

## PELL AND PELL-LUCAS POLYNOMIALS

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### 1. INTRODUCTION

The object of this paper is to record some properties of *Pell polynomials*  $P_n(x)$  and *Pell-Lucas polynomials*  $Q_n(x)$  defined by the recurrence relations

$$P_{n+2}(x) = 2xP_{n+1}(x) + P_n(x) \quad P_0(x) = 0, P_1(x) = 1 \quad (1.1)$$

and

$$Q_{n+2}(x) = 2xQ_{n+1}(x) + Q_n(x) \quad Q_0(x) = 2, Q_1(x) = 2x. \quad (1.2)$$

Initially, the polynomials are defined for  $n \geq 0$  but their existence for  $n < 0$  is readily extended, yielding

$$P_{-n}(x) = (-1)^{n+1}P_n(x) \quad (1.3)$$

and

$$Q_{-n}(x) = (-1)^n Q_n(x). \quad (1.4)$$

Some of these polynomials are:

$$\begin{cases} P_2(x) = 2x, & P_3(x) = 4x^2 + 1, & P_4(x) = 8x^3 + 4x, \\ P_5(x) = 16x^4 + 12x^2 + 1, & P_6(x) = 32x^5 + 32x^3 + 6x, \dots; \end{cases} \quad (1.5)$$

$$\begin{cases} Q_2(x) = 4x^2 + 2, & Q_3(x) = 8x^3 + 6x, & Q_4(x) = 16x^4 + 16x^2 + 2, \\ Q_5(x) = 32x^5 + 40x^3 + 10x, & Q_6(x) = 64x^6 + 96x^4 + 36x^2 + 2, \dots. \end{cases} \quad (1.6)$$

Important special numerical cases are:  $P_n(1) = P_n$ , the  $n^{\text{th}}$  *Pell number*;  $Q_n(1) = Q_n$ , the  $n^{\text{th}}$  *Pell-Lucas number*;  $P_n(\frac{1}{2}) = F_n$ , the  $n^{\text{th}}$  *Fibonacci number*; and  $Q_n(\frac{1}{2}) = L_n$ , the  $n^{\text{th}}$  *Lucas number*. Furthermore,  $P_n(\frac{1}{2}x) = F_n(x)$ , the  $n^{\text{th}}$  *Fibonacci polynomial*, and  $Q_n(\frac{1}{2}x) = L_n(x)$ , the  $n^{\text{th}}$  *Lucas polynomial* (see [2]).

Following standard procedures, we easily obtain the *Binet forms*

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (1.7)$$

and

$$Q_n(x) = \alpha^n + \beta^n, \quad (1.8)$$

where

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \quad (1.9)$$

are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0, \quad (1.10)$$

so that

$$\alpha + \beta = 2x, \alpha - \beta = 2\sqrt{x^2 + 1}, \alpha\beta = -1. \quad (1.11)$$

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The generating functions for the infinite sets of polynomials  $\{P_n(x)\}$  and  $\{Q_n(x)\}$  are found in the usual way to be

$$\sum_{r=0}^{\infty} P_{r+1}(x)y^r = \frac{1}{1 - 2xy - y^2} \quad (1.12)$$

and

$$\sum_{r=0}^{\infty} Q_{r+1}(x)y^r = \frac{2x + 2y}{1 - 2xy - y^2}. \quad (1.13)$$

Results involving these generating functions are not developed here.

2. ELEMENTARY PROPERTIES OF  $P_n(x)$ ,  $Q_n(x)$

Important elementary relationships involving  $P_n(x)$  and  $Q_n(x)$  follow without difficulty with the aid of (1.7)-(1.11). Some of these are:

$$P_{n+1}(x) + P_{n-1}(x) = Q_n(x) = 2xP_n(x) + 2P_{n-1}(x) \quad (2.1)$$

$$Q_{n+1}(x) + Q_{n-1}(x) = 4(x^2 + 1)P_n(x) \quad (2.2)$$

$$P_n(x)Q_n(x) = P_{2n}(x) \quad (2.3)$$

$$Q_{2n}(x) = \frac{1}{2}\{Q_n^2(x) + 4(x^2 + 1)P_n^2(x)\} \quad (2.4)$$

$$P_{n+1}(x)P_{n-1}(x) - P_n^2(x) = (-1)^n \quad (2.5)$$

$$Q_{n+1}(x)Q_{n-1}(x) - Q_n^2(x) = (-1)^{n-1}4(x^2 + 1) \quad (2.6)$$

$$P_{n+1}^2(x) - P_{n-1}^2(x) = 2xP_{2n}(x) \text{ by (1.1), (2.1), (2.3)} \quad (2.7)$$

$$4(x^2 + 1)P_n^2(x) - Q_n^2(x) = 4(-1)^{n-1} \quad (2.8)$$

Formula (2.3) is useful in establishing divisibility properties of the polynomials. Geometrical paradoxes can be constructed from (2.5) when numerical values of  $x$  are inserted.

Summations of an elementary nature are obtained in the usual manner. The simplest are:

$$\sum_{r=1}^n P_{2r}(x) = (P_{2n+1}(x) - 1)/2x \quad (2.9)$$

$$\sum_{r=1}^n P_{2r-1}(x) = P_{2n}(x)/2x \quad (2.10)$$

$$\sum_{r=1}^n P_r(x) = (P_{n+1}(x) + P_n(x) - 1)/2x \text{ by (2.9), (2.10)} \quad (2.11)$$

$$\sum_{r=1}^n Q_{2r}(x) = (Q_{2n+1}(x) - 2x)/2x \quad (2.12)$$

$$\sum_{r=1}^n Q_{2r-1}(x) = (Q_{2n}(x) - 2)/2x \quad (2.13)$$

$$\sum_{r=1}^n Q_r(x) = (Q_{n+1}(x) + Q_n(x) - 2 - 2x)/2x \text{ by (2.12), (2.13)} \quad (2.14)$$

Extensions and variations of these finite summations, e.g.,  $\sum_{r=1}^n r^2 P_r(x)$  and  $\sum_{r=1}^n (-1)^r Q_r(x)$ , are omitted in this treatment of the polynomials.

Induction can be used, with a little effort, to establish the explicit expressions

$$P_n(x) = \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-m-1}{m} (2x)^{n-2m-1} \quad (2.15)$$

and

$$Q_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-m} \binom{n-m}{m} (2x)^{n-2m}, \quad n \neq 0, \quad (2.16)$$

where, in (2.16) we used the combinatorial identity

$$\frac{n}{n-m} \binom{n-m}{m} + \frac{n-1}{n-m} \binom{n-m}{m-1} = \frac{n+1}{n-m+1} \binom{n-m+1}{m}.$$

We proceed to prove (2.15).

Proof of (2.15): The formula is trivially true for  $n = 1$  and  $n = 2$ . Assume it is true for  $n = k$  and  $n = k - 1$  where  $k \geq 3$ . Then we have

$$\begin{aligned} P_{k+1}(x) &= 2xP_k(x) + P_{k-1}(x) \quad \text{by (1.1)} \\ &= \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k-m-1}{m} (2x)^{k-2m} + \sum_{m=0}^{\lfloor \frac{k-2}{2} \rfloor} \binom{k-m-2}{m} (2x)^{k-2m-2}. \end{aligned}$$

If  $k = 2t$ , this becomes

$$\begin{aligned} &\sum_{m=0}^{t-1} \binom{2t-m-1}{m} (2x)^{2t-2m} + \sum_{m=0}^{t-1} \binom{2t-m-2}{m} (2x)^{2t-2m-2} \\ &= \binom{2t-1}{0} (2x)^{2t} + \binom{2t-2}{1} (2x)^{2t-2} + \binom{2t-3}{2} (2x)^{2t-4} + \dots + \binom{t}{t-1} (2x)^2 \\ &\quad + \binom{2t-2}{0} (2x)^{2t-2} + \binom{2t-3}{1} (2x)^{2t-4} + \dots + \binom{t}{t-2} (2x)^2 + \binom{t-1}{t-1} \\ &= \sum_{m=0}^t \binom{2t-m}{m} (2x)^{2t-2m} = \sum_{m=0}^{\lfloor k/2 \rfloor} \binom{k-m}{m} (2x)^{k-2m} \end{aligned}$$

by using Pascal's formula. Similarly, it holds if  $k$  is odd, and the proof is completed.

Basic relationships involving  $P_n(x)$  and  $Q_n(x)$  may be obtained from these combinatorial formulas, but the calculations required are tedious. Binet forms produce the same results more quickly.

In passing, we note the differential calculus result:

$$\frac{dQ_n(x)}{dx} = 2nP_n(x). \quad (2.17)$$

Later, in (6.20), we shall see that the first derivative of  $P_n(x)$  is given in terms of a (complex) Gegenbauer polynomial.

Because  $P_n(x)$  and  $Q_n(x)$  are generalizations of  $F_n$  and  $L_n$ , the collection of miscellaneous results for  $F_n$  and  $L_n$  given in [7] may be generalized; e.g.,

$$Q_{4n}(x) - 2 = 4(x^2 + 1)P_{2n}^2(x), \quad (2.18)$$

$$P_{n-1}(x)P_{n+1}(x) + Q_{n-1}(x)Q_{n+1}(x) = (4x^2 + 5)P_n^2(x) + (-1)^{n-1}(4x^2 - 1), \quad (2.19)$$

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and

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k} P_{2k+p}(x) = [4(x^2 + 1)]^n Q_{2n+p+1}(x). \quad (2.20)$$

### 3. MATRIX GENERATION OF FORMULAS

We demonstrate that the matrix

$$P = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix} \quad (3.1)$$

generates Pell polynomials and Pell-Lucas polynomials, and use it to establish some elementary properties of these polynomials.

Induction, with (1.1), leads to

$$P^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix} \quad (3.2)$$

whence

$$\begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} = P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (3.3)$$

and

$$P_n(x) = [1 \quad 0] P^{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (3.4)$$

The characteristic equation of  $P$  is

$$\lambda^2 - 2x\lambda - 1 = 0 \quad (3.5)$$

with eigenvalues

$$\begin{cases} \alpha = x + \sqrt{x^2 + 1} \\ \beta = x - \sqrt{x^2 + 1} \end{cases} \quad (3.6)$$

By the division algorithm for polynomials,

$$\lambda^n = (\lambda^2 - 2x\lambda - 1)f(\lambda) + m\lambda + k, \quad (3.7)$$

where  $f(\lambda)$  is of degree  $n-2$  in  $\lambda$  and  $m, k$  are functions of  $x$ .

Put  $\lambda = \alpha$  in (3.7). Then

$$\alpha^n = m\alpha + k. \quad (3.8)$$

Similarly,

$$\beta^n = m\beta + k. \quad (3.9)$$

Solving (3.8) and (3.9) yields

$$m = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad k = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}. \quad (3.10)$$

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From (3.8)

$$P^n = mP + kI. \tag{3.11}$$

Equate the top right elements in (3.11) to obtain  $m = P_n(x)$  so that the Binet form (1.7) for  $P_n(x)$  is again produced from (3.10).

Use of (2.1) gives

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix} \tag{3.12}$$

and

$$Q_n(x) = [1 \quad 0]P^{n-1} \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.13}$$

To illustrate the matrix technique, we prove

$$P_{m+n}(x) = P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) \tag{3.14}$$

for

$$\begin{aligned} P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x) &= [P_m(x), P_{m-1}(x)] \begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [P_m(x), P_{m-1}(x)]P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3)} \\ &= [1 \quad 0]P^{m+n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.3) and } P^m P^n = P^{m+n} \\ &= P_{m+n}(x) \text{ by (3.4)}. \end{aligned}$$

Similarly

$$Q_{m+n}(x) = P_{m-1}(x)Q_n(x) + P_m(x)Q_{n+1}(x). \tag{3.15}$$

From (3.14) and (3.15) with (3.2) and (3.12), we derive

$$\begin{bmatrix} P_{n+r}(x) \\ P_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{3.16}$$

and

$$\begin{bmatrix} Q_{n+r}(x) \\ Q_n(x) \end{bmatrix} = \begin{bmatrix} P_r(x) & P_{r-1}(x) \\ 0 & 1 \end{bmatrix} P^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}. \tag{3.17}$$

Equation (3.14), including an interchange of  $m$  and  $n$ , in conjunction with (2.1) gives

$$P_{m+n}(x) = \frac{1}{2}\{P_m(x)Q_n(x) + P_n(x)Q_m(x)\}, \tag{3.18}$$

while (3.15), including a replacement of  $m$  by  $m + 1$  and  $n$  by  $n - 1$ , with (2.1) and (2.2) gives

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$$Q_{m+n}(x) = \frac{1}{2}\{Q_m(x)Q_n(x) + 4(x^2 + 1)P_m(x)P_n(x)\}. \quad (3.19)$$

Putting  $m = n$  in (3.18) and (3.19) yields (2.3) and (2.4). Further,

$$P_{n+1}^2(x) + P_n^2(x) = P_{2n+1}(x) \quad (3.20)$$

since

$$\begin{aligned} P_{n+1}^2(x) + P_n^2(x) &= [P_{n+1}(x), P_n(x)] \begin{bmatrix} P_{n+1}(x) \\ P_n(x) \end{bmatrix} \\ &= [1 \quad 0]P^{2n} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ by (3.2) and (3.3)} \\ &= P_{2n+1}(x) \text{ by 3.4.} \end{aligned}$$

Result (3.20) also follows directly from (3.14) with  $m = n + 1$ .

Similarly,

$$Q_{n+1}^2(x) + Q_n^2(x) = 4(x^2 + 1)P_{2n+1}(x). \quad (3.21)$$

All the above results can, of course, be derived by using the Binet forms (1.7) and (1.8). Techniques employed in these sections give rise to the following formulas:

$$P_{n+r}(x) + P_{n-r}(x) = \begin{cases} P_n(x)Q_r(x) & \text{if } r \text{ is even} \\ Q_n(x)P_r(x) & \text{if } r \text{ is odd} \end{cases} \quad (3.22)$$

$$Q_{n+r}(x) + Q_{n-r}(x) = \begin{cases} Q_n(x)Q_r(x) & r \text{ even} \\ 4(x^2 + 1)P_n(x)P_r(x) & r \text{ odd} \end{cases} \quad (3.23)$$

$$P_{n+r}(x) - P_{n-r}(x) = \begin{cases} Q_n(x)P_r(x) & r \text{ even} \\ P_n(x)Q_r(x) & r \text{ odd} \end{cases} \quad (3.24)$$

$$Q_{n+r}(x) - Q_{n-r}(x) = \begin{cases} 4(x^2 + 1)P_n(x)P_r(x) & r \text{ even} \\ Q_n(x)Q_r(x) & r \text{ odd} \end{cases} \quad (3.25)$$

$$P_{n+r}^2(x) - P_{n-r}^2(x) = P_{2n}(x)P_{2r}(x) \quad \text{by (3.22), (3.24) and (2.3)} \quad (3.26)$$

$$Q_{n+r}^2(x) - Q_{n-r}^2(x) = 4(x^2 + 1)P_{2n}(x)P_{2r}(x) \quad \text{by (3.23), (3.25), and (2.3)} \quad (3.27)$$

$$P_{mn+r}(x) = \begin{cases} P_n(x)Q_{(m-1)n+r}(x) + (-1)^n P_{(m-2)n+r}(x) \\ P_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1} P_{(m-2)n+r}(x) \end{cases} \quad (3.28)$$

$$Q_{mn+r}(x) = Q_{(m-1)n+r}(x)Q_n(x) + (-1)^{n-1} Q_{(m-2)n+r}(x) \quad (3.29)$$

$$P_n^2(x) - P_{n+r}(x)P_{n-r}(x) = (-1)^{n-r} P_r^2(x) \quad (3.30)$$

$$Q_n^2(x) - Q_{n+r}(x)Q_{n-r}(x) = (-1)^{n-r+1} 4(x^2 + 1)P_r^2(x) \quad (3.31)$$

$$P_{n+h}(x)P_{n+k}(x) - P_n(x)P_{n+h+k}(x) = (-1)^n P_h(x)P_k(x) \quad (3.32)$$

} *Simson formulas*

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$$Q_{n+h}(x)Q_{n+k}(x) - Q_n(x)Q_{n+h+k}(x) = (-1)^{n-1}4(x^2 + 1)P_h(x)P_k(x) \quad (3.33)$$

$$P_{n+h}(x)Q_{n+k}(x) - P_n(x)Q_{n+h+k}(x) = (-1)^n P_h(x)Q_k(x) \quad (3.34)$$

Finally, we offer two relationships that can be described as being of the *de Moivre type*:

$$\{Q_n(x) + 2\sqrt{x^2 + 1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) + 2\sqrt{x^2 + 1}P_{nr}(x)\} \quad (3.35)$$

and

$$\{Q_n(x) - 2\sqrt{x^2 + 1}P_n(x)\}^r = 2^{r-1}\{Q_{nr}(x) - 2\sqrt{x^2 + 1}P_{nr}(x)\}. \quad (3.36)$$

When  $x = \frac{1}{2}$ , (3.35) and (3.36) reduce to

$$\left\{\frac{L_n + \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} + \sqrt{5}F_{nr}}{2} \quad (3.37)$$

and

$$\left\{\frac{L_n - \sqrt{5}F_n}{2}\right\}^r = \frac{L_{nr} - \sqrt{5}F_{nr}}{2}, \quad (3.38)$$

respectively, the first of which is given in [7, p. 60].

Results involving  $P_n(x)$  and  $Q_n(x)$  are as multitudinous as the sands of the seashore, and one can gather these grains *ad infinitum*, *ad nauseam*.

4. PASCAL ARRAYS GENERATING  $P_n(x)$ ,  $Q_n(x)$

Consider the following table.

Table 1: Pell Polynomials from Rising Diagonals

$\begin{matrix} m \\ \backslash \\ n \end{matrix}$	1	2	3	4	5	6 ...
1	<del>1</del>					
2	<del>2x</del>	1				
3	<del>4x<sup>2</sup></del>	<del>4x</del>	1			
4	<del>8x<sup>3</sup></del>	<del>12x<sup>2</sup></del>	<del>6x</del>	1		
5	<del>16x<sup>4</sup></del>	<del>32x<sup>3</sup></del>	<del>24x<sup>2</sup></del>	8x	1	
6	<del>32x<sup>5</sup></del>	<del>80x<sup>4</sup></del>	<del>80x<sup>3</sup></del>	40x <sup>2</sup>	10x	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Denote the coefficient of the power of  $x$  in the  $m^{\text{th}}$  row and  $n^{\text{th}}$  column by  $(m, n)$ .

It is now shown that the rising diagonals presented in Table 1 produce the Pell polynomial (1.5).

Define the entries in row  $m$  as the terms in the expansion  $(2x + 1)^{m-1}$ , that is

$$\sum_{n=1}^m (m, n)x^{m-n} = (2x + 1)^{m-1} \quad m \geq n. \quad (4.2)$$

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Hence,

$$(m, n) = \binom{m-1}{m-n} 2^{m-n} \quad m \geq n. \tag{4.3}$$

Now the rising diagonal function  $R_m(x)$  of degree  $m$  in  $x$  in Table 1 is:

$$\begin{aligned} R_m(x) &= \sum_{n=1}^{\lfloor \frac{m+1}{2} \rfloor} (m+1-n, n) x^{m+1-2n} \quad (m \geq 1) \tag{4.4} \\ &= \sum_{n=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-n}{m+1-2n} (2x)^{m+1-2n} \quad \text{by (4.3)} \\ &= \sum_{n=1}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-n}{n-1} (2x)^{m+1-2n} \\ &= \sum_{n=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-n-1}{n} (2x)^{m-1-2n} \\ &= P_m(x) \quad \text{from (2.15)} \end{aligned}$$

Now consider Table 2.

Table 2: Pell-Lucas Polynomials from Rising Diagonals

$m$ $n$	1	2	3	4	5	6	7 ...
1	<del><math>2x</math></del>	<del><math>2</math></del>					
2	<del><math>4x^2</math></del>	<del><math>6x</math></del>	<del><math>2</math></del>				
3	<del><math>8x^3</math></del>	<del><math>16x^2</math></del>	<del><math>10x</math></del>	<del><math>2</math></del>			
4	<del><math>16x^4</math></del>	<del><math>40x^3</math></del>	<del><math>36x^2</math></del>	<del><math>14x</math></del>	2		
5	<del><math>32x^5</math></del>	<del><math>96x^4</math></del>	<del><math>112x^3</math></del>	<del><math>64x^2</math></del>	18x	2	
6	<del><math>64x^6</math></del>	<del><math>224x^5</math></del>	<del><math>320x^4</math></del>	<del><math>240x^3</math></del>	100x <sup>2</sup>	22x	2
⋮							

Let  $[m, n]$  denote the coefficient of the power of  $x$  in the  $m^{\text{th}}$  row and  $n^{\text{th}}$  column.

We may define the entries in row  $m$  as the terms in the expansion of

$$(2x + 1)^m + (2x + 1)^{m-1} = (2x + 1)^{m-1}(2x + 2),$$

that is,

$$\sum_{n=1}^{m+1} [m, n] x^{m+1-n} = (2x + 1)^{m-1}(2x + 2) \tag{4.6}$$

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and so

$$\begin{aligned}
 [m, n] &= 2(m, n) + 2(m, n - 1) = 2(m, n) + (m, n - 1) + (m, n - 1) \\
 &= (m + 1, n) + (m, n - 1). \tag{4.7}
 \end{aligned}$$

Denote the rising diagonal function of degree  $m$  in  $x$  in Table 2 by  $S_m(x)$ . Then

$$\begin{aligned}
 S_m(x) &= \sum_{n=1}^{\lfloor \frac{m+2}{2} \rfloor} [m + 1 - n, n] x^{m+2-2n} \\
 &= \sum_{n=1}^{\lfloor \frac{m+2}{2} \rfloor} \{ (m + 2 - n, n) + (m + 1 - n, n - 1) \} x^{m+2-2n} \quad \text{by (4.7)} \\
 &= \sum_{n=1}^{\lfloor \frac{m+2}{2} \rfloor} \left\{ \binom{m+1-n}{n-1} + \binom{m-n}{n-2} \right\} (2x)^{m+2-2n} \quad \text{by (4.3)} \\
 &= \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m}{m-n} \binom{m-n}{n} (2x)^{m-2n} \quad \text{on simplification} \\
 &= Q_m(x) \quad \text{by (2.16)}
 \end{aligned}$$

Thus, we have demonstrated that Pell and Pell-Lucas polynomials are generated by the rising diagonals in Table 1 and Table 2, respectively.

Next, arrange the coefficients of the powers of  $x$  in  $P_n(x)$ , (1.5), in the following Pascal-like display.

Table 3: Pell Polynomial Coefficients

Coeffs. in $P_n(x)$	Powers	0	1	2	3	4	5	6	7	8	9 ...
1		1									
2		0	2								
3		1	0	4							
4		0	4	0	8						
5		1	0	12	0	16					
6		0	6	0	32	0	32				
7		1	0	24	0	80	0	64			
8		0	8	0	80	0	192	0	128		
9		1	0	40	0	240	0	448	0	256	
10		0	10	0	160	0	672	0	1024	0	512
⋮											

Designate the entry in the  $r^{\text{th}}$  row and  $c^{\text{th}}$  column of Table 3 by  $\{r, c\}$ . From the table and (2.15), we have:

$$\{2r, 2c\} = 0 \tag{4.8}$$

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$$\{2r, 2c - 1\} = \begin{cases} \binom{r+c-1}{r-c} 2^{2c-1} & c = 1, 2, \dots, r \\ 0 & c > r \end{cases} \quad (4.9)$$

$$\{2r - 1, 2c - 1\} = 0 \quad (4.10)$$

$$\{2r - 1, 2c\} = \begin{cases} \binom{r+c-1}{r-c-1} 2^{2c} & c = 0, 1, 2, \dots, r - 1 \\ 0 & c \geq r \end{cases} \quad (4.11)$$

Using (4.8)-(4-11), we can prove:

$$\sum_{i=0}^{r-1} \{2r - 1 - i, i\} = 3^{r-1} \quad (4.12)$$

$$\sum_{i=1}^{2r} \{i, 2c - 1\} = \frac{1}{2}\{2r + 1, 2c\} \quad (4.13)$$

$$\sum_{i=1}^{2r} \{i, 2c\} = \frac{1}{2}\{2r, 2c + 1\} \quad (4.14)$$

$$\sum_{i=1}^{2r-1} \{i, 2c - 1\} = \frac{1}{2}\{2r - 1, 2c\} \quad (4.15)$$

$$\sum_{i=1}^{2r-1} \{i, 2c\} = \frac{1}{2}\{2r, 2c + 1\} \quad (4.16)$$

Proof of (4.12)

$$\begin{aligned} \sum_{i=0}^{r-1} \{2r - 1 - i, i\} &= \{2r - 1, 0\} + \{2r - 2, 1\} + \dots + \{r, r - 1\} \\ &= \binom{r-1}{r-1} 2^0 + \binom{r-1}{r-2} 2^1 + \dots + \binom{r-1}{0} 2^{r-1} \quad \text{by (4.9)} \\ &\quad \text{and (4.11)} \\ &= (1 + 2)^{r-1} = 3^{r-1} \end{aligned}$$

Proof of (4.13)

$$\begin{aligned} \sum_{i=1}^{2r} \{i, 2c - 1\} &= \{2, 2c - 1\} + \{4, 2c - 1\} + \dots + \{2r, 2c - 1\} \quad \text{by (4.10)} \\ &= \{2c, 2c - 1\} + \{2c + 2, 2c - 1\} + \dots + \{2r, 2c - 1\} \\ &\quad \text{by (4.9)} \\ &= 2^{2c-1} \left( \binom{2c-1}{0} + \binom{2c}{1} + \dots + \binom{r+c-1}{r-c} \right) \quad \text{by (4.9)} \\ &= 2^{2c-1} \left( \binom{2c-1}{2c-1} + \binom{2c}{2c-1} + \dots + \binom{r+c-1}{2c-1} \right) \\ &= 2^{2c-1} \binom{r+c}{2c} \quad \text{by identity (1.52) in [6]} \\ &= \frac{1}{2}\{2r + 1, 2c\} \quad \text{by (4.11)} \end{aligned}$$

If a similar table for  $Q_n(x)$  is constructed, and if we designate the element in row  $r$  and column  $c$  by  $\overline{r, c}$ , we have from (2.1) that

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$$\overline{r, c} = \{r + 1, c\} + \{r - 1, c\} = 2\{r, c - 1\} + 2\{r - 1, c\}. \quad (4.17)$$

Properties of  $\overline{r, c}$  may then be developed on the basis of (4.8)-(4.11).

From (2.2), we derive

$$\overline{r + 1, c} + \overline{r - 1, c} = 4\{r, c\} + 4\{r, c - 2\}. \quad (4.18)$$

To conclude this section, we establish a relationship between  $(m, n)$  and  $\{r, c\}$  in Tables 1 and 3, respectively (both relating to the Pell polynomials). A relationship between  $[m, n]$  and  $\overline{r, c}$  will also be formulated for the Pell-Lucas polynomials.

Now in (4.9),  $2c - 1$  is the power of  $x$  in  $P_{2r}(x)$ . Comparing the coefficient of the term  $x^{2c-1}$  in (2.15) with that in (4.3), where we recall that

$$\binom{m-1}{m-n} = \binom{m-1}{n-1}$$

we deduce that

$$\{2r, 2c - 1\} = (r + c, r - c + 1) \quad (4.19)$$

and so

$$(r, c) = \{r + c - 1, r - c\}. \quad (4.20)$$

A similar argument applied to (2.15) and (4.3) for (4.1) yields

$$\{2r - 1, 2c\} = (r + c, r - c)$$

whence (4.20) results again.

Lastly, consider  $\overline{2r, 2c}$ , the coefficient of  $x^{2c}$  in  $Q_{2r}(x)$ . From (4.17),

$$\overline{2r, 2c} = \left( \binom{r+c}{r-c} + \binom{r+c-1}{r-c-1} \right) 2^{2c}.$$

Using (4.7) with (4.3), we find

$$[m, n] = \left( \binom{m}{n-1} + \binom{m-1}{n-2} \right) 2^{m-n+1}$$

whence, by comparison of the two forms,

$$\overline{2r, 2c} = [r + c, r - c + 1]. \quad (4.21)$$

Reversely,

$$[r, c] = \overline{r + c - 1, r - c + 1}. \quad (4.22)$$

A similar formula to (4.21) is

$$\overline{2r - 1, 2c + 1} = [r + c, r - c]$$

whence (4.22) results again.

5. DETERMINANTAL GENERATION OF  $P_n(x)$ ,  $Q_n(x)$

Write  $d_{ij}$  for the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of an  $n \times n$  determinant.

Let  $\Delta_n(x)$  be the  $n \times n$  determinant defined by

$$\Delta_n(x) : \begin{cases} d_{ii} = 2x & i = 1, 2, \dots, n \\ d_{i, i+1} = 1 & i = 1, \dots, n - 1 \\ d_{i, i-1} = -1 & i = 2, \dots, n \\ d_{ij} = 0 & \text{otherwise} \end{cases} \quad (5.1)$$

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From  $\Delta_n(x)$ , the determinants  $\delta_n(x)$ ,  $\Delta_n^*(x)$ , and  $\delta_n^*(x)$  are defined as follows:

$$\delta_n(x): \text{ as for } \Delta_n(x) \text{ except that } d_{i, i+1} = -1, d_{i, i-1} = 1 \quad (5.2)$$

$$\Delta_n^*(x): \text{ as for } \Delta_n(x) \text{ except that } d_{12} = 2, d_{i, i+1} = 1 \quad (5.3)$$

$(i = 2, \dots, n - 1)$

$$\delta_n^*(x): \text{ as for } \Delta_n(x) \text{ except that } d_{12} = -2, d_{i, i+1} = -1 \quad (5.4)$$

$(i = 2, \dots, n - 1) \quad d_{i, i-1} = 1.$

Induction and expansion along the first row, together with basic properties of  $P_n(x)$  and  $Q_n(x)$ , e.g., (1.1), (2.1), yield

$$\Delta_n(x) = P_{n+1}(x) \quad (5.5)$$

$$\delta_n(x) = P_{n+1}(x) \quad (5.6)$$

$$\Delta_n^*(x) = Q_n(x) \quad (5.7)$$

$$\delta_n^*(x) = Q_n(x). \quad (5.8)$$

In the process of expansion, we derive recurrence relations such as

$$\Delta_k(x) = 2x\Delta_{k-1}(x) + \Delta_{k-2}(x) \quad k \geq 3 \quad (5.9)$$

and

$$\Delta_k^*(x) = 2x\Delta_{k-1}^*(x) + 2\Delta_{k-2}^*(x) \quad k \geq 3. \quad (5.10)$$

6. RELATIONS OF  $P_n(x)$ ,  $Q_n(x)$  TO OTHER FUNCTIONS

Perhaps the simplest results relating  $P_n(x)$  to other functions are found in [4]:

$$P_{2n}(x) = \sinh 2nt / \cosh t \quad \left. \vphantom{P_{2n}(x)} \right\} x = \sinh t \quad (6.1)$$

$$P_{2n+1}(x) = \cosh(2n + 1)t / \cosh t \quad \left. \vphantom{P_{2n+1}(x)} \right\} x = \sinh t \quad (6.2)$$

Hence

$$Q_{2n}(x) = 2 \cosh 2nt \quad \left. \vphantom{Q_{2n}(x)} \right\} x = \sinh t \quad (6.3)$$

$$Q_{2n+1}(x) = 2 \sinh(2n + 1)t \quad \left. \vphantom{Q_{2n+1}(x)} \right\} x = \sinh t \quad (6.4)$$

Comparison of the explicit summation formulas for  $P_n(x)$  and  $Q_n(x)$  given in (2.15) and (2.16) with the explicit summation formulas for  $U_n(x)$  and  $T_n(x)$ , the Chebyshev polynomials of the second and first kinds, respectively (see [11]), shows that

$$P_n(x) = (-i)^{n-1} U_{n-1}(ix) \quad (6.5)$$

and

$$Q_n(x) = 2(-i)^n T_n(ix) \quad (6.6)$$

i.e.,  $P_n(x)$  and  $Q_n(x)$  are modified Chebyshev polynomials in a complex variable. To reconcile the form in [11] with (2.16) we had to replace the Gamma function, namely,  $\Gamma(n - m) = (n - m - 1)!$

Because of (6.5) and (6.6),  $P_n(x)$  and  $Q_n(x)$  would have [9] complex hypergeometric representations. Other representations also exist in view of the many forms the expressions for  $U_n(x)$  and  $T_n(x)$  can take.

In particular, we may record that

$$P_n(i \cosh x) = i^{n-1} \sinh nx / \sinh x \quad (6.7)$$

and

$$Q_n(i \cosh x) = 2i^n \cosh nx. \quad (6.8)$$

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From (1.1) we observe that

$$P_{n+1}(ix) + P_{n-1}(ix) = Q_n(ix)$$

leads, with the help of (6.5) and (6.6), to

$$U_n(ix) - U_{n-2}(ix) = 2T_n(ix), \tag{6.9}$$

which is a complex version of a basic relationship between the two kinds of Chebyshev polynomials. Similarly, other Chebyshev relationships may be tied to corresponding relationships involving  $P_n(x)$  and  $Q_n(x)$ .

Finally, we allude to the *Gegenbauer (ultraspherical) polynomial* of degree  $n$  and order  $\nu$ ,  $C_n^\nu(x)$ , defined by

$$\sum_{n=0}^{\infty} C_n^\nu(x) t^n = (1 - 2xt + t^2)^{-\nu} \quad (\nu > 0, \quad |t| < 1). \tag{6.10}$$

with explicit forms

$$C_n^0(x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^r}{n-r} \binom{n-r}{r} (2x)^{n-2r} \quad C_0^0(x) = 1 \quad (\nu = 0) \tag{6.11}$$

and

$$C_n^\nu(x) = \frac{1}{\Gamma(\nu)} \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \frac{\Gamma(n-r+\nu)}{\Gamma(n-r+1)} \binom{n-r}{r} (2x)^{n-2r} \quad (\nu > -\frac{1}{2}; \nu \neq 0). \tag{6.12}$$

A recurrence relation for  $C_n^\nu(x)$  is

$$(n+2)C_{n+2}^\nu(x) = 2(n+\nu+1)x C_{n+1}^\nu(x) - (n+2\nu)C_n^\nu(x) \tag{6.13}$$

which, for  $\nu = 1$ , reduces to

$$C_{n+2}^1(x) = 2xC_{n+1}^1(x) - C_n^1(x) \tag{6.14}$$

with

$$C_0^1(x) = 1, \quad C_1^1(x) = 2x. \tag{6.15}$$

Clearly,  $C_n^1(x) = U_n(x)$ , and by (6.5),

$$P_n(x) = (-i)^{n-1} C_{n-1}^1(ix). \tag{6.16}$$

When  $\nu = 0$ , (6.11), where  $C_1^0(x) = 2x$ , gives

$$C_n^0(x) = \frac{2}{n} T_n(x),$$

so that (6.6) gives

$$Q_n(x) = n(-i) C_n^0(ix) \quad (n \geq 1) \tag{6.17}$$

i.e.,  $P_n(x)$ ,  $Q_n(x)$  are modified Gegenbauer polynomials in a complex variable.

As the Fibonacci and Lucas numbers arise from  $P_n(x)$  and  $Q_n(x)$  when  $x = \frac{1}{2}$ , we have, from (6.16) and (6.17),

$$F_1 = C_0^1\left(\frac{i}{2}\right) = 1, \quad F_n = (-i)^{n-1} C_{n-1}^1\left(\frac{i}{2}\right) \tag{6.18}$$

and

$$L_0 = 2C_0^0\left(\frac{i}{2}\right) = 2, \quad L_n = n(-1)^n C_n^0\left(\frac{i}{2}\right) \quad n \geq 1. \tag{6.19}$$

Using the known [9] result  $dT_n(x)/dx = nU_{n-1}(x)$  from [11] with (6.5) and (6.6), we can arrive back at (2.17), viz.,  $dQ_n(x)/dx = 2nP_n(x)$ .

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Differentiating in (2.15) and applying (6.12) in the case  $v = 2$ , we deduce that

$$\frac{dP_n(x)}{dx} = 2(-i)^{n-2}C_{n-2}^2(ix). \quad (6.20)$$

Alternatively, we may differentiate in (6.16) and invoke the result [11]

$$\frac{dC_n^v(x)}{dx} = 2vC_{n-1}^{v+1}(x)$$

to obtain (6.20).

Some of the above results, e.g., (6.16), were generalized in [12] for the sequence of polynomials  $\{A_n(x)\}$  defined by

$$A_{n+2}(x) = 2xA_{n+1}(x) + A_n(x) \quad A_0(x) = s, \quad A_1(x) = r. \quad (6.21)$$

Of course,  $\{A_n(x)\}$  is a special case of the sequence  $\{W_n(p, q; a, b)\}$ , some of whose properties are documented in [8].

Information related to some aspects of the above ideas can be found in [1], [2], [3], [4], [5], [9], and [10].

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