PELL WALKS AND RIORDAN MATRICES

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ABSTRACT

The purpose of this paper is twofold. As the first goal, we show that three different classes of random walks are counted by the Pell numbers. The calculations are done using a convenient technique that involves the Riordan group. This leads to the second goal, which is to demonstrate this convenient technique. We also construct bijections among Pell, certain Motzkin, and certain Schröder walks. As a consequence of using these Riordan group methods, we also find unexpected connections to a special class of Riordan matrices called Schröder matrices.

1. INTRODUCTION

We count three related kinds of walks and call them Pell walks of length n . The first problem, for $0 \leq n \leq 6$, has a count starting with

$$1, 3, 7, 17, 41, 99, 239. \tag{1}$$

The two other related problems yield the initial values

$$1, 2, 5, 12, 29, 70, 169 \tag{2}$$

which gives another version of Pell numbers. The sequence (1) is defined by

$$p_n = 2p_{n-1} + p_{n-2},\ n \geq 2,\ p_0 = 1,\ \text{and}\ p_1 = 3. \tag{3}$$

The generating function is $p(z) := \sum_{n\geq 0} p_n z^n = (1+z)/\left(1-2z-z^2\right)$. The companion sequence (2) is defined by

$$q_n = 2q_{n-1} + q_{n-2},\ n \geq 2,\ q_0 = 1,\ \text{and}\ q_1 = 2 \tag{4}$$

and $q(z) := \sum_{n\geq 0} q_n z^n = 1/\left(1-2z-z^2\right)$. Both sequences are called Pell numbers and they are closely related to the Fibonacci and Lucas numbers. For more details and other properties of the Pell numbers, see Sellers [12] and Hoggatt and Bicknell-Johnson [5].

The walks counted in this paper lead to three infinite lower-triangular arrays. When considered as infinite lower-triangular matrices (i.e., Riordan matrices) all three arrays are elements of the Riordan group. The set of all Riordan matrices forms the Riordan group under matrix multiplication [14]. For instance, Pascal's triangle written in lower-triangular matrix form denoted by $\boldsymbol{P}^*$ is an example of a Riordan matrix (see Section 4).

The notion of a Riordan matrix is formally defined in Section 2 in conjuction with counting the Pell walks. The group structure enables us to count these walks and is of considerable independent interest. One elementary use of the Riordan group is to prove and invert combinatorial identities as discussed by Shapiro, *et. al.* [14]. Another use is to compute expected values. We will see examples of both uses later in Sections 2 and 4. Connections to the

Fibonacci numbers are also given in these sections. Pell walk bijections are constructed in Section 3. These include bijections with the sets of bounded grand Motzkin and modified Schröder walks given by Ferrari, *et. al.* [3]. The Riordan group multiplication is defined in Section 4. By this multiplication, the walk arrays generated in Section 2 lead to unexpected connections to a special class of Riordan matrices called Schröder matrices.

2. THE PELL WALKS

2.1. Pell Walk #1: We start by counting walks which start at the origin $(0,0)$ and take unit steps $(1,0) = \text{E}$ (east), $(0,1) = \text{N}$ (north), and $(-1,0) = \text{W}$ (west) with the restriction that no E step immediately follows a W step and vice versa. We call this class of walks ENW walks. The restriction has the effect of making the walks self-avoiding. It is a major unsolved problem to enumerate all self-avoiding walks. See Madras and Slade [6].

A typical example of an ENW walk is denoted by the steps NENW. Let $p(n,k)$ denote the number of ENW walks where n is the number of steps and k is the final height. We then get the following array for the first few values.

$n \backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	2	1	0	0	0
2	2	4	1	0	0
3	2	8	6	1	0
4	2	12	18	8	1

By convention $p(0,0) = 1$. Entry $p(4,2) = 18$ and the given example is one these 18 walks. We see that $p(n,0) = 2$ when $n \geq 1$ since a walk ending at height 0 always goes east or always goes west. It is obvious that $p(n,n) = 1$ since the only way to be at height n after n steps is to always go north.

It is convenient to consider the array as part of an infinite lower-triangular matrix which we denote by $\boldsymbol{P} := (p(n,k))_{n,k\geq 0}$. The formation rule of the array is

$$p(n+1,k) = p(n,k-1) + 2\sum_{j\geq 1} p(n-j,k-1). \tag{5}$$

The row sums of the array give the Pell numbers of (1). To prove this holds in general use induction on n and sum (5) over k. We will prove that the ENW walks are counted by the array $\boldsymbol{P}$ at the end of Section 2.1.

We now want to prove $\boldsymbol{P}$ is a Riordan matrix. We observe the leftmost (zeroth) column of $\boldsymbol{P}$ is $(1,2,2,\dots)^T$. These are the coefficients of the generating function $(1+z)/(1-z)$. This gives the first generating function needed to prove that $\boldsymbol{P}$ is a Riordan matrix. A Riordan matrix is an infinite lower-triangular matrix such that the generating function of the k^{th} column for $k = 0,1,2,\dots$ is $g(z) f^k(z)$ where $g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ and

$$f(z) = f_1 z + f_2 z^2 + f_3 z^3 + \cdots, \; f_1 \neq 0 \tag{6}$$

and $g(z)$ and $f(z)$ belong to the ring of formal power series $\mathbb{C}[[z]]$. We insert this definition here since we are about to show that $\boldsymbol{P}$ is Riordan. We make the strong assumption that the generating function of the k^{th} column of $\boldsymbol{P}$ is $((1+z)/(1-z)) f^k(z)$ where $f(z)$ satisfies (6).

If the k^{th} column is of this form, we can easily solve for $f(z)$ and conclude $\boldsymbol{P}$ is Riordan. By the assumption made on the formation rule, we find for (5) to hold we must have

$$\left(\frac{1+z}{1-z}\right) f^{k}(z) = \left(z + 2z^2 + 2z^3 + \cdots\right)\left(\frac{1+z}{1-z}\right) f^{k-1}(z). \tag{7}$$

This follows by applying the definition of a Riordan matrix. The equation then simplifies to $f(z) = z(1+z)/(1-z) = z + 2z^2 + 2z^3 + \cdots$ which gives the second generating function needed to prove $\boldsymbol{P}$ is Riordan.

Given $g(z)$ and $f(z)$, we then say an infinite lower-triangular array is a Riordan matrix. More precisely, let $\boldsymbol{M} := (m(n,k))_{n,k\geq 0}$ be a Riordan matrix then $\boldsymbol{M} = (g(z), f(z))$ is the pair form notation. From $f(z)$ given above and the zeroth column generating function, the Riordan pair of the ENW walk array is $\boldsymbol{P} = ((1+z)/(1-z), z(1+z)/(1-z))$. Thus, the assumed formation rules are indeed defined for $\boldsymbol{P}$ and $\boldsymbol{P}$ is Riordan.

Theorem 1: *Given a Riordan matrix* $\boldsymbol{M}$ *and column vector* $h = (h_0, h_1, \dots)^T$, *then the product of* $\boldsymbol{M}$ *and* h *gives a column vector whose generating function is* $g(z)\,h(f(z))$.

Proof: See Shapiro, *et. al.* [14]. □

We call this theorem The Fundamental Theorem of the Riordan Group. When multiplying with Riordan matrices we switch freely among column vectors, sequences, and generating functions. If $\circledast$ denotes matrix multiplication, then with these identifications we can express the fundamental theorem as $(g(z), f(z)) \circledast h(z) = g(z)\,h(f(z))$. This leads directly to the Riordan group multiplication which is defined in Section 4 where more details and other examples are treated.

Our first problem is to find the number of ENW walks of length n. We want to connect all this by showing that the ENW walks satisfy (5). Consider the following combinatorial arguments. Let $p(n,k)$ denote the number of ENW walks of length n and height k. To form such a walk, we consider the following cases. First, if the last step is N, then we have walks of the form $\star\ldots\star$N and there are $p(n,k-1)$ possibilities. If the last step is not N, then we have walks of the form $\star\ldots\star$NE...E and there are $p(n-j,k-1)$ possibilities where $n > j$. Walks of the form $\star\ldots\star$NW...W give the same result. Thus, summing over the cases gives (5). The boundary condition $p(n,k) = 0$ $(k > n)$ is trivial. This proves the formation rule and gives $\boldsymbol{P}$ an ENW lattice walk interpretation.

To find the total number of all ENW walks of length n, we multiply by $(1,1,\dots)^T$ which has $1/(1-z)$ as its generating function. The fundamental theorem then tells us that the generating function for the total number of ENW walks is $\boldsymbol{P} \circledast (1/(1-z)) = p(z)$. Thus, the total number of ENW walks of length n is the Pell number p_n (see (3)). Note that Richard Stanley discusses this same problem in [17], pp. 203-204.

2.2. Pell Walk #2: We now consider ENW walks with the additional restriction no walk ends with a W step. We call this class of walks EN$\overline{\text{W}}$ walks. A typical example of an EN$\overline{\text{W}}$ walk is denoted by the steps ENWN. Let $q(n,k)$ denote the number of EN$\overline{\text{W}}$ walks with no ending W steps, then we get the following array for the first few values.

$n\backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	3	1	0	0
3	1	5	5	1	0
4	1	7	13	7	1

For instance $q(4,2)=13$ and the given example is one these 13 walks. For the zeroth column, we have $q(n,0)=1$ since a walk ending at height 0 always goes east. Like the previous problem $q(n,n)=1$ is obvious. By symmetry we can also consider walks with no last E steps or walks with no beginning E or no beginning W steps (see Proposition 5).

Counting the EN$\overline{\text{W}}$ walks, we again observe a Pascal-like array and consider the array as part of an infinite lower-triangular matrix $\boldsymbol{Q} := (q(n,k))_{n,k\geq 0}$. An interesting observation is the entries of $\boldsymbol{Q}$ are the Delannoy numbers [1], [18], [19]. We also observe the central numbers $\{q(2k,k)\}_{k\geq 0}=\{1,3,13,63,\dots\}$ are the coefficients of

$$1/\sqrt{1-6z+z^2}=1+3z+13z^2+63z^3+\cdots=\sum_{n\geq 0}G_n(3)\,z^n$$

where $\sum_{n\geq 0}G_n(t)\,z^n=1/\sqrt{1-2tz+z^2}$ and $G_n(t)$ is the n^{th} Legeńdre polynomial [1], p. 81. We will prove the EN$\overline{\text{W}}$ walks are counted by the array $\boldsymbol{Q}$ at the end of Section 2.2.

We now prove that $\boldsymbol{Q}$ is a Riordan matrix. The formation rule of $\boldsymbol{Q}$ is

$$q(n+1,k)=q(n,k)+q(n,k-1)+q(n-1,k-1). \tag{8}$$

The row sums here give the first few Pell numbers of (2). Like Problem 1, we prove this holds in general by observing the columns of $\boldsymbol{Q}$. The zeroth column is $(1,1,\dots)^T$. Thus, it can be easily shown that the k^{th} column generating function is $(1/(1-z))\,f^k(z)$ where $f(z)=z(1+z)/(1-z)$. Hence, $\boldsymbol{Q}=(1/(1-z),\,z(1+z)/(1-z))$ is a Riordan matrix.

Remark 2: *Let G be a graph with n vertices. Then a k-matching in G is a set of k edges of G, no two of which have a vertex in common. Following Godsil [4], the number of k-matchings in G is denoted by $\rho(G,k)$. We set $\rho(G,0)=1$ and define the matching polynomial of G by $\mu(G,x)=\sum_{k\geq 0}\rho(G,k)\,x^{n-2k}$. Thus, the matching polynomial counts the matchings in a graph. The connection here is that $\boldsymbol{Q}$ has rows approaching a normal distribution. This is obtained from comb graphs (i.e., graphs $|,\sqcup,\sqcup\!\sqcup,\cdots$) whose coefficients are given by the following matching polynomials*

$$\begin{aligned}&1+1x\\&1+3x+1x^2\\&1+5x+5x^2+1x^3\\&\cdots\quad\cdots\end{aligned}$$

For instance if $G=\sqcup$ then $\mu(G,x)=1+3x+1x^2$ and the matchings are

$$\underset{k=0}{\sqcup,}\qquad\underset{k=1}{\sqcup\sqcup\sqcup,}\qquad\underset{k=2}{\sqcup}.$$

Thus, Godsil's Theorem in [4] (Chapter 6) implies normality and the mean is $n/2$ by symmetry. By Riordan group methods, the variance approaches $n\sqrt{2}/4$. See Schmidt and Shapiro [11] for more details.

Our second problem is to find the number of EN$\overline{\text{W}}$ walks of length n. We want to show that the EN$\overline{\text{W}}$ walks satisfy (8). Let $q(n,k)$ denote the number of EN$\overline{\text{W}}$ walks of length n ending at height k. To count such walks, we combine $p(n,k)$ and $q(n,k)$ as follows. First $p(n+1,k)=p(n,k-1)+2q(n,k)$ where we recall that $p(n,k)$ denotes ENW walks of length n and height k. Then $p(n,k-1)$ counts walks with last step N and $2q(n,k)$ counts walks with last step E or W. Next, by letting the last step be N or E we have $q(n+1,k) = p(n,k-1) + q(n,k)$. Now combining and manipulating the two equations yields

$$\begin{aligned} q(n+1,k) &= p(n-1,k-2) + 2q(n-1,k-1) + q(n,k) \\ &= p(n-1,k-2) + q(n-1,k-1) + q(n-1,k-1) + q(n,k) \\ &= q(n,k-1) + q(n-1,k-1) + q(n,k) \end{aligned}$$

which gives (8). This proves the formation rule and gives $\boldsymbol{Q}$ an EN$\overline{\text{W}}$ lattice walk interpretation.

Again utilizing the fundamental theorem, the generating function for the total number of EN$\overline{\text{W}}$ walks is $\boldsymbol{Q} \circledast (1/(1-z)) = q(z)$. Thus, the total number of EN$\overline{\text{W}}$ walks of length n is the Pell number q_n (see (4)).

2.3. Pell Walk #3: As the final walk problem, let $c(n,k)$ denote walks of length n with final height k that start at $(0,0)$ and take unit steps $(1,0) = \text{E}$ (east) and $(0,1) = \text{N}$ (north) and double east steps of length 2 denoted by D. We call this class of walks END walks.

Counting the first few END walks of length n where column k again has the number of walks ending at height k gives another Pascal-like array.

$n\backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	2	2	1	0	0
3	3	5	3	1	0
4	5	10	9	4	1

We consider this array as part of an infinite lower-triangular matrix $\boldsymbol{C} := (c(n,k))_{n,k\geq 0}$. The matrix $\boldsymbol{C}$ is called a Fibonacci matrix since its zeroth column contains the Fibonacci numbers. It is well known that the Fibonacci numbers count the number of ways one can ascend a staircase one or two steps at a time. We change this interpretation to unit steps or double steps east to interpret the END walks $c(n,k)$. The formation rule of $\boldsymbol{C}$ is

$$c(n+1,k) = c(n,k) + c(n,k-1) + c(n-1,k). \tag{9}$$

Casting (9) into Riordan matrix form gives

$$\boldsymbol{C} = (F(z), zF(z)) = \left(1/\left(1-z-z^2\right), z/\left(1-z-z^2\right)\right). \tag{10}$$

Thus $\boldsymbol{C}$ is Riordan and the k^{th} column generating function is $z^{k-1}\left(F^k(z)\right)$. This leads to our third problem which is to find the number of END walks of length n. By arguments similar to those given in the previous problems the END walks satisfy (9). Once again by the fundamental theorem, the closed form generating function for the total number of END walks is $\boldsymbol{C} \circledast (1/(1-z)) = 1/\left(1-2z-z^2\right)$. This is the same generating function as given by the solution of the second walk problem. This means the sets of END and EN$\overline{\text{W}}$ walks have the same number of walks, q_n (again, see (4)).

The natural question now is to find a bijection between the walks. This and several other bijections are the topics of the next section. We conclude this section by giving an algebraic interpretation of the array $\boldsymbol{C}$. We observe that the columns of $\boldsymbol{C}$ are the coefficients of the powers of the Fibonacci generating function

$$F(z) := 1/\left(1 - z - z^2\right) = 1 + z + 2z^2 + 3z^3 + 5z^4 + \cdots = \sum_{n\geq 0} F_{n+1} z^n$$

where $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0, F_1 = 1$, $n \geq 2$. The coefficients of the generating functions $F^k(z) = z^{k-1}/\left(1 - z - z^2\right)^k$ $(k \geq 1)$ are called the convolved Fibonacci numbers. An interesting related Riordan matrix is given by $\left(F^k(z), z(1+z)\right)$. We call this matrix a convolved Fibonacci array since the leftmost column equals the convolved Fibonacci numbers. See [2], [5], and [9] for more on these numbers.

3. BIJECTIONS

The Pell numbers appear in two other settings of walks called bounded grand Motzkin and modified Schröder walks. The set of bounded grand Motzkin walks, denoted by M_n, consists of walks bounded in the strip $[-1, 1]$ with $(1, 0) =$ H (horizontal), $(1, 1) =$ R (rise), and $(-1, 1) =$ F (fall) steps running from $(0, 0)$ to $(n, 0)$. The set of modified Schröder walks, denoted by ν_n, are made up of walks with rise, fall, and two-length horizontal steps (H $= (2, 0)$) that run from $(0, 0)$ to the line $x = n$ and do not end with a rise step. RHFFRH is denoted as an example of a bounded grand Motzkin walk of M_6 and RHFHF is denoted as an example of a modified Schröder walk of ν_7. Other examples of these walks are given in [3] as well as bijections among the sets of ENW, bounded grand Motzkin, and modified Schröder walks. Using certain pipe configurations we also construct bijections among these walks.

Here we recall the general heuristic for ordinary generating functions where we have the equation $a(z) = 1/(1 - b(z))$. This equation often means that $a(z)$ is the generating function for all configurations while $b(z)$ is the generating function of the connected components. For geometric reasons, we call our connected components pipes and we can set up bijections. Consider the following pipe configurations:

I. Bent down
II. Bent up
III. Straight

Bijections are then constructed between walks as illustrated in the table below where the symbol $(*)$ means the end of the word or that the next letter is N. The symbol $(\#)$ means the previous letter is not R.

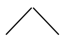
Lengths \ Walks	ENW	Motzkin	Schröder
1	N	H	F
2	NE, NW	RF, FR	H, RF
3	NE^2, NW^2	RHF, FHR	RH, R^{2}F
⋮	⋮	⋮	⋮
k	NE$^k(*)$, NW$^k(*)$	RH^{k-1}F, FH^{k-1}R	$(\#)$R^{k-1}H, $(\#)$R^{k}F

Table 1.

Proposition 3: *(Ferrari, et. al. [3]) For all $n \geq 0$, there is a bijection $\phi : P_n \to M_{n+1}$.*

Proof: (Sketch) Let P_n denote the set of ENW walks of length n counted by the array $\boldsymbol{P}$ (recall Section 2). A bijection ϕ between P_n and M_{n+1} is constructed using the above types of pipe configurations. An example should make this bijection easy to see. Consider the ENW walk denoted by EEENNEEENWW. We start by adjoining an N step at the beginning of the walk. Then we decompose the walk as NEEE·N·NEE ·NWW and make the following associations. Each maximal block NE^k, NW^k, and N^k forms a type I, II, and III pipe, respectively. For $k \geq 1$ we have the following rules.

Types	Rules
I	$\mathrm{NE}^k \leftrightarrow \mathrm{RH}^{k-1}\mathrm{F}$
II	$\mathrm{NW}^k \leftrightarrow \mathrm{FH}^{k-1}\mathrm{R}$
III	$\mathrm{N}^k \leftrightarrow \mathrm{F}^k$

The sharp bent pipes  and  are associated with NE and NW steps, respectively. For the above walk we have the following pipes:

NEEE $\leftrightarrow$
N $\leftrightarrow$
NEE $\leftrightarrow$
NWW $\leftrightarrow$

Connecting these pipes, according to the above blocks of steps, starting at the origin gives the bounded grand Motzkin walk denoted by RHHFHRHFFHR. By the way the pipes are configured, the walks always remain in the strip $[-1, 1]$. The bijection is easily reversed. □

Proposition 4: *(Ferrari, et. al. [3]) For all $n \geq 0$, there is a bijection $\psi : P_n \to \nu_{n+1}$.*

Proof: (Sketch) Following similar reasoning as given above a bijection ψ between P_n and ν_{n+1} is constructed according to the following pipe types and rules:

Types	Rules
I	$\mathrm{NE}^k \leftrightarrow \mathrm{R}^{k-1}\mathrm{H}$
II	$\mathrm{NW}^k \leftrightarrow \mathrm{R}^k\mathrm{F}$
III	$\mathrm{N}^k \leftrightarrow \mathrm{F}^k$

By the way the rules are defined, the constructed walks never end with an R step. Here we also note that an N step is adjoined at the beginning of the walk. Following these rules and considering the example given above, EEENNEEENWW is decomposed into NE^3·N·$\mathrm{NE}^2\mathrm{NW}^2$. The maximal blocks here give the walk denoted by RRH·F·RH· RF. This bijection is also easily reversed. □

We now consider ENW walks with the restriction that no walk begins with W. By symmetry, these walks are bijective with the EN$\overline{\mathrm{W}}$ walks and they are also counted by $\boldsymbol{Q}$. This leads to a bijection between walks counted by $\boldsymbol{Q}$ and $\boldsymbol{C}$.

Proposition 5: *For all $n \geq 0$, there is a bijection $\mu : C_n \to Q_{n+1}$.*

Proof: Let Q_n denote the set of ENW walks of length n that begin with E, and let C_n denote the set of END walks of length n. A bijection μ between walks counted by C_n and Q_{n+1} is illustrated by the following example. Consider the walk denoted by DNEENDE in $\boldsymbol{C}$. The bijection is then constructed as follows. Change all D steps to T and all E steps to C. The N steps remain unchanged. This leads to the word TNCCNTC. Here T means to turn back in the sense that we start by reading the word from left to right and change T to NW then the next T to NE and so on alternating between NW and NE. The letter C means to continue in the same direction in the sense of E or W. This means if we first read an E, change all the C's to E's until reading the first W. Upon reading the first W, now change all C's to W's until reading the next E. This process continues and alternates between E and W until all the C's are changed to E or W. Likewise if the first read for C is W then the process alternates between W and E. All of this leads to the next step of the construction. Now read TNCCNTC from left to right and change the first T to NW and the second T to NE. This gives NWNCCNNEC. To complete the construction, adjoin E to the beginning of the word and change the C's. This puts us in Q_{n+1}. For our example this gives the walk denoted by ENWNWWNNEE of the array $\boldsymbol{Q}$. The bijection is easily reversed after first observing the turn backs and determining which NE and NW pairs change to T. □

4. MATRIX RELATIONS

We find interesting matrix relations connecting $\boldsymbol{P}$ and $\boldsymbol{Q}$ to Pascal's matrix and certain Schröder matrices. We also find interesting relations connecting $\boldsymbol{C}$ to a generalized version of Pascal's matrix. All of this leads to connections with the Associated, Appell, and Bell subgroups of the Riordan group. The Bell subgroup consists of all pairs of the form $(g(z), zg(z))$. The Appell subgroup, which is normal, consists of all pairs of the form $(g(z), z)$. The Associated subgroup consists of all pairs of the form $(1, f(z))$.

As previously mentioned, the set of all Riordan matrices is a group under the operation of matrix multiplication. The group multiplication, denoted by "$*$" , is defined for two Riordan matrices $\boldsymbol{M} = (g(z), f(z))$ and $\boldsymbol{N} = (h(z), l(z))$ by $\boldsymbol{M} * \boldsymbol{N} = (g(z)\, h(f(z)), l(f(z)))$. This is proved by applying The Fundamental Theorem of the Riordan Group to $\boldsymbol{N}$, one column of $\boldsymbol{N}$ at a time. The inverse of $\boldsymbol{M}$ is $\boldsymbol{M}^{-1} = \left(1/g\left(\overline{f}(z)\right), \overline{f}(z)\right)$ where $\overline{f}(z)$ is the compositional inverse of $f(z)$.

Example 6: *Consider the matrix* $\boldsymbol{F} = (1, z(1+z))$ *with Fibonacci row sums. Then by the inverse formula* $\boldsymbol{F}^{-1} = (1, zC(-z))$ *where*

$$C(z) := \left(1 - \sqrt{1-4z}\right)/2z = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots = \sum_{n \geq 0} c_n z^n$$

is the Catalan generating function and $c_n = \frac{1}{1+n}\binom{2n}{n}$ *are the Catalan numbers. See [13] for more details. We note that both* $\boldsymbol{F}$ *and* $\boldsymbol{F}^{-1}$ *are elements of the Associated subgroup.*

Recall $\boldsymbol{P} = ((1+z)/(1-z), z(1+z)/(1-z))$ and this gives $\boldsymbol{P}^{-1} = (s_1(-z), zs_1(-z))$ where $s_1(-z) = \left((1+z) - \sqrt{1+6z+z^2}\right)/(-2z)$. We call this matrix a Schröder matrix since (surprisingly) the entries of the zeroth column are the "large" Schröder numbers $\{1, 2, 6, 22, 90, \dots\}$ with alternating signs. The companion sequence of the "large" Schröder numbers are the "little" Schröder numbers $\{1, 1, 3, 11, 45, 197, \dots\}$. See [8] and [10] for combinatorial interpretations of these numbers and [16] for a short history and summary. The Schröder generating functions are

$$s_1(z) := \left((1-z) - \sqrt{1-6z+z^2}\right)/2z = 1 + 2z + 6z^2 + 22z^3 + \cdots \text{ , and}$$

$$s_2(z) := \left((1+z) - \sqrt{1-6z+z^2}\right)/4z = 1 + z + 3z + 11z^2 + 45z^3 + \cdots .$$

We now consider Schröder-type matrices $\boldsymbol{S} = (s_2(-z), z)$ and $\boldsymbol{S}^{-1} = (1/s_2(-z), z)$. The matrices $\boldsymbol{S}$**,** $\boldsymbol{S}^{-1}$, and $\boldsymbol{P}^{-1}$ are all of interest, in part since their columns are closely related to the Schröder numbers. Here we note that both $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$ are elements of the Appell subgroup. We can also show the following relation.

Proposition 7: $\boldsymbol{P}$, $\boldsymbol{Q}$, *and* $\boldsymbol{S}$ *satisfy* $\boldsymbol{P} * \boldsymbol{S} = \boldsymbol{Q}$**.**

Proof: Since we are in the Riordan group verifications are done via composition of generating functions.

$$\boldsymbol{P} * \boldsymbol{S} = \left(\left(\frac{1+z}{1-z}\right) s_2\left(-z\left(\frac{1+z}{1-z}\right)\right), z\left(\frac{1+z}{1-z}\right)\right)$$

$$= \left(\left(\frac{1+z}{1-z}\right)\left(\frac{1-2z-z^2-\sqrt{(1+2z-z^2)^2}}{-4z(1+z)}\right), z\left(\frac{1+z}{1-z}\right)\right) = \boldsymbol{Q},$$

as desired. A more combinatorial proof would be of interest. □

Next we look at relations involving the Pascal matrices $\boldsymbol{P}^* = (1/(1-z), z/(1-z))$ and $(\boldsymbol{P}^*)^{-1} = (1/(1+z), z/(1+z))$. These matrices establish relations between $\boldsymbol{P}$ and $\boldsymbol{Q}$ and certain Schröder matrices. However before giving the matrix relations, we give a property for computing the diagonal sums of a Riordan matrix. We insert this property since it leads to connections to the Fibonacci numbers.

Definition 8: *The diagonal sums of a matrix are defined by*

$$d_k = m_{k,0} + m_{k-1,1} + m_{k-2,2} + \cdots + m_{2,k-2} + m_{1,k-1} + m_{0,k} = \sum_{j=0}^{k} m_{k-j,j}.$$

Lemma 9: *The generating function of the sequence* $\{d_k\}_{k\geq 0}$ *of diagonal sums of a Riordan matrix* $\boldsymbol{M} = (g(z), f(z))$ *is* $(m(z))_{diag} = g(z)/(1 - zf(z))$.

Proof: See Nkwanta [7]. □

Proposition 10: *The diagonal sums of* $\boldsymbol{C}$ *are given by the sequence* $\{1, 1, 3, 5, 11, \dots, r_n, \dots\}$ *with* $r_{n+1} = r_n + 2r_{n-1}$.

Proof: By Lemma 9

$$(c(z))_{diag} = F(z)/\left(1 - z^2 F(z)\right) = 1/\left(1 - z - 2z^2\right) = 1 + z + 3z^2 + 5z^3 + \cdots .$$

See [15] (sequence M2482) for interpretations of the sequence $\{r_n\}_{n\geq 0}$. □

Now we see what happens if we multiply $\boldsymbol{P}$ and $\boldsymbol{Q}$ by $\boldsymbol{P}^*$. We get

$$\boldsymbol{F}_1 = \boldsymbol{P}^* * \boldsymbol{P} = \left(\frac{1}{1-3z+2z^2}, \frac{z}{1-3z+2z^2}\right) \text{ and}$$

$$\boldsymbol{F}_2 = \boldsymbol{P}^* * \boldsymbol{Q} = \left(\frac{1}{1-2z}, \frac{z}{1-3z+2z^2}\right).$$

This leads to more Fibonacci results.

Proposition 11: *The diagonal sums of* $\boldsymbol{F}_1$ *and* $\boldsymbol{F}_2$ *are* $\{F_{2n+2}\}_{n\geq 0} = \{1,3,8,21,55,\dots\}$ *and* $\{F_{2n+1}\}_{n\geq 0} = \{1,2,5,13,34,\dots\}$, *respectively.*

Proof: Use Lemma 9 to get the corresponding generating functions. □

We can interpret the entries of $\boldsymbol{F}_2$ as follows. Take the walks counted by $\boldsymbol{Q}$ in Section 2 but now allow level steps to be colored a second color, say red, if they are in the "prevailing direction." By prevailing direction we mean the same direction as the last previous E or W as we read from left to right. The red level steps are denoted R_E for prevailing direction E and R_W for prevailing direction W. If only N steps have occurred when we want to insert a red level step, then we require the new red step to be R_E. For instance the five walks of length 2 of height 1 of $\boldsymbol{F}_2$ are EN, R_EN, NE, NR_E, and WN. We don't get R_WN or NR_W since with no prevailing direction we must insert R_E. Thus, we obtained a new kind of level step. We note that it is possible for $\boldsymbol{F}_2$ to include in its' count walks ending with R_W though not with W.

It is easy to show that left multiplication by Pascal's matrix $\boldsymbol{P}^*$ introduces a new kind of level step for the type of walks counted by $\boldsymbol{Q}$. This also holds for walks counted by $\boldsymbol{P}$ and thus leads to a similar interpretation for the entries of $\boldsymbol{F}_1$. In general, left multiplication by $\boldsymbol{P}^*$ leads to one more kind of level step for a variety of lattice walk counts.

Finally we want to see what happens when $\boldsymbol{C}$ is multiplied by the generalized Pascal matrix

$$(\boldsymbol{P}^*)^k = (1/(1-kz), z/(1-kz)). \tag{11}$$

This leads to general forms of (10).

Proposition 12: *For* $k \geq 0$

$$\text{(a)}\ \ (\boldsymbol{C}_L)^k = (\boldsymbol{P}^*)^k * \boldsymbol{C} = \left(\frac{(1-z)^k}{(1-z)^{2k} - z(1-z)^k - z^2}, \frac{z(1-z)^k}{(1-z)^{2k} - z(1-z)^k - z^2}\right)$$

$$\text{(b)}\ \ (\boldsymbol{C}_R)^k = \boldsymbol{C} * (\boldsymbol{P}^*)^k = \left(\frac{1}{1-(k+1)z-z^2}, \frac{z}{1-(k+1)z-z^2}\right).$$

Proof: Use matrix multiplication and simplify. □

A walk interpretation for $(\boldsymbol{C}_L)^k$ can be given by allowing k additional different kinds of level steps for the walks defined by $\boldsymbol{C}$. In this case repeated left multiplication of $\boldsymbol{C}$ by $\boldsymbol{P}^*$ introduces k new kinds of level steps, say using k different colors.

We conclude with mentioning that $\boldsymbol{F}_1$, $\boldsymbol{P}$, $\boldsymbol{P}^*$, $(\boldsymbol{P}^*)^k$, $\boldsymbol{C}$, $(\boldsymbol{C}_R)^k$, and $(\boldsymbol{C}_L)^k$ are all elements of the Bell subgroup. Also, the various sequences mentioned in this paper can be found in [15].

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