PELL POLYNOMIALS AND A CONJECTURE OF MAHON AND HORADAM

PAUL DUVALL and THERESA VAUGHAN
University of North Carolina at Greensboro, NC 27412

(Submitted December 1986)

1. INTRODUCTION

In [1], Horadam and Mahon define a family of $n \times n$ matrices V_n in connection with the Pell polynomials $U_n(x)$. They conjecture that the characteristic polynomial of V_n is given by

$$C_n(\lambda) = \sum_{k=0}^{n} (-1)^{(k^2+k)/2}\{n,\ k\}\lambda^{n-k}, \tag{1.1}$$

where

$$\{n,\ k\} = \prod_{i=1}^{n} U_i(x) \Big/ \prod_{i=1}^{k} U_i(x) \prod_{i=1}^{n-k} U_i(x). \tag{1.2}$$

In this paper we prove the conjecture of Horadam and Mahon and also derive various other results concerning the structure of V_n and $C_n(\lambda)$.

2. NOTATION

The Pell polynomials are defined recursively by

$$U_0(x) = 0, \quad U_1(x) = 1,$$
$$U_n(x) = 2xU_{n-1}(x) + U_{n-2}(x) \qquad (n \geqslant 2)$$

and the associated Pell-Lucas polynomials by

$$W_0(x) = 2, \quad W_1(x) = 2x,$$
$$W_n(x) = 2xW_{n-1}(x) + W_{n-2}(x) \qquad (n \geqslant 2).$$

In this paper, to keep the notation as simple as possible, we shall work with the following closely related polynomials in the indeterminate t:

$$P_0(t) = 0, \quad P_1(t) = 1,$$
$$P_n(t) = tP_{n-1}(t) + P_{n-2}(t) \qquad (n \geqslant 2)$$

and

$$Q_0(t) = 2, \quad Q_1(t) = t,$$
$$Q_n(t) = tQ_{n-1}(t) + Q_{n-2}(t) \qquad (n \geqslant 2).$$

Standard manipulations with difference equations give the *Binet formulas*:

$$P_n(t) = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \text{and} \quad Q_n(t) = \alpha^n + \beta^n,$$

where α, β are the roots of the polynomial $y^2 - ty - 1$;

$$= \frac{1}{2}[t + \sqrt{t^2 + 4}] \quad \text{and} \quad = \frac{1}{2}[t - \sqrt{t^2 + 4}].$$

We shall require the easily proven identity

$$P_n(t) = \sum_{k=1}^{[n/2]} \binom{n - k - 1}{k} t^{n-1-2k}. \tag{2.1}$$

V_n is defined to be the $n \times n$ matrix whose (i, j) entry is

$$(V_n)_{ij} = \binom{j - 1}{j + i - n - 1} t^{i+j-n-1},$$

for example,

$$V_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3t \\ 0 & 1 & 2t & 3t^2 \\ 1 & t & t^2 & t^3 \end{bmatrix}.$$

3. A SIMILARITY TRANSFORMATION ON V_n

The main result of this section (Theorem 3.2) shows that V_n is similar to a particularly nice matrix in block upper triangular form. This form will lead to a recursion for the characteristic polynomial of V_n.

Let T_n be the $n \times n$ matrix whose columns carry the recurrence satisfied by $P_n(-t)$, i.e.,

$$(T_n)_{ij} = \begin{cases} 1, & \text{if } i = j \\ t, & \text{if } i = j + 1 \\ -1, & \text{if } i = j + 2 \\ 0, & \text{otherwise.} \end{cases}$$

Then we have

Lemma 3.1: The inverse of T_n is given by

$$(T_n^{-1})_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i < j \\ P_{k+1}(-t), & \text{if } i = j + k. \end{cases}$$

Proof: Let A denote the matrix defined in the statement of the Lemma, and let $B = T_nA$. Then B is lower triangular, with diagonal elements all equal to one. A typical element below the diagonal has the form

$$P_i(-t) + tP_{i-1}(-t) - P_{i-2}(-t) = P_i(-t) - (-t)P_{i-1}(-t) - P_{i-2}(-t) = 0,$$

since this is the recursion defining $P_i(-t)$. Thus, $B = I$ and $A = T_n^{-1}$. ∎

Theorem 3.2: The matrix $T_n^{-1}V_nT_n$ has the block form $\begin{bmatrix} -V_{n-2} & X \\ 0 & Y \end{bmatrix}$, where X is $(n - 2) \times 2$, Y is 2×2, and

$$Y = \begin{bmatrix} P_n(t) & P_{n-1}(t) \\ P_{n-1}(t) & P_{n-2}(t) \end{bmatrix}.$$

Proof: First we show, by induction, that the first $n - 2$ columns of the matrix

$$A = (a_{ij}) = T_n^{-1} V_n T_n$$

have the desired form.

The i^{th} row of T_n^{-1} is

$$R_i = [P_i(-t), P_{i-1}(-t), \ldots, P_2(-t), 1, 0, \ldots, 0]$$

and the j^{th} column of $V_n T_n$ is $C_j = \text{col}(x_1, \ldots, x_n)$, where

$$\begin{aligned} x_k &= 0 \ (k = 1, 2, \ldots, n - j - 2) \\ x_{n-j-1} &= -1 \\ x_{n-j} &= -\binom{j+1}{1}t + t \\ x_{n-j+k} &= -\binom{j+1}{k+1}t^{k+1} + \binom{j}{k}t^{k+1} + \binom{j-1}{k-1}t^{k-1}. \end{aligned}$$

Then a_{ij} is the dot product $R_i \cdot C_j$, and to start the induction, we have:

$$\begin{aligned} a_{ij} &= 0 \text{ if } n - j - 2 \geqslant i \\ a_{ij} &= -1 \text{ if } n - j - 2 = i - 1 \\ a_{ij} &= -\binom{j-1}{1}t \text{ if } n - j - 2 = i - 2 \\ a_{ij} &= -\binom{j-1}{2}t^2 \text{ if } n - j - 2 = i - 3. \end{aligned}$$

Now suppose that, if $0 \leqslant s < r$ and $n - j - 2 = i - s$, then

$$a_{ij} = -\binom{j-1}{s-1}t^{s-1}.$$

Then, for $n - j - 2 = i - r$,

$$\begin{aligned} a_{ij} &= \sum_{k=1}^{i} P_{i+1-k}(-t)x_k = \sum_{k=i-r+1}^{i} P_{i+1-k}(-t)x_k \\ &= \sum_{k=i-r+1}^{i-1} P_{i+1-k}(-t)x_k + P_1(-t)x_i \\ &= \sum_{k=i-r+1}^{i-1} [(-t)P_{i-k}(-t) + P_{i-k-1}(-t)]x_k + P_1(-t)x_i \\ &= (-t)\left[-t^{r-2}\binom{j-1}{r-2}\right] + \left[-t^{r-3}\binom{j-1}{r-3}\right] - \binom{j+1}{r-1}t^{r-1} \\ &\quad + \binom{j}{r-2}t^{r-1} + \binom{j-1}{r-3}t^{r-3} \end{aligned}$$

(continued)

$$= -t^{r-1}\binom{j-1}{r-1}.$$

This completes the induction.

From the definition of V_n, the j^{th} column of V_{n-2} must be

$$\text{col}\left[0, 0, \ldots, 0, 1, \binom{j-1}{1}t, \binom{j-1}{2}t, \ldots, \binom{j-1}{j-2}t^{j-2}, t^{j-1}\right];$$

therefore, the upper left diagonal $(n-2)\times(n-2)$ block of $T_n^{1}V_nT_n$ is indeed $-V_{n-2}$.

The entries $a_{n-1,j}$ and $a_{n,j}$ for $1 \leqslant j \leqslant n-2$ are all zero because, if $i = n-1$, then $n - j - 2 = i - r$ implies $r = j + 1$. Then the term

$$-t^{r-1}\binom{j-1}{r-1} = -t^{r-1}\binom{j-1}{j} = 0.$$

If $i = n$ and $n - j - 2 = i - r$, then $r = j + 2$ and we have

$$-t^{r-1}\binom{j-1}{r-1} = -t^{r-1}\binom{j-1}{j+1} = 0.$$

It remains to show that the lower right diagonal 2×2 block of $T_n^{-1}V_nT_n$ is given by

$$\begin{bmatrix} P_n(t) & P_{n-1}(t) \\ P_{n-1}(t) & P_{n-2}(t) \end{bmatrix}.$$

We shall compute $a_{n,n}$ in detail. The other three cases are similar. Recalling that

$$R_n = [P_n(-t), P_{n-1}(-t), \ldots, P_2(-t), 1]$$

and

$$C_n = \text{col}\left[1, \binom{n-1}{1}t, \binom{n-1}{2}t^2, \ldots, t^{n-1}\right],$$

we have

$$\begin{aligned} a_{n,n} &= \sum_{k=0}^{n-1}\binom{n-1}{k}t^kP_{n-k}(-t) \\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}t^k\sum_{j=0}^{[(n-k)/2]}\binom{n-k-1-j}{j}(-t)^{n-k-1-2j}, \end{aligned}$$

by (2.1). Reversing the order of summation gives

$$a_{n,n} = \sum_{j=0}^{[n/2]}t^{n-1-2j}\sum_{k=0}^{n-2j}\binom{n-1}{k}\binom{n-j-k-1}{j}(-1)^{n-k-1-2j}.$$

Consider the inner sum

$$S = \sum_{k=0}^{n-2j}\binom{n-1}{k}\binom{n-j-k-1}{j}(-1)^{n-k-1-2j}.$$

When $k = n - 2j$, the binomial coefficient $\binom{n-j-k-1}{j} = \binom{j-1}{j} = 0$, so we may take the upper limit to be $n - 2j - 1$.

Now, make the substitution $p = n - 2j - 1$ in S to get

$$S = \sum_{k=0}^{p} \binom{p + 2j}{k}\binom{p + j - k}{j}(-1)^{p-k} = \sum_{k=0}^{p} \binom{p + 2j}{k}\binom{p + j - k}{p - k}(-1)^{p-k}.$$

Note that $\binom{p + 2j}{k}$ is the coefficient of x^k in the expansion of $(1 + x)^{p+2j}$ and that $\binom{p + j - k}{p - k}(-1)^{p-k}$ is the coefficient of x^{p-k} in the expansion of $(1 + x)^{-j-1}$. Then S is the coefficient of x^p in the expansion of

$$(1 + x)^{p+2j-j-1} = (1 + x)^{n-j-2},$$

that is,

$$S = \binom{n - j - 2}{n - 2j - 1} = \binom{n - j - 2}{j - 1}.$$

Returning to the calculation of $a_{n,n}$, we have

$$a_{n,n} = \sum_{j=0}^{[n/2]} t^{n-1-2j}\binom{n - j - 2}{j - 1} = \sum_{k=0}^{[(n-2)/2]} \binom{n - 3 - k}{k} t^{n-3-2k}$$

(eliminating zero terms and replacing $j - 1$ by k). Thus, $a_{n,n} = P_{n-2}(t)$, by (2.1). The sums for $a_{n,n-1}$, $a_{n-1,n}$, and $a_{n-1,n-1}$ can be evaluated by the same methods, but we omit the proofs here. ■

4. THE CHARACTERISTIC POLYNOMIAL OF $V_n(t)$

Let A_n denote the matrix $T_n^{-1}V_nT_n$ and let $C_n(\lambda)$ be the characteristic polynomial of V_n. As before, let $Y = Y_n$ be the matrix

$$Y_n = \begin{bmatrix} P_n(t) & P_{n-1}(t) \\ P_{n-1}(t) & P_{n-2}(t) \end{bmatrix}.$$

In this section, we establish some basic properties of $C_n(\lambda)$ and prove the conjecture of Mahon and Horadam.

Lemma 4.1: The characteristic polynomial $C_n(\lambda)$ of V_n satisfies the recurrence:

$$C_2(\lambda) = \lambda^2 - t\lambda - 1$$
$$C_3(\lambda) = (\lambda + 1)(\lambda^2 + Q_2(t)\lambda + 1)$$
$$C_n(\lambda) = (-1)^{n-2}C_{n-2}(-\lambda)(\lambda^2 - Q_{n-1}(t)\lambda + (-1)^{n-1}).$$

Proof: Since A_n and V_n are similar, $C_n(\lambda) = |\lambda I - A_n|$. By the block form of A_n,

$$|\lambda I - A_n| = |\lambda I + V_{n-2}| \cdot |\lambda I - Y_n|.$$

Since $P_n(t)P_{n-2}(t) - P_{n-1}(t)^2 = (-1)^{n-1}$ and $P_n(t) + P_{n-2}(t) = Q_{n-1}(t)$,

$$|\lambda I - Y_n| = \lambda^2 - Q_{n-1}(t)\lambda + (-1)^{n-1}.$$

Since $|\lambda I + V_{n-2}| = (-1)^{n-2}C_{n-2}(-\lambda)$, Lemma 4.1 follows. ∎

Corollary 4.2:

a) If n is even, say $n = 2k$, then

$$C_{2k}(\lambda) = \prod_{j=0}^{k-1}(\lambda^2 - Q_{n-1-2j}(t)\cdot(-1)^j\lambda - 1),$$

and the characteristic roots of $C_{2k}(\lambda)$ are

$$\{(-1)^j\alpha^{n-1-2j},\ (-1)^j\beta^{n-1-2j} : j = 0, 1, \ldots, k-1\}.$$

b) If n is odd, say $n = 2k + 1$, then

$$C_{2k+1}(\lambda) = (\lambda - (-1)^k)\prod_{j=0}^{k-1}(\lambda^2 - Q_{n-1-2j}(t)\cdot(-1)^j\lambda + 1),$$

and the characteristic roots of $C_{2k+1}(\lambda)$ are

$$\{(-1)^k,\ (-1)^j\alpha^{n-1-2j},\ (-1)^j\beta^{n-1-2j} : j = 0, 1, \ldots, k-1\}.$$

Proof: We prove b); the proof of a) is similar. From Lemma 4.1, we get

$$C_5(\lambda) = (\lambda^2 - Q_4(t)\lambda + 1)(\lambda^2 - Q_2(t)(-\lambda) + 1)(\lambda - 1),$$

and from the recurrence, for $n \geqslant 5$, we derive

$$C_n(\lambda) = (\lambda^2 - Q_{n-1}(t)\lambda + 1)(\lambda^2 - Q_{n-3}(t)(-\lambda) + 1)C_{n-4}(\lambda).$$

Since $C_3(\lambda)$ has the factor $(\lambda + 1)$, if $n \equiv 3 \pmod 4$, $C_n(\lambda)$ will also have the the factor

$$(\lambda + 1) = \lambda + (-1)^{(n-1)/2}.$$

Since $C_5(\lambda)$ has the factor $(\lambda - 1)$, if $n \equiv 1 \pmod 4$, $C_n(\lambda)$ will also have the factor

$$(\lambda - 1) = \lambda + (-1)^{(n-1)/2}.$$

The rest of b) is clear.

The characteristic roots of $C_n(\lambda)$ are the roots of its factors. We have

$$(\lambda - \alpha^j)(\lambda - \beta^j) = \lambda^2 - (\alpha^j + \beta^j)\lambda + (\alpha\beta)^j = \lambda^2 - Q_j(t) + (-1)^j$$

and

$$(\lambda + \alpha^j)(\lambda + \beta^j) = \lambda^2 - Q_j(t)(-\lambda) + (-1)^j,$$

and this completes the proof. ∎

Define the coefficient $\{n, k\}$ by

$$\{n, k\} = \prod_{i=1}^{n} P_i(t) \Big/ \prod_{i=1}^{k} P_i(t) \prod_{i=1}^{n-k} P_i(t)$$

and define the polynomial $R_n(\lambda)$ by

$$R_n(\lambda) = \sum_{k=0}^{n} (-1)^{(k^2+k)/2}\{n, k\}\lambda^{n-k}.$$

The next theorem states that $R_n(\lambda) = C_n(\lambda)$. Then the conjecture of Mahon and Horadam follows by making the substitution $t = 2x$.

Theorem 4.3: For all $n \geqslant 2$, $R_n(\lambda) = C_n(\lambda)$.

Proof: It is easy to verify the cases $n = 2, 3$. Thus, we need only show that $R_n(\lambda)$ satisfies the recurrence of Lemma 4.1; that is, we must show that

$$R_n(\lambda) = (-1)^n R_{n-2}(-\lambda)\cdot(\lambda^2 - Q_{n-1}(t)\lambda + (-1)^{n-1}). \qquad (*)$$

Let $F(\lambda)$ denote the right-hand side of (*), let a_j denote the coefficient of λ^j in $R_n(\lambda)$, and b_j the coefficient of λ^j in $F(\lambda)$. Then, from the definition of $R_n(\lambda)$, $a_n = 1$, $a_{n-1} = -P_n$, $a_1 = (-1)^{(n^2-n)/2}P_n$, and $a_0 = (-1)^{(n^2+n)/2}$.

The n^{th} term in $F(\lambda)$ is

$$(-1)^n(-\lambda)^{n-2}\lambda^2 = \lambda^n,$$

so $b_n = 1 = a_n$.

The $(n-1)^{\text{th}}$ term in $F(\lambda)$ is

$$(-1)^n\lambda^2(-\lambda)^{n-2}(-1)\{n-2, 1\} + (-1)^n(-Q_{n-1}(t)\lambda)(-\lambda)^{n-2}$$
$$= \lambda^{n-1}(P_{n-2}(t) - Q_{n-1}(t)) = \lambda^{n-1}(-P_{n-1}(t)),$$

so $b_{n-1} = a_{n-1}$.

The constant term of $F(\lambda)$ is

$$(-1)^n(-1)^{n-1}(-1)^{(n-1)(n-2)/2} = (-1)^{(n+1)n/2},$$

so $a_0 = b_0$.

For b_1, we have

$$b_1 = (-1)^n(-Q_{n-1}(t))\lambda(-1)^{(n-1)(n-2)/2}$$
$$+ (-1)^n(-1)^{n-1}(-\lambda)(-1)^{(n-2)(n-3)/2}\{n-2, n-3\}$$
$$= (-1)^{n(n-1)/2}(Q_{n-1}(t) - P_{n-2}(t))\lambda$$
$$= (-1)^{n(n-1)/2}P_n(t),$$

giving $a_1 = b_1$.

For the remaining coefficients we need to show that, for $2 \leqslant k \leqslant n-2$, $a_{n-k} = b_{n-k}$; that is,

$$(-1)^{(k+1)k/2}\{n, k\} = (-1)^n(-1)^{n-k-2}(-1)^{(k+1)k/2}\{n-2, k\}$$
$$+ (-1)^n(-1)^{n-k-1}(-1)^{k(k-1)/2}\{n-2, k-1\}(-Q_{n-1}(t))$$
$$+ (-1)^n(-1)^{n-k}(-1)^{(k-1)(k-2)/2}\{n-2, k-2\}(-1)^{n-1}.$$

Clearing signs, this reduces to

$$\{n, k\} = (-1)^k\{n - 2, k\} + Q_{n-1}(t)\{n - 2, k - 1\} + (-1)^{n+k}\{n - 2, k - 2\}. \qquad (**)$$

Factoring out $\{n - 2, k - 1\}$ reduces (**) to

$$\frac{P_n(t)P_{n-1}(t)}{P_k(t)P_{n-k}(t)} = (-1)^k \frac{P_{n-k-1}(t)}{P_k(t)} + Q_{n-1}(t) + (-1)^{n+k} \frac{P_{k-1}(t)}{P_{n-k}(t)}.$$

Thus, it suffices to show that for $2 \leqslant k \leqslant n - 2$,

$$P_n(t)P_{n-1}(t) - P_k(t)P_{n-k}(t)Q_{n-1}(t)$$
$$= (-1)^k P_{n-k}(t)P_{n-k-1}(t) + (-1)^{n-k}P_k(t)P_{k-1}(t).$$

This last identity is proven using the Binet formulas and the properties of α and β. For convenience, denote $P_n(t)$ by P_n and so on. First,

$$P_nP_{n-1} = (\alpha^n - \beta^n)(\alpha^{n-1} - \beta^{n-1})/(\alpha - \beta)^2 = Q_{2n-1} + (-1)^n Q_1,$$

and

$$Q_{n-1}P_kP_{n-k} = (\alpha^{n-1} + \beta^{n-1})(\alpha^n + \beta^n - \beta^k\alpha^{n-k} - \alpha^k\beta^{n-k})/(\alpha - \beta)^2$$
$$= (\alpha^{2n-1} + \beta^{2n-1} + (-1)^{n-1}(\beta + \alpha) - (-1)^k(\alpha^{2n-2k-1} + \beta^{2n-2k-1}) - (-1)^{n-k}(\alpha^{2k-1} + \beta^{2k-1}))/(\alpha - \beta)^2$$
$$= (Q_{2n-1} + (-1)^{n-1}Q_1 + (-1)^{k+1}Q_{2n-2k-1} + (-1)^{n-k-1}Q_{2k-1})/(\alpha - \beta)^2.$$

Then

$$P_nP_{n-1} - P_kP_{n-k}Q_{n-1}$$
$$= ((-1)^kQ_{2n-2k-1} + (-1)^{n-k}Q_{2k-1} + 2(-1)^nQ_1)/(\alpha - \beta)^2.$$

On the other side,

$$(-1)^kP_{n-k}P_{n-k-1} + (-1)^{n-k}P_kP_{k-1}$$
$$= (-1)^k(Q_{2n-2k-1} + (-1)^{n-k}Q_1)/(\alpha - \beta)^2 + (-1)^{n-k}(Q_{2k-1} + (-1)^kQ_1)/(\alpha - \beta)^2$$
$$= ((-1)^kQ_{2n-2k-1} + (-1)^{n-k}Q_{2k-1} + 2(-1)^nQ_1)/(\alpha - \beta)^2.$$

Thus, the identity is true, and (**) is true; that is, $a_{n-k} = b_{n-k}$ for all k, $2 \leqslant k \leqslant n - 2$. Then $R_n(\lambda)$ satisfies the recurrence and initial conditions of Lemma 4.1, and it follows that $R_n(\lambda) = C_n(\lambda)$. ∎

5. THE EIGENVECTORS OF V_n

The eigenvectors of V_n can be computed in a recursive way. The initial cases are given below.

Lemma 5.1: V_2 has eigenvalues α, β. Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to α and β are given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ \alpha \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ \beta \end{bmatrix}.$$

The matrix V_3 has eigenvalues -1, α^2, β^2 with corresponding eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ t \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2\alpha \\ \alpha^2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2\beta \\ \beta^2 \end{bmatrix}. \quad \blacksquare$$

Lemma 5.2: Let $\mathbf{u} = \mathrm{col}(u_1, u_2, \ldots, u_n)$ and $\mathbf{w} = \mathrm{col}(w_1, w_2, \ldots, w_n)$ be adjacent columns of V_n, with $\mathbf{u}$ to the left of $\mathbf{w}$. Then

$$tu_n = w_n$$
$$tu_i + u_{i+1} = w_i \quad (i = 1, 2, \ldots, n-1).$$

Proof: If $\mathbf{u}$ is column j, then for $i = 1, 2, \ldots, n - j - 1$ we have u = 0 and $tu_i + u_{i+1} = w_i$. If $i = n - j + k$ for some k, $0 \leqslant k < j$, then

$$tu_i + u_{i+1} = t\binom{j-1}{i-1}t^{i-1} + \binom{j-1}{i}t^i = \binom{j}{i}t^i = w_i.$$

Since $u_n = t^{j-1}$ and $w_n = t^j$, we have $tu_n = w_n$. $\blacksquare$

Corollary 5.3: Define vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} = \mathrm{col}(\underbrace{0, \ldots, 0}_{j}, x_1, \ldots, x_t, \underbrace{0, \ldots, 0}_{k})$$

$$\mathbf{y} = \mathrm{col}(\underbrace{0, \ldots, 0}_{j+1}, x_1, \ldots, x_t, \underbrace{0, \ldots, 0}_{k-1})$$

where $j + t + k = n$ and $k > 0$. Put

$$\mathbf{u} = V_n\mathbf{x} \quad \text{and} \quad \mathbf{v} = V_n\mathbf{y}$$

with $\mathbf{u} = \mathrm{col}(u_1, \ldots, u_n)$ and $\mathbf{v} = \mathrm{col}(v_1, \ldots, v_n)$. Then $tu_i + u_{i+1} = v_i$.

Proof: Let $\mathbf{e}_k$ denote the column vector with 1 in the k^{th} place and 0 everywhere else. By Lemma 5.2, the result is true for

$$\mathbf{x} = \mathbf{e}_{j+1} \quad \text{and} \quad \mathbf{y} = \mathbf{e}_{j+2} \quad (j + 2 \leqslant n),$$

and hence is true in general by linearity. $\blacksquare$

Theorem 5.4: Let $n > 1$ be odd, so that V_n has

$$\varepsilon = (-1)^{(n-1)/2}$$

as an eigenvalue. Let

$$\mathbf{v} = \mathrm{col}(v_1, \ldots, v_n)$$

be an eigenvector corresponding to ε. Put

$$\begin{aligned} \mathbf{w} &= \mathrm{col}(v_1, \ldots, v_n, 0, 0) \\ &\quad + \mathrm{col}(0, tv_1, \ldots, tv_n, 0) \\ &\quad + \mathrm{col}(0, 0, -v_1, \ldots, -v_n). \end{aligned}$$

Then $\mathbf{w}$ is an eigenvector for V_{n+2}, corresponding to the eigenvalue

$$-\varepsilon = (-1)^{(n+1)/2}.$$

Proof: Put $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$, where the $\mathbf{w}_i$ are the summands in the statement of the Theorem. From the form of V_n (it has V_{n-2} in the lower left block, with zeros above it), it is clear that

$$V_{n+2}\mathbf{w}_1 = \varepsilon(0, 0, v_1, \ldots, v_n)$$

since $\mathbf{v}$ is an eigenvector for V_n corresponding to ε. Then by Corollary 5.3,

$$V_{n+2}\mathbf{w}_2 = t\varepsilon[(0, v_1, \ldots, v_n, 0) + t(0, 0, v_1, \ldots, v_n)]$$

$$V_{n+2}\mathbf{w}_3 = -\varepsilon[\mathbf{w}_1 + 2\mathbf{w}_2 - t^2\mathbf{w}_3]$$

so

$$V_{n+2}\mathbf{w} = \varepsilon(-\mathbf{w}_1 - \mathbf{w}_2 - \mathbf{w}_3) = -\varepsilon\mathbf{w}. \blacksquare$$

Theorem 5.5: Suppose that $\mathbf{v} = \text{col}(v_1, \ldots, v_{n-1})$ is an eigenvector for V_{n-1} corresponding to the eigenvalue α^i ($i \geqslant 0$). Put

$$\mathbf{w} = \text{col}(v_1, \ldots, v_{n-1}, 0) + \alpha\,\text{col}(0, v_1, \ldots, v_n) = \mathbf{x} + \alpha\mathbf{y}.$$

Then $\mathbf{w}$ is an eigenvector for V_n corresponding to the eigenvalue α^{i+1}.

Proof: We have

$$V_n\mathbf{x} = \alpha^i\mathbf{y}$$

$$V_n\mathbf{y} = \alpha^i\mathbf{x} + \alpha^i t\mathbf{y}$$

so that

$$V_n(\mathbf{x} + \alpha\mathbf{y}) = \alpha^i(\mathbf{y} + \alpha\mathbf{x} + \alpha t\mathbf{y}).$$

Since $\alpha^2 = 1 + \alpha t$,

$$V_n(\mathbf{x} + \alpha\mathbf{y}) = \alpha^i(\alpha\mathbf{x} + \alpha^2\mathbf{y}) = \alpha^{i+1}(\mathbf{x} + \alpha\mathbf{y})$$

as required. $\blacksquare$

Remark: The analogous result also holds for the eigenvectors corresponding to the eigenvalues β^i.

Corollary 5.6: All of the eigenvectors of V_n can be computed in terms of the eigenvectors of V_{n-1} and V_{n-2}. $\blacksquare$

REFERENCE

1. J. M. Mahon & A. F. Horadam. "Matrix and Other Summation Techniques for Pell Polynomials." *The Fibonacci Quarterly* 24, no. 4 (1986):290-308.

◆◇◆◇◆