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The log-concavity and log-convexity properties associated to hyperpell and hyperpell-lucas sequences

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Abstract

We establish the log-concavity and the log-convexity properties for the hyperpell, hyperpell-lucas and associated sequences. Further, we investigate the q-log-concavity property.

Keywords: hyperpell numbers; hyperpell-lucas numbers; log-concavity; q-log-concavity, log-convexity.

MSC: 11B39; 05A19; 11B37.

1. Introduction

Zheng and Liu [13] discuss the properties of the hyperfibonacci numbers $F_n^{[r]}$ and the hyperlucas numbers $L_n^{[r]}$. They investigate the log-concavity and the log convexity property of hyperfibonacci and hyperlucas numbers. In addition, they extend their work to the generalized hyperfibonacci and hyperlucas numbers.

The *hyperfibonacci* numbers $F_n^{[r]}$ and *hyperlucas* numbers $L_n^{[r]}$, introduced by Dil and Mező [9] are defined as follows. Put

$$F_n^{[r]} = \sum_{k=0}^{n} F_k^{[r-1]}, \quad \text{with} \quad F_n^{[0]} = F_n,$$

$$L_n^{[r]} = \sum_{k=0}^{n} L_k^{[r-1]}, \quad \text{with} \quad L_n^{[0]} = L_n,$$

where r is a positive integer, and F_n and L_n are the Fibonacci and Lucas numbers, respectively.

Belbachir and Belkhir [1] gave a combinatorial interpretation and an explicit formula for hyperfibonacci numbers,

$$F_{n+1}^{[r]} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n+r-k}{k+r}. \tag{1.1}$$

Let $\{U_n\}_{n\geq 0}$ and $\{V_n\}_{n\geq 0}$ denote the generalized Fibonacci and Lucas sequences given by the recurrence relation

$$W_{n+1} = pW_n + W_{n-1} \quad (n \geq 1), \quad \text{with} \;\; U_0 = 0,\; U_1 = 1,\; V_0 = 2,\; V_1 = p. \tag{1.2}$$

The Binet forms of U_n and V_n are

$$U_n = \frac{\tau^n - (-1)^n \tau^{-n}}{\sqrt{\Delta}} \;\; \text{and} \;\; V_n = \tau^n + (-1)^n \tau^{-n}; \tag{1.3}$$

with $\Delta = p^2 + 4$, $\tau = (p + \sqrt{\Delta})/2$, and $p \geq 1$.

The generalized hyperfibonacci and generalized hyperlucas numbers are defined, respectively, by

$$U_n^{[r]} := \sum_{k=0}^{n} U_k^{[r-1]}, \quad \text{with} \quad U_n^{[0]} = U_n,$$

$$V_n^{[r]} := \sum_{k=0}^{n} V_k^{[r-1]}, \quad \text{with} \quad V_n^{[0]} = V_n.$$

The paper of Zheng and Liu [13] allows us to exploit other relevant results. More precisely, we propose some results on log-concavity and log-convexity in the case of $p = 2$ for the hyperpell sequence and the hyperpell-lucas sequence.

Definition 1.1. Hyperpell numbers $P_n^{[r]}$ and hyperpell-lucas numbers $Q_n^{[r]}$ are defined by

$$P_n^{[r]} := \sum_{k=0}^{n} P_k^{[r-1]}, \quad \text{with} \quad P_n^{[0]} = P_n,$$

$$Q_n^{[r]} := \sum_{k=0}^{n} Q_k^{[r-1]}, \quad \text{with} \quad Q_n^{[0]} = Q_n,$$

where r is a positive integer, and $\{P_n\}$ and $\{Q_n\}$ are the Pell and the Pell-Lucas sequences respectively.

Now we recall some formulas for Pell and Pell-Lucas numbers. It is well know that the Binet forms of P_n and Q_n are

$$P_n = \frac{\alpha^n - (-1)^n \alpha^{-n}}{2\sqrt{2}} \ \text{ and } \ Q_n = \alpha^n + (-1)^n \alpha^{-n}, \tag{1.4}$$

where $\alpha = (1 + \sqrt{2})$. The integers

$$P(n,k) = 2^{n-2k} \binom{n-k}{k} \ \text{ and } \ Q(n,k) = 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k}, \tag{1.5}$$

are linked to the sequences $\{P_n\}$ and $\{Q_n\}$. It is established [2] that for each fixed n these two sequences are log-concave and then unimodal. For the generalized sequence given by (1.2) , also the corresponding associated sequences are log-concave and then unimodal, see [3, 4].

The sequences $\{P_n\}$ and $\{Q_n\}$ satisfy the recurrence relation (1.2), for $p = 2$, and for $n \geq 0$ and $n \geq 1$ respectively, we have

$$P_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-k}{k} \ \text{ and } \ Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k}. \tag{1.6}$$

It follows from (1.4) that the following formulas hold

$$P_n^2 - P_{n-1}P_{n+1} = (-1)^{n+1}, \tag{1.7}$$

$$Q_n^2 - Q_{n-1}Q_{n+1} = 8(-1)^n. \tag{1.8}$$

It is easy to see, for example by induction, that for $n \geq 1$

$$P_n \geq n \ \text{ and } \ Q_n \geq n. \tag{1.9}$$

Let $\{x_n\}_{n\geq 0}$ be a sequence of nonnegative numbers. The sequence $\{x_n\}_{n\geq 0}$ is *log-concave* (respectively *log-convex*) if $x_j^2 \geq x_{j-1}x_{j+1}$ (respectively $x_j^2 \leq x_{j-1}x_{j+1}$) for all $j > 0$, which is equivalent (see [5]) to $x_i x_j \geq x_{i-1}x_{j+1}$ (respectively $x_i x_j \leq x_{i-1}x_{j+1}$) for $j \geq i \geq 1$.

We say that $\{x_n\}_{n\geq 0}$ is *log-balanced* if $\{x_n\}_{n\geq 0}$ is log-convex and $\{x_n/n!\}_{n\geq 0}$ is log-concave.

Let q be an indeterminate and $\{f_n(q)\}_{n\geq 0}$ be a sequence of polynomials of q. If for each $n \geq 1$, $f_n^2(q) - f_{n-1}(q)f_{n+1}(q)$ has nonnegative coefficients, we say that $\{f_n(q)\}_{n\geq 0}$ is *q-log-concave.*

In section 2, we give the generating functions of hyperpell and hyperpell-lucas sequences. In section 3, we discuss their log-concavity and log-convexity. We investigate also the q-log-concavity of some polynomials related to hyperpell and hyperpell-lucas numbers.

2. The generating functions

The generating function of Pell numbers and Pell-Lucas numbers denoted $G_P(t)$ and $G_Q(t)$, respectively, are

$$G_P(t) := \sum_{n=0}^{+\infty} P_n t^n = \frac{t}{1-2t-t^2}, \tag{2.1}$$

and

$$G_Q(t) := \sum_{n=0}^{+\infty} Q_n t^n = \frac{2-2t}{1-2t-t^2}. \tag{2.2}$$

So, we establish the generating function of hyperpell and hyperpell-lucas numbers using respectively

$$P_n^{[r]} = P_{n-1}^{[r]} + P_n^{[r-1]} \quad \text{and} \quad Q_n^{[r]} = Q_{n-1}^{[r]} + Q_n^{[r-1]}. \tag{2.3}$$

The generating functions of hyperpell numbers and hyperlucas numbers are

$$G_P^{[r]}(t) = \sum_{n=0}^{\infty} P_n^{[r]} t^n = \frac{t}{(1-2t-t^2)(1-t)^r}, \tag{2.4}$$

and

$$G_Q^{[r]}(t) = \sum_{n=0}^{\infty} Q_n^{[r]} t^n = \frac{2-2t}{(1-2t-t^2)(1-t)^r}. \tag{2.5}$$

3. The log-concavity and log-convexity properties

We start the section by some useful lemmas.

Lemma 3.1. [12] *If the sequences $\{x_n\}$ and $\{y_n\}$ are log-concave, then so is their ordinary convolution $z_n = \sum_{k=0}^{n} x_k y_{n-k}$, $n = 0, 1, \ldots$.*

Lemma 3.2. [12] *If the sequence $\{x_n\}$ is log-concave, then so is the binomial convolution $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k$, $n = 0, 1, \ldots$.*

Lemma 3.3. [8] *If the sequence $\{x_n\}$ is log-convex, then so is the binomial convolution $z_n = \sum_{k=0}^{n} \binom{n}{k} x_k$, $n = 0, 1, \ldots$.*

The following result deals with the log-concavity of hyperpell numbers and hyperlucas sequences.

Theorem 3.4. *The sequences $\left\{P_n^{[r]}\right\}_{n\geq 0}$ and $\left\{Q_n^{[r]}\right\}_{n\geq 0}$ are log-concave for $r \geq 1$ and $r \geq 2$ respectively.*

Proof. We have

$$P_n^{[1]} = \frac{1}{4}\left(Q_{n+1} - 2\right) \quad \text{and} \quad Q_n^{[1]} = 2P_{n+1}. \tag{3.1}$$

When $n = 1$, $\left(P_n^{[1]}\right)^2 - P_{n-1}^{[1]} P_{n+1}^{[1]} = 1 > 0$. When $n \geq 2$, it follows from (3.1) and (1.8) that

$$\begin{aligned}\left(P_n^{[1]}\right)^2 - P_{n-1}^{[1]} P_{n+1}^{[1]} &= \frac{1}{16}\left[\left(Q_{n+1} - 2\right)^2 - \left(Q_n - 2\right)\left(Q_{n+2} - 2\right)\right] \\ &= \frac{1}{16}\left(Q_{n+1}^2 - Q_n Q_{n+2} - 4Q_{n+1} + 2Q_n + 2Q_{n+2}\right) \\ &= \frac{1}{4}\left(2(-1)^{n-1} + Q_{n+1}\right) \geq 0.\end{aligned}$$

Then $\left\{P_n^{[1]}\right\}_{n\geq 0}$ is log-concave. By Lemma 3.1, we know that $\left\{P_n^{[r]}\right\}_{n\geq 0}$ $(r \geq 1)$ is log-concave.

It follows from (3.1) and (1.7) that

$$\left(Q_n^{[1]}\right)^2 - Q_{n-1}^{[1]} Q_{n+1}^{[1]} = 4\left(P_{n+1}^2 - P_n P_{n+2}\right) = 4\left(-1\right)^n = \pm 4 \tag{3.2}$$

Hence $\left\{Q_n^{[1]}\right\}_{n\geq 0}$ is not log-concave.

One can verify that

$$Q_n^{[2]} = \frac{1}{2}\left(Q_{n+2} - 2\right) = 2P_{n+1}^{[1]}. \tag{3.3}$$

Then $\left\{Q_n^{[2]}\right\}_{n\geq 0}$ is log-concave. By Lemma 3.1, we know that $\left\{Q_n^{[r]}\right\}_{n\geq 0}$ $(r \geq 2)$ is log-concave. This completes the proof of Theorem 3.4. □

Then we have the following corollary.

Corollary 3.5. *The sequences* $\left\{\sum_{k=0}^{n}\binom{n}{k}P_k^{[r]}\right\}_{n\geq 0}$ *and* $\left\{\sum_{k=0}^{n}\binom{n}{k}Q_k^{[r]}\right\}_{n\geq 0}$ *are log-concave for* $r \geq 1$ *and* $r \geq 2$ *respectively.*

Proof. Use Lemma 3.2.

Now we establish the *log-concavity of order two* of the sequences $\left\{P_n^{[1]}\right\}_{n\geq 0}$ and $\left\{Q_n^{[2]}\right\}_{n\geq 0}$ for some special sub-sequences. □

Theorem 3.6. *Let be for* $n \geq 1$

$$T_n := \left(P_n^{[1]}\right)^2 - P_{n-1}^{[1]} P_{n+1}^{[1]} \quad \text{and} \quad R_n := \left(Q_n^{[2]}\right)^2 - Q_{n-1}^{[2]} Q_{n+1}^{[2]}.$$

Then $\{T_{2n}\}_{n\geq 1}$, $\{R_{2n+1}\}_{n\geq 0}$ *are log-concave, and* $\{T_{2n+1}\}_{n\geq 0}$, $\{R_{2n}\}_{n\geq 1}$ *are log-convex.*

Proof. Using respectively (3.3) and (1.8), we get

$$\left(Q_n^{[2]}\right)^2 - Q_{n-1}^{[2]}Q_{n+1}^{[2]} = 2(-1)^n + Q_{n+1},$$

and thus, for $n \geq 1$,

$$T_n = \frac{1}{4}\left(2\left(-1\right)^{n-1} + Q_n\right) \quad \text{and} \quad R_n = 2(-1)^n + Q_{n+1}. \tag{3.4}$$

By applying (3.4) and (1.8), for $n \geq 1$ we get

$$Q_{2n}^2 - Q_{2n-2}Q_{2n+2} = -32 \quad \text{and} \quad Q_{2n+1}^2 - Q_{2n-1}Q_{2n+3} = 32. \tag{3.5}$$

Then

$$\begin{aligned} T_{2n}^2 - T_{2(n-1)}T_{2(n+1)} &= \frac{1}{16}\left(Q_{2n}^2 - Q_{2n-2}Q_{2n+2} - 4Q_{2n} + 2Q_{2n-2} + 2Q_{2n+2}\right) \\ &= 4(Q_{2n} - 4) > 0. \end{aligned}$$

and

$$\begin{aligned} R_{2n+1}^2 - R_{2n-1}R_{2n+3} &= \left(Q_{2n+2}^2 - Q_{2n}Q_{2n+2} - 4Q_{2n+2} + 2Q_{2n} + 2Q_{2n+4}\right) \\ &= 64(Q_{2n+2} - 4) > 0. \end{aligned}$$

Then $\{T_{2n}\}_{n\geq 1}$ and $\{R_{2n+1}\}_{n\geq 0}$ are log-concave.

Similarly by applying (3.4) and (3.5), we have

$$T_{2n+1}^2 - T_{2n-1}T_{2n+3} = -\frac{1}{2}Q_{2n+1} < 0,$$

and

$$R_{2n}^2 - R_{2(n-1)}R_{2(n+1)} = -8Q_{2n+1} < 0.$$

Then $\{T_{2n+1}\}_{n\geq 0}$ and $\{R_{2n}\}_{n\geq 1}$ are log-convex. This completes the proof. □

Corollary 3.7. *The sequences* $\left\{\sum_{k=0}^{n}\binom{n}{k}T_{2k}\right\}_{n\geq 0}$ *and* $\left\{\sum_{k=0}^{n}\binom{n}{k}R_{2k+1}\right\}_{n\geq 0}$ *are log-concave.*

Proof. Use Lemma 3.2. □

Corollary 3.8. *The sequences* $\left\{\sum_{k=0}^{n}\binom{n}{k}T_{2k+1}\right\}_{n\geq 1}$ *and* $\left\{\sum_{k=0}^{n}\binom{n}{k}R_{2k}\right\}_{n\geq 1}$ *are log-convex.*

Proof. Use Lemma 3.3. □

Lemma 3.9. *Let* $a_n := \sum_{k=0}^{n}\binom{n}{k}P_{k+1}$, *where* $\{P_n\}_{n\geq 0}$ *is the Pell sequence. Then* $\{a_n\}_{n\geq 0}$ *satisfy the following recurrence relations*

$$a_n = 3a_{n-1} + \sum_{k=0}^{n-2} a_k \quad \text{and} \quad a_n = 4a_{n-1} - 2a_{n-2}.$$

Proof. Let be $b_n := \sum_{k=0}^{n} \binom{n}{k} P_k$, where $\{P_n\}_{n\geq -1}$ is the Pell sequence extended to $P_{-1} = 1$.

Using Pascal formula and the recurrence relation of Pell sequence together into the development $\sum_{k=0}^{n} \binom{n}{k} P_{k+1}$ we get $a_n = 3a_{n-1} + b_{n-1}$, then by $b_n = b_{n-1} + a_{n-1}$. By iterated use of this relation with the precedent one, we get $a_n = 3a_{n-1} + \sum_{k=0}^{n-2} a_k$ (with $b_0 = 0$ and $a_0 = 1$), thus $a_n = 4a_{n-1} - 2a_{n-2}$. □

Theorem 3.10. *The sequences* $\left\{nQ_n^{[1]}\right\}_{n\geq 0}$ *and* $\left\{\sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]}\right\}_{n\geq 0}$ *are log-concave and log-convex, respectively.*

Proof. Let be

$$S_n := n^2 \left(Q_n^{[1]}\right)^2 - (n^2-1) Q_{n-1}^{[1]} Q_{n+1}^{[1]} \quad \text{and} \quad K_n := \sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]},$$

with the convention that $K_{<0} = 0$.

From (3.2), we have

$$\begin{aligned} S_n &= 4(n^2-1)\,(-1)^n + \left(Q_n^{[1]}\right)^2 \\ &= 4\left[(n^2-1)\,(-1)^n + P_{n+1}^2\right] \geq 4\left[(n^2-1)\,(-1)^n + (n+1)^2\right] > 0. \end{aligned}$$

Then $\left\{nQ_n^{[1]}\right\}_{n\geq 0}$ is log-concave.

Using Lemma 3.9, we can verify that

$$K_n = 4K_{n-1} - 2K_{n-2}. \tag{3.6}$$

The associated Binet-formula is

$$K_n = \frac{\left(1+\sqrt{2}\right)\alpha^n - \left(1-\sqrt{2}\right)\beta^n}{\alpha - \beta}, \quad \text{with} \quad \alpha, \beta = 2 \pm \sqrt{2},$$

which provides

$$K_n^2 - K_{n-1}K_{n+1} = -2^{n+1} < 0.$$

Then $\left\{\sum_{k=0}^{n} \binom{n}{k} Q_k^{[1]}\right\}_{n\geq 0}$ is log-convex. □

Remark 3.11. The terms of the sequence $\{K_n\}_n$ satisfy $K_n = 2^{(n+2)/2} P_{n+1}$ if n is even, and $K_n = 2^{(n-1)/2} Q_{n+1}$ if n is odd.

Theorem 3.12. *The sequences* $\left\{n!P_n^{[1]}\right\}_{n\geq 0}$ *and* $\left\{n!Q_n^{[2]}\right\}_{n\geq 0}$ *are log-balanced.*

Proof. By Theorem 3.4, in order to prove the log-balanced property of $\left\{n!P_n^{[1]}\right\}_{n\geq 0}$ and $\left\{n!Q_n^{[2]}\right\}_{n\geq 0}$ we only need to show that they are log-convex. It follows from the proof of Theorem 3.4 that

$$\left(P_n^{[1]}\right)^2 - P_{n-1}^{[1]} P_{n+1}^{[1]} = \frac{1}{4}\left(2\,(-1)^{n-1} + Q_{n+1}\right), \tag{3.7}$$

and from the proof of Theorem 3.6 that

$$\left(Q_n^{[2]}\right)^2 - Q_{n-1}^{[2]}Q_{n+1}^{[2]} = 2\,(-1)^n + Q_{n+1}. \tag{3.8}$$

Let

$$\begin{aligned} M_n &:= n\left(P_n^{[1]}\right)^2 - (n+1)P_{n-1}^{[1]}P_{n+1}^{[1]},\\ B_n &:= n\left(Q_n^{[2]}\right)^2 - (n+1)Q_{n-1}^{[2]}Q_{n+1}^{[2]}, \end{aligned}$$

from (3.3), (3.7) and (3.8), we get

$$\begin{aligned} M_n &= \frac{(n+1)}{4}\left(2\,(-1)^{n-1} + Q_{n+1}\right) - \frac{1}{4}(Q_{n+1}-2)^2,\\ B_n &= (n+1)\,(2\,(-1)^n + Q_{n+1}) - \frac{1}{4}(Q_{n+2}-2)^2. \end{aligned}$$

Clearly $B_n \leq 0$ for $n = 0, 1, 2$. We have by induction that for $n \geq 1$, $Q_n \geq n+1$. This gives

$$B_n \leq (Q_{n+1}-1)\,(2\,(-1)^n + Q_{n+1}) - \frac{1}{4}(2Q_{n+1}+Q_n-2)^2 < 0.$$

Also, $M_n \leq 0$ for $n = 2$ and for $n \geq 3$, $Q_n \geq n+6$. This gives $n+1 \leq Q_{n+1}-6$, and

$$\begin{aligned} M_n &\leq \frac{1}{4}\left[(Q_{n+1}-6)\left(2\,(-1)^{n-1} + Q_{n+1}\right) - (Q_{n+1}-2)^2\right]\\ &= \frac{1}{4}\left[\left(-2 + 2\,(-1)^{n-1}\right)Q_{n+1} - 4 - 12\,(-1)^{n-1}\right] < 0. \end{aligned}$$

Hence $\{n!P_n^{[1]}\}_{n\geq 0}$ and $\{n!Q_n^{[2]}\}_{n\geq 0}$ are log-convex. As the sequences $\{P_n^{[1]}\}_{n\geq 0}$ and $\{Q_n^{[2]}\}_{n\geq 0}$ are log-concave, so the sequences $\{n!P_n^{[1]}\}_{n\geq 0}$ and $\{n!Q_n^{[2]}\}_{n\geq 0}$ are log-balanced. □

Theorem 3.13. *Define, for $r \geq 1$, the polynomials*

$$P_{n,r}(q) := \sum_{k=0}^{n} P_k^{[r]} q^k \quad \text{and} \quad Q_{n,r}(q) := \sum_{k=0}^{n} Q_k^{[r]} q^k.$$

The polynomials $P_{n,r}(q)$ $(r \geq 1)$ and $Q_{n,r}(q)$ $(r \geq 2)$ are q-log-concave.

Proof. When $n \geq 1$, $r \geq 1$,

$$\begin{aligned} &P_{n,r}^2(q) - P_{n-1,r}(q)P_{n+1,r}(q)\\ &= \left(\sum_{k=0}^{n} P_k^{[r]} q^k\right)^2 - \left(\sum_{k=0}^{n-1} P_k^{[r]} q^k\right)\left(\sum_{k=0}^{n+1} P_k^{[r]} q^k\right) \end{aligned}$$

$$= \left(\sum_{k=0}^{n} P_k^{[r]} q^k\right)^2 - \left(\sum_{k=0}^{n} P_k^{[r]} q^k - P_n^{[r]} q^n\right)\left(\sum_{k=0}^{n} P_k^{[r]} q^k + P_{n+1}^{[r]} q^{n+1}\right)$$

$$= \left(P_n^{[r]} q^n - P_{n+1}^{[r]} q^{n+1}\right) \sum_{k=0}^{n} P_k^{[r]} q^k + P_n^{[r]} P_{n+1}^{[r]} q^{2n+1}$$

$$= \sum_{k=1}^{n} \left(P_k^{[r]} P_n^{[r]} - P_{k-1}^{[r]} P_{n+1}^{[r]}\right) q^{k+n}.$$

When $n \geq 1$, $r \geq 2$, through computation, we get

$$Q_{n,r}^2(q) - Q_{n-1,r}(q) Q_{n+1,r}(q) = \sum_{k=1}^{n} \left(Q_k^{[r]} Q_n^{[r]} - Q_{k-1}^{[r]} Q_{n+1}^{[r]}\right) q^{k+n} + Q_n^{[r]} q^n.$$

As $\left\{P_n^{[r]}\right\}$ and $\left\{Q_n^{[r]}\right\}$ $(r \geq 2)$ are log-concave, then the polynomials $P_{n,r}(q)$ $(r \geq 1)$ and $Q_{n,r}(q)$ $(r \geq 2)$ are q-log-concave. □

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References

[1] Belbachir, H., Belkhir, A., Combinatorial Expressions Involving Fibonacci, Hyperfibonacci, and Incomplete Fibonacci Numbers, *J. Integer Seq.,* Vol. 17 (2014), Article 14.4.3.

[2] Belbachir, H., Bencherif, F., Unimodality of sequences associated to Pell numbers, *Ars Combin.,* 102 (2011), 305–311.

[3] Belbachir, H., Bencherif, F., Szalay, L., Unimodality of certain sequences connected with binomial coefficients, *J. Integer Seq.,* 10 (2007), Article 07. 2. 3.

[4] Belbachir, H., Szalay, L., Unimodal rays in the regular and generalized Pascal triangles, *J. Integer Seq.,* 11 (2008), Article. 08.2.4.

[5] Brenti, F., Unimodal, log-concave and Pólya frequency sequences in combinatorics, *Mem. Amer. Math. Soc.,* no. 413 (1989).

[6] Cao, N. N., Zhao, F. Z, Some Properties of Hyperfibonacci and Hyperlucas Numbers, *J. Integer Seq.,* 13 (8) (2010), Article 10.8.8.

[7] Chen, W. Y. C., Wang, L. X. W., Yang, A. L. B., Schur positivity and the q-log-convexity of the Narayana polynomials, *J. Algebr. Comb.,* 32 (2010), 303–338.

[8] Davenport, H., Pólya, G., On the product of two power series, *Canadian J. Math.,* 1 (1949), 1–5.

[9] Dil, A., Mező, I., A symmetric algorithm for hyperharmonic and Fibonacci numbers, *Appl. Math. Comput.,* 206 (2008), 942–951.

[10] Liu, L., Wang, Y., On the log-convexity of combinatorial sequences, *Advances in Applied Mathematics,* vol. 39, Issue 4, (2007), 453–476.

[11] Sloane, N. J. A., *On-line Encyclopedia of Integer Sequences,* http://oeis.org, (2014).

[12] Wang, Y., Yeh, Y. N., Log-concavity and LC-positivity, *Combin. Theory Ser. A,* 114 (2007), 195–210.

[13] Zheng, L. N., Liu, R., On the Log-Concavity of the Hyperfibonacci Numbers and the Hyperlucas Numbers, *J. Integer Seq.,* Vol. 17 (2014), Article 14.1.4.