-1}) - (k+2)(P_{k,2n} - P_{k,n}))$$

and using the definition of k -Pell numbers this is equivalent to

$$\|E\|_1 = \frac{1}{(k+1)} (P_{k,2n+1} + kP_{k,2n} - P_{k,n+1} - kP_{k,n}).$$

Now using Proposition 4 of [2] we have exactly the same as in (15).

Now we will consider the Hadamard product, denoted by \circ , of the previously defined matrices, that is the entrywise product, see [8].

Consider the Hankel matrices $G = A \circ B = (g_{ij})$ where

$$g_{ij} = P_{k,i+j-1} Q_{k,i+j-1},$$

$H = A \circ C = (h_{ij})$ where

$$h_{ij} = P_{k,i+j-1} q_{k,i+j-1}$$

and $L = B \circ C = (l_{ij})$ where

$$l_{ij} = Q_{k,i+j-1} q_{k,i+j-1}$$

Since $Q_{k,j} = 2q_{k,j}$ we have that

$$\|H\|_1 = \frac{1}{2} \|G\|_1.$$

Now following the same steps of Theorem 3.1 and using Proposition 6 item 1. of [10],

$$\sum_{i=0}^n (P_{k,i} Q_{k,i}) = \frac{1}{3-k} \left[\frac{2}{k+1} (-1 + P_{k,2n+1} + kP_{k,2n}) - kP_{k,2n} \right],$$

we obtain

$$\|G\|_1 = \frac{1}{3-k} \left[\frac{2}{k+1} (P_{k,4n-1} + kP_{k,4n-2} - P_{k,2n-1} - kP_{k,2n-2}) - kP_{k,4n-2} + kP_{k,2n-2} \right] \quad (16)$$

and using Proposition 6 item 3. of [10],

$$\sum_{i=0}^n (Q_{k,i} q_{k,i}) = (1-k) (q_{k,2} + q_{k,2}q_{k,3} + \cdots + q_{k,n-1}q_{k,n}) + q_{k,n}q_{k,n+1} - k,$$

we obtain

$$\|L\|_1 = (1-k) (q_{k,n-1}q_{k,n} + q_{k,n}q_{k,n+1} + \cdots + q_{k,2n-2}q_{k,2n-1}) + q_{k,2n-1}q_{k,2n} - q_{k,n-1}q_{k,n}. \quad (17)$$

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