The sum and product of Fibonacci numbers and Lucas numbers, Pell numbers and Pell-Lucas numbers representation by matrix method

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Abstract: Denote by $\{F_n\}$ and $\{L_n\}$ the Fibonacci numbers and Lucas numbers, respectively. Let $\mathcal{F}_n = F_n \times L_n$ and $\mathcal{L}_n = F_n + L_n$. Denote by $\{P_n\}$ and $\{Q_n\}$ the Pell numbers and Pell-Lucas numbers, respectively. Let $\mathcal{P}_n = P_n \times Q_n$ and $\mathcal{Q}_n = P_n + Q_n$. In this paper, we give some determinants and permanent representations of $\mathcal{P}_n$, $\mathcal{Q}_n$, $\mathcal{F}_n$ and $\mathcal{L}_n$. Also, complex factorization formulas for those numbers are presented.

Key–Words: Fibonacci number; Lucas number; Pell numbers; Pell-Lucas number; matrix.

1. Introduction

The Fibonacci and Lucas sequences are defined by the following recurrence relations:

$$F_{n+2} = F_{n+1} + F_n \text{ where } F_0 = 0, F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n \text{ where } L_0 = 2, L_1 = 1.$$

The Pell and Pell-Lucas sequences are defined by the following recurrence relations, respectively:

$$P_{n+2} = 2P_{n+1} + P_n \text{ where } P_0 = 0, P_1 = 1;$$

$$Q_{n+2} = 2Q_{n+1} + Q_n \text{ where } Q_0 = 2, Q_1 = 2.$$

Let $A = [a_{i,j}]$ be an $n \times n$ matrix. The permanent of A is defined by

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where the summation extends over all permutations σ of the symmetric group S_n [15].

It is known that there are lots of relations between determinants or permanents of matrices and the famous sequences. By the determinant of tridiagonal matrix, an identity of Fibonacci number is proved [5]. Demirtürk [4] derived some Fibonacci and Lucas sums by matrix method. In [9], the authors present a result involving the permanent of an $(-1, 0, 1)$-matrix and the Fibonacci number F_{n+1}. Kilic and Stakhov [12] considered certain generalizations of the well-known Fibonacci and Lucas numbers, the generalized Fibonacci and Lucas p-numbers. They also determined certain matrices whose permanents generate the Lucas p-numbers and their sums.

By the determinant of tridiagonal matrix, an identity of Pell identities is presented [19]. Yilmaz and Bozkurt [20] derived some relationships between Pell sequence and permanents and determinants of one type of Hessenberg matrices. Li [14] gave another proofs of two results relative to the Pell and Perrin numbers by constructing new Fibonacci-Hessenberg matrices. Fu and Zhou [6] derived the relation between Pell numbers and its companion sequence by matrix representations of them. Gulec and Taskara [7] gave new generalizations for (s, t)- Pell and (s, t)-Pell-Lucas sequences for Pell and Pell-Lucas numbers, and defined the matrix sequences which have elements of them and investigated their properties. Kilic [11] gave the definition of generalized Pell $(p, i)-$numbers and then presented their generating matrix. He obtained relationships between the generalized Pell $(p, i)-$numbers and their sums and permanents of certain matrices. Civciv and Türkmen [1] defined a new matrix generalization of the Lucas numbers, and using essentially a matrix approach they showed properties of these matrix sequences.

A. Nalli [17] present a family of tridiagonal matrices given by

$$A_n^1 = \begin{bmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3 \end{bmatrix}_{n\times n},$$

such that the determinant $|A_n^1|$ is the Fibonacci number F_{2n+2}. Another example in [2] is the family of tridiagonal matrices given by:

$$\begin{bmatrix} 1 & i & 0 & \cdots & 0 & 0 \\ i & 1 & i & 0 & \cdots & 0 \\ 0 & i & 1 & i & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i & 1 & i \\ 0 & 0 & \cdots & 0 & i & 1 \end{bmatrix}_{n\times n}.$$

Strang [18] presents the tridiagonal matrix

$$A_n^2 = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}_{n\times n},$$

the determinant $|A_n^2|$ is the Fibonacci number F_{n+1}. It can be checked that

$$F_{n+1} = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ & 0 & 1 & 1 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{vmatrix}_{n\times n},$$

$$P_{n+1} = \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & 0 \\ & 0 & 1 & 2 & 1 \\ 0 & \cdots & 0 & 1 & 2 \end{vmatrix}_{n\times n},$$

$$L_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ -2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{vmatrix}_{n\times n}.$$

The Jacobsthal and Jacobsthal-Lucas sequences are defined by the following recurrence relations, respectively:

$$J_{n+2} = J_{n+1} + 2J_n \text{ where } J_0 = 0, J_1 = 1;$$

$$j_{n+2} = j_{n+1} + 2j_n \text{ where } j_0 = 2,\ j_1 = 1.$$

Let

$$A_n^3 = \begin{bmatrix} 3 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 2 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 2 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

be an n-square matrix.

F. Yilmaz and D. Bozkurt [21] showed that

$$\text{per } A_n^3 = J_{n+2}$$

Let

$$A_n^4 = \begin{bmatrix} 1 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 3 & 2 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 2 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 2 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

be an n-square matrix.

F. Yilmaz and D. Bozkurt [21] showed that

$$\text{per } A_n^4 = j_n.$$

Also, they gave a complex factorization formulas for Jacobsthal numbers. That is

$$J_n = \prod_{k=1}^{n} (1 + 2\sqrt{2} i \cos \frac{k\pi}{n+1}).$$

Let

$$A_n^5 = \begin{bmatrix} 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

be an n-square matrix. F. Yilmaz and D. Bozkurt [20] showed that

$$\text{per } A_n^5 = P_n.$$

N. D. Cahill, D. A. Narayan [3] showed that the symmetric tridiagonal family of matrices $M_{a,b}(k)(k = 1, 2, \cdots)$, whose elements are given by:

$$m_{1,1} = F_{a+b}, m_{2,2} = \left\lceil \frac{F_{2a+b}}{F_{a+b}} \right\rceil,$$

$$m_{i,i} = L_a, 3 \le i \le k,$$

$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}F_{a+b} - F_{2a+b}},$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^a}, 2 \le j < k,$$

where $a \in Z^+$ and $b \in N$, has determinants

$$|M_{a,b}(k)| = F_{ak+b}.$$

For example,

$$F_{4k-2} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 8 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 7 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 7 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 7 \end{vmatrix}_{k\times k},$$

$$F_{3k+3} = \begin{vmatrix} 8 & \sqrt{6} & 0 & \cdots & 0 & 0 \\ \sqrt{6} & 5 & i & 0 & \cdots & 0 \\ 0 & i & 4 & i & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i & 4 & i \\ 0 & 0 & \cdots & 0 & i & 4 \end{vmatrix}_{k\times k},$$

and

$$F_{2k+5} = \begin{vmatrix} 13 & -\sqrt{5} & 0 & \cdots & 0 & 0 \\ -\sqrt{5} & 3 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 3 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{vmatrix}_{k\times k}.$$

In the same paper, they proved that the symmetric tridiagonal family of matrices $T_{a,b}(k)(k = 1, 2, \cdots)$, whose elements are given by:

$$t_{1,1} = L_{a+b}, t_{2,2} = \left\lceil \frac{L_{2a+b}}{L_{a+b}} \right\rceil,$$

$$t_{i,i} = L_a, 3 \le i \le k,$$

$$t_{1,2} = t_{2,1} = \sqrt{t_{2,2}L_{a+b} - L_{2a+b}},$$

$$t_{j,j+1} = t_{j+1,j} = \sqrt{(-1)^a}, 2 \le j < k,$$

where $a \in Z^+$ and $b \in N$, has determinants

$$|T_{a,b}(k)| = L_{ak+b}.$$

For example,

$$L_{4k-2} = \begin{vmatrix} 3 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 6 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 7 & -1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 7 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 7 \end{vmatrix}_{k\times k},$$

$$L_{3k+3} = \begin{vmatrix} 18 & \sqrt{14} & 0 & \cdots & 0 & 0 \\ \sqrt{14} & 5 & i & 0 & \cdots & 0 \\ 0 & i & 4 & i & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i & 4 & i \\ 0 & 0 & \cdots & 0 & i & 4 \end{vmatrix}_{k\times k},$$

and

$$L_{2k+5} = \begin{vmatrix} 29 & \sqrt{11} & 0 & \cdots & 0 & 0 \\ \sqrt{11} & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3 \end{vmatrix}_{k\times k}.$$

N. D. Cahill and D. A. Narayan [3] also derived the following factorization

$$F_{2mn} = F_{2m} \prod_{k=1}^{n-1} (L_{2m} - 2cos\frac{k\pi}{n}).$$

A. Nalli and H. Civciv [17] generalized the result of N. D. Cahill and D. A. Narayan [3]. The tridiagonal family of matrices $M_{-a,-b}(k), k = 1, 2, \cdots$ whose elements are given by:

$$m_{1,1} = F_{-a-b}, m_{2,2} = \left\lceil \frac{F_{-2a-b}}{F_{-a-b}} \right\rceil,$$

$$m_{i,i} = L_{-a}, 3 \le i \le k,$$

$$m_{1,2} = m_{2,1} = \sqrt{m_{2,2}F_{-a-b} - F_{-2a-b}},$$

$$m_{j,j+1} = m_{j+1,j} = \sqrt{(-1)^a}, 2 \le j < k,$$

where $a \in Z^+$ and $b \in N$. Then
(1) if a and b are odd, then

$$\det(M_{-a,-b}(k)) = \begin{cases} -F_{ak+b} & k \text{ is odd} \\ F_{ak+b} & k \text{ is even} \end{cases};$$

(2) if a is odd and b is even, then

$$\det(M_{-a,-b}(k)) = \begin{cases} F_{ak+b} & k \text{ is odd} \\ -F_{ak+b} & k \text{ is even} \end{cases};$$

(3) if a and b are even, then

$$\det(M_{-a,-b}(k)) = -F_{ak+b};$$

(4) if a is even and b is odd, then

$$\det(M_{-a,-b}(k)) = F_{ak+b}.$$

For example,

$$M_{-4,-2}(k) = \begin{bmatrix} -8 & i & 0 & \cdots & 0 & 0 \\ i & 7 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 7 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 7 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 7 \end{bmatrix}_{k\times k},$$

$|M_{-4,-2}(1)| = -F_6$,
$|M_{-4,-2}(2)| = -F_{10}$,
$|M_{-4,-2}(3)| = -F_{14}$,
$|M_{-4,-2}(4)| = -F_{18}$,
$|M_{-4,-2}(k)| = -F_{4k+2}$.

$$M_{-1,-1}(k) = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -2 & i & 0 & \cdots & 0 \\ 0 & i & -1 & i & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i & -1 & i \\ 0 & 0 & \cdots & 0 & i & -1 \end{bmatrix}_{k\times k},$$

$|M_{-1,-1}(1)| = -F_2$,
$|M_{-1,-1}(2)| = -F_3$,
$|M_{-1,-1}(3)| = -F_4$,
$|M_{-1,-1}(4)| = -F_5$,
$|M_{-1,-1}(k)| = \begin{cases} -F_{k+1} & k \text{ is odd} \\ F_{k+1} & k \text{ is even} \end{cases}$.

Also, they proved if the symmetric tridiagonal family of matrices $T_{-a,-b}(k), k = 1, 2, \cdots$ whose elements are given by:

$$t_{1,1} = L_{-a-b}, t_{2,2} = \left\lceil \frac{L_{-2a-b}}{L_{-a-b}} \right\rceil,$$

$$t_{i,i} = L_{-a}, 3 \le i \le k,$$

$$t_{1,2} = t_{2,1} = \sqrt{t_{2,2}L_{-a-b} - L_{-2a-b}},$$

$$t_{j,j+1} = t_{j+1,j} = \sqrt{(-1)^a}, 2 \le j < k,$$

where $a \in Z^+$ and $b \in N$. Then
(1) if a and b are odd, then

$$\det(T_{-a,-b}(k)) = \begin{cases} L_{ak+b} & k \text{ is odd} \\ -L_{ak+b} & k \text{ is even} \end{cases};$$

(2) if a is odd and b is even, then

$$\det(T_{-a,-b}(k)) = \begin{cases} -L_{ak+b} & k \text{ is odd} \\ L_{ak+b} & k \text{ is even} \end{cases};$$

(3) if a and b are even, then

$$\det(T_{-a,-b}(k)) = L_{ak+b};$$

(4) if a is even and b is odd, then

$$\det(T_{-a,-b}(k)) = -L_{ak+b}.$$

For example,

$$T_{-3,-4}(k) = \begin{bmatrix} -29 & 6i & 0 & \cdots & 0 & 0 \\ 6i & -3 & i & 0 & \cdots & 0 \\ 0 & i & -4 & i & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & i & -4 & i \\ 0 & 0 & \cdots & 0 & i & -4 \end{bmatrix}_{k\times k},$$

$|T_{-3,-4}(1)| = -L_7$,
$|T_{-3,-4}(2)| = L_{10}$,
$|T_{-3,-4}(3)| = -L_{13}$,
$|T_{-3,-4}(4)| = L_{16}$,
$|T_{-3,-4}(k)| = \begin{cases} -L_{3k+4} & k \text{ is odd} \\ L_{3k+4} & k \text{ is even} \end{cases}$.

$$T_{-2,-5}(k) = \begin{bmatrix} -29 & \sqrt{11}i & 0 & \cdots & 0 & 0 \\ \sqrt{11}i & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3 \end{bmatrix}_{k\times k},$$

$|T_{-2,-5}(1)| = -L_7$,
$|T_{-2,-5}(2)| = -L_9$,
$|T_{-2,-5}(3)| = -L_{11}$,
$|T_{-2,-5}(4)| = -L_{13}$,
$|T_{-2,-5}(k)| = -L_{2k+5}$.

Let $A = [a_{i,j}]$ be an $n \times n$ matrix with row vectors $r_1, r_2, \cdots, r_n$. Suppose column k contains exactly two nonzero elements $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(n-1)\times(n-1)$ matrix $A_{ij:k}$ obtained from A by replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j, and we say A is contractible on column k. Similarly, if row k contains exactly two nonzero elements, $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A^T_{ij:k}]^T$ is called the contraction of A on row k relative to columns i and j. If A is a nonnegative matrix and B is a contraction of A, then per A = per B [10].

A directed pseudo graph $G = (V, E)$, with set of vertices $V(G) = \{1, 2, \cdots, n\}$ and set of edges

n	0	1	2	3	4
$\mathcal{F}_n$	0	1	3	8	21
$\mathcal{L}_n$	2	2	4	6	10
$\mathcal{P}_n$	0	2	12	70	408
$\mathcal{Q}_n$	2	3	8	19	46

Table 1: The first few values of the sequences.

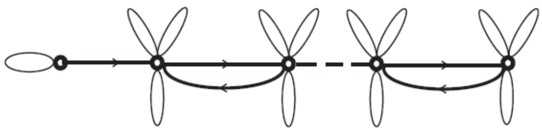

Figure 1.

$E(G) = \{e_1, e_2, \cdots, e_m\}$, is a graph in which loops and multiple edges are allowed. A directed graph represented with arrows on its edges, each arrow pointing towards the head of the corresponding arc. The adjacency matrix $A(G) = [a_{i,j}]$ is $n \times n$ matrix, defined by the rows and the columns of $A(G)$ are indexed by $V(G)$, in which $a_{i,j}$ is the number of edges jointing v_i and v_j [13].

Let $\mathcal{F}_n = F_n \times L_n$ and $\mathcal{L}_n = F_n + L_n$. Then we can get the following recurrence relations:

$$\mathcal{F}_{n+2} = 3\mathcal{F}_{n+1} - \mathcal{F}_n \text{ where } \mathcal{F}_0 = 0, \mathcal{F}_1 = 1;$$

$$\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n \text{ where } \mathcal{L}_0 = 2,\ \mathcal{L}_1 = 2.$$

The graph gotten by join a single vertex to every vertices of a path is called a fan. By [16], the number of spanning tree of a fan with $n+1$ vertices is $\mathcal{F}_n$.

Let $\mathcal{P}_n = P_n \times Q_n$ and $\mathcal{Q}_n = P_n + Q_n$. Then we can get the following recurrence relations:

$$\mathcal{P}_{n+2} = 6\mathcal{P}_{n+1} - \mathcal{P}_n \text{ where } \mathcal{P}_0 = 0, \mathcal{P}_1 = 2;$$

$$\mathcal{Q}_{n+2} = 2\mathcal{Q}_{n+1} + \mathcal{Q}_n \text{ where } \mathcal{Q}_0 = 2,\ \mathcal{Q}_1 = 3.$$

The first few values of the sequences are in Table 1.

In this paper, we investigate relationships between adjacency matrices of graphs and the $\mathcal{F}_n$, the $\mathcal{L}_n$, the $\mathcal{P}_n$ and the $\mathcal{Q}_n$ sequences. We also give complex factorization formulas for the numbers.

1 Determinant representations of $\mathcal{F}_n$ and $\mathcal{L}_n$

In this section, we consider a class of pseudo graph given in Figure 1 and Figure 2, respectively. Then we investigate relationships between permanents of the adjacency matrices of the graphs and $\mathcal{F}_n$ and $\mathcal{L}_n$.

Let $A_n = [a_{ij}]_{n\times n}$ be the adjacency matrix of the graph given by Figure 1, in which

$$a_{11} = a_{t,t+1} = 1, a_{ss} = 3, a_{l,l-1} = 1$$

for $t = 1, 2, \cdots, n-1$, $s = 2, 3, \cdots, n$ and $l = 3, 4, \cdots, n$ and otherwise 0. That is

$$A_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3 \end{bmatrix}.$$

Let S_n be a $(1, -1)$ matrix of order n, defined as

$$S_n = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 \end{bmatrix}.$$

Denote the matrices $A_n \circ S_n$ by H_n, where $A_n \circ S_n$ denotes Hadamard product of A_n and S_n. Thus

$$H_n = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 3 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}. \quad (1)$$

Theorem 1 *Let H_n be an matrix as in (1). Then*

$$per\ H_n = per\ H_n^{(n-2)} = \mathcal{F}_n.$$

Proof. By definition of the matrix H_n, it can be contracted on column 1. Let $H_n^{(r)}$ be the rth contraction of H_n. If $r = 1$, then

$$H_n^{(1)} = \begin{bmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}.$$

Note that $H_n^{(1)}$ also can be contracted by the first column, then

$$H_n^{(2)} = \begin{bmatrix} 8 & 3 & 0 & \cdots & 0 & 0 \\ -1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}.$$

Figure 2.

Similarly,

$$H_n^{(3)} = \begin{bmatrix} 21 & 8 & 0 & \cdots & 0 & 0 \\ -1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 3 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 3 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 3 \end{bmatrix}.$$

Going with this induction process, we have

$$H_n^{(n-2)} = \begin{bmatrix} \mathcal{F}_{n-1} & \mathcal{F}_{n-2} \\ -1 & 3 \end{bmatrix},$$

so

$$\text{per } H_n = \text{per } H_n^{(n-2)} = \mathcal{F}_n.$$

□

Let $K_n = [k_{ij}]_{n\times n}$ be the adjacency matrix of the pseudo graph given in Figure 2, with

$$k_{11} = k_{12} = k_{t,t-1} = 2, k_{ss} = 1, k_{l,l+1} = 1$$

for $t = 2, \cdots, n$, $s = 2, 3, \cdots, n$ and $l = 2, 3, \cdots, n$, and otherwise 0. That is

$$K_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}. \quad (2)$$

Theorem 2 *Let K_n be a matrix as in (2). Then per K_n = per $K_n^{(n-2)} = \mathcal{L}_n$.*

Proof. By definition of the matrix K_n, it can be contracted on column 1. Let $K_n^{(r)}$ be the rth contraction of K_n. If $r = 1$, then

$$K_n^{(1)} = \begin{bmatrix} 4 & 2 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Note that $K_n^{(1)}$ also can be contracted by the first column, then

$$K_n^{(2)} = \begin{bmatrix} 6 & 4 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Similarly,

$$K_n^{(3)} = \begin{bmatrix} 10 & 6 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}.$$

Going with this induction process, we have

$$K_n^{(n-2)} = \begin{bmatrix} \mathcal{L}_{n-1} & \mathcal{L}_{n-2} \\ 1 & 1 \end{bmatrix},$$

so

$$\text{per } K_n = \text{per } K_n^{(n-2)} = \mathcal{L}_n.$$

□

Denote the matrices $K_n \circ S_n$ by B_n. That is

$$B_n = \begin{bmatrix} 2 & 2 & 0 & \cdots & 0 & 0 \\ -2 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Then we have

$$\det A_n = \text{per } H_n = \mathcal{F}_n$$

and

$$\det B_n = \text{per } K_n = \mathcal{L}_n.$$

Let C_{n+1} be an $(n+1) \times (n+1)$ matrix defined as

$$C_{n+1} = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

Similar to Theorem 2, it can be proofed that

$$\text{per } (C_{n+1} \circ S_{n+1}) = \det C_{n+1} = \mathcal{L}_n.$$

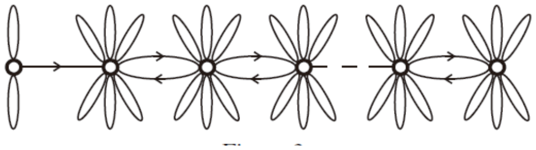

Figure 3.

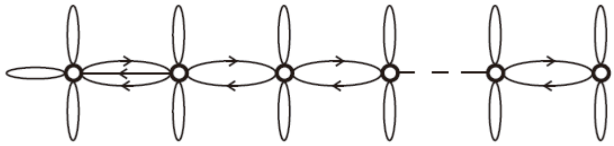

Figure 4.

2 Determinant representations of $\mathcal{P}_n$ and $\mathcal{Q}_n$

In this section, we consider a class of pseudo graph given in Figure 1 and Figure 2, respectively. Then we investigate relationships between permanents of the adjacency matrices of the graphs and $\mathcal{P}_n$ and $\mathcal{Q}_n$.

Let $A'_n = [a_{ij}]_{n\times n}$ be the adjacency matrix of the graph given by Figure 1, in which

$$a'_{11} = 2, a'_{tt+1} = a'_{ll-1} = 1, a'_{ss} = 6,$$

for $t = 1, 2, \cdots, n-1$, $s = 2, 3, \cdots, n$ and $l = 3, 4, \cdots, n$ and otherwise 0. That is

$$A'_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 6 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 6 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 6 \end{bmatrix}.$$

Denote the matrices $A'_n \circ S_n$ by U_n, where $U_n \circ S_n$ denotes Hadamard product of U_n and S_n. Thus

$$U_n = \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 6 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 6 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 6 \end{bmatrix}. \quad (3)$$

Theorem 3 *Let U_n be an matrix as in (3). Then per U_n = per $U_n^{(n-2)} = \mathcal{P}_n$.*

Proof. By definition of the matrix U_n, it can be contracted on column 1. Let $U_n^{(r)}$ be the rth contraction of U_n. If $r = 1$, then

$$U_n^{(1)} = \begin{bmatrix} 12 & 2 & 0 & \cdots & 0 & 0 \\ -1 & 6 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 6 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 6 \end{bmatrix}.$$

Note that $U_n^{(1)}$ also can be contracted by the first column, then

$$U_n^{(2)} = \begin{bmatrix} 70 & 12 & 0 & \cdots & 0 & 0 \\ -1 & 6 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 6 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 6 \end{bmatrix}.$$

Similarly,

$$U_n^{(3)} = \begin{bmatrix} 408 & 70 & 0 & \cdots & 0 & 0 \\ -1 & 6 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 6 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 6 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 6 \end{bmatrix}.$$

Going with this induction process, we have

$$U_n^{(n-2)} = \begin{bmatrix} \mathcal{P}_{n-1} & \mathcal{P}_{n-2} \\ -1 & 6 \end{bmatrix},$$

so

$$\text{per } U_n = \text{per } U_n^{(n-2)} = \mathcal{P}_n$$

. □

Let $V_n = [v_{ij}]_{n\times n}$ be the adjacency matrix of the pseudo graph given in Figure 2, with

$$v_{11} = 3, v_{21} = v_{ss} = 2, v_{tt-1} = v_{ll+1} = 1$$

for

$$t = 3, \cdots, n, s = 2, 3, \cdots, n$$

and

$$l = 1, 2, \cdots, n-1,$$

and otherwise 0. That is

$$V_n = \begin{bmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ 2 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}. \quad (4)$$

Theorem 4 *Let V_n be an matrix as in (4). Then per V_n = per $V_n^{(n-2)} = \mathcal{Q}_n$.*

Proof. By definition of the matrix V_n, it can be contracted on column 1. Let $V_n^{(r)}$ be the rth contraction of V_n. If $r = 1$, then

$$V_n^{(1)} = \begin{bmatrix} 8 & 3 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

Note that $V_n^{(1)}$ also can be contracted by the first column, then

$$V_n^{(2)} = \begin{bmatrix} 19 & 8 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

Similarly,

$$V_n^{(3)} = \begin{bmatrix} 46 & 19 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 2 \end{bmatrix}.$$

Going with this induction process, we have

$$V_n^{(n-2)} = \begin{bmatrix} \mathcal{Q}_{n-1} & \mathcal{Q}_{n-2} \\ 1 & 2 \end{bmatrix},$$

so

$$\text{per } V_n = \text{per } V_n^{(n-2)} = \mathcal{Q}_n.$$

□

Denote the matrices $V_n \circ S_n$ by B'_n. That is

$$B'_n = \begin{bmatrix} 3 & 1 & 0 & \cdots & 0 & 0 \\ -2 & 2 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

Then we have

$$\text{det}A'_n = \text{per } U_n = \mathcal{P}_n$$

and

$$\det B'_n = \text{per } V_n = \mathcal{Q}_n.$$

Let C'_{n+1} be an $(n+1) \times (n+1)$ matrix defined as

$$C'_{n+1} = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 3/2 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 2 & 1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

Similar to Theorem 2, it can be proofed that

$$\text{per } (C'_{n+1} \circ S_{n+1}) = \det C'_{n+1} = \mathcal{Q}_n.$$

3 Complex factorization formulas

In this section, we give complex factorization formulas for $\mathcal{F}_n$, $\mathcal{L}_n$ and $\mathcal{P}_n$.

Theorem 5 $\mathcal{F}_n = \prod_{k=1}^{n-1}(3 + \cos\frac{k\pi}{n})$.

Proof. The characteristic equation of A_n is

$$0 = \det(A_n - \lambda I)$$

$$= \begin{vmatrix} 1-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 3-\lambda & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3-\lambda & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 3-\lambda & 1 \\ 0 & 0 & \cdots & 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 3-\lambda & 1 & & & \\ 1 & 3-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3-\lambda & 1 \\ & & & 1 & 3-\lambda \end{vmatrix},$$

here I is the identity matrix. By [22], the eigenvalues of the matrix

$$\begin{bmatrix} 3-\lambda & 1 & & & \\ 1 & 3-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 3-\lambda & 1 \\ & & & 1 & 3-\lambda \end{bmatrix}$$

are $3 + \cos\frac{k\pi}{n} (k = 1, 2, \cdots, n-1)$.
So the result follows. □

Theorem 6 $\mathcal{L}_n = 2\prod_{k=1}^{n}(1 + i\cos\frac{k\pi}{n+1})$.

Proof. The characteristic equation of C_{n+1} is

$$0 = \det(C_{n+1} - \lambda I)$$

$$= \begin{vmatrix} 2-\lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1-\lambda & 1 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1-\lambda & 1 \\ 0 & 0 & \cdots & 0 & -1 & 1-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 1-\lambda & 1 & & & \\ -1 & 1-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1-\lambda & 1 \\ & & & -1 & 1-\lambda \end{vmatrix}.$$

By [22], the eigenvalues of the matrix

$$\begin{bmatrix} 1-\lambda & 1 & & & \\ -1 & 1-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1-\lambda & 1 \\ & & & -1 & 1-\lambda \end{bmatrix}$$

are $1 + i\cos\frac{k\pi}{n+1}(k = 1, 2, \cdots, n)$. So the result follows. □

Then, we give complex factorization formulas for $\mathcal{P}_n$.

Theorem 7 $\mathcal{P}_n = 2\prod_{k=1}^{n-1}(6 + \cos\frac{k\pi}{n})$.

Proof. The characteristic equation of A'_n is

$$0 = \det(A'_n - \lambda I)$$

$$= \begin{vmatrix} 2-\lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & 6-\lambda & 1 & 0 & \cdots & 0 \\ 0 & 1 & 6-\lambda & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 6-\lambda & 1 \\ 0 & 0 & \cdots & 0 & 1 & 6-\lambda \end{vmatrix}$$

$$= (2-\lambda) \begin{vmatrix} 6-\lambda & 1 & & & \\ 1 & 6-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6-\lambda & 1 \\ & & & 1 & 6-\lambda \end{vmatrix},$$

here I is the identity matrix. By [22], the eigenvalues of the matrix

$$\begin{bmatrix} 6-\lambda & 1 & & & \\ 1 & 6-\lambda & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 6-\lambda & 1 \\ & & & 1 & 6-\lambda \end{bmatrix}$$

are $6 + \cos\frac{k\pi}{n}(k = 1, 2, \cdots, n-1)$.
So the result follows. □

4 Conclusion

We give some determinant and permanent representations of $\mathcal{F}_n, \mathcal{L}_n, \mathcal{P}_n$ and $\mathcal{Q}_n$ and complex factorization formulas for $\mathcal{F}_n, \mathcal{L}_n$ and $\mathcal{P}_n$. So, it is natural to ask the question: what is the complex factorization formula for $\mathcal{Q}_n$?

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