The Fibonacci Quarterly 1981 (19,5): 450-451

$$2x^2 \equiv x^2 + y^2 \equiv z^2 \pmod{p}.$$

By definition, x^2 is a quadratic residue of p. The above congruence implies $2x^2$ is also a quadratic residue of p. If p were of the form $8t \pm 3$, then 2 would be a quadratic nonresidue of p and since x^2 is a quadratic residue of p, $2x^2$ would be a quadratic nonresidue of p, a contradiction. Thus p must be of the form $8t \pm 1$.

Now, if we assume that there is a finite number of primes of the form $8t \pm 1$, and if we let m be the product of these primes, then we obtain a contradiction by imitating the above proof that there are infinitely many primes.

REFERENCE

1. W. Sierpinski. "Pythagorean Triangles." *Scripta Mathematica Studies*, No. 9. New York: Yeshiva University, 1964.

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AN APPLICATION OF PELL'S EQUATION

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The following problem solution is a good classroom presentation or exercise following a discussion of Pell's equation.

Statement of the Problem

Find all natural numbers a and b such that

$$\frac{a(a+1)}{2} = b^2.$$

An alternate statement of the problem is to ask for all triangular numbers which are squares.

Solution of the Problem

$$\frac{a(a+1)}{2} = b^2 \iff a^2 + a = 2b^2 \iff a^2 + a - 2b^2 = 0 \iff a = \frac{-1 \pm \sqrt{1 + 8b^2}}{2} \iff \exists$$

an odd integer t such that $t^2 - 2(2b)^2 = 1$.

This is Pell's equation with fundamental solution [1, p. 197] $t = 3$ and $2b = 2$ or, equivalently, $t = 3$ and $b = 1$. Note that $t = 3$ implies

$$a = \frac{-1 \pm 3}{2},$$

but, according to the following theorem, we may discard $a = -2$. Also note that t is odd.

Theorem 1: If D is a natural number that is not a perfect square, the Diophantine equation $x^2 - Dy^2 = 1$ has infinitely many solutions x, y.

All solutions with positive x and y are obtained by the formula

$$x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n,$$

where x_1, y_1 is the fundamental solution of $x^2 - Dy^2 = 1$ and where n runs through all natural numbers.

A comparison of $(x_n + y_n\sqrt{2})(3 + 2\sqrt{2})$ and $\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}x_n\\ y_n\end{pmatrix}$ shows that all solutions of $t^2 - 2(2b)^2 = 1$ are obtained by

$$\begin{pmatrix}3 & 4\\ 2 & 3\end{pmatrix}\begin{pmatrix}t_n\\ 2b_n\end{pmatrix} = \begin{pmatrix}t_{n+1}\\ 2b_{n+1}\end{pmatrix}$$

and hence all solutions of $\dfrac{a(a+1)}{2} = b^2$ are obtained from $a_n = \dfrac{t_n - 1}{2}$, $b_n = \dfrac{2b_n}{2}$. Note that t_n is odd for all n so a_n is an integer.

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CENTRAL FACTORIAL NUMBERS AND RELATED EXPANSIONS

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1. INTRODUCTION

The central factorials have been introduced and studied by Stephensen; properties and applications of these factorials have been discussed among others and by Jordan [3], Riordan [5], and recently by Roman and Rota [4].

For positive integer m,

$$x^{[m,b]} = x\left(x + \frac{1}{2}mb - b\right)\left(x + \frac{1}{2}mb - 2b\right) \cdots \left(x - \frac{1}{2}mb + b\right)$$

defines the generalized central factorial of degree m and increment b. This definition can be extended to any integer m as follows:

$$x^{[0,b]} = 1$$
$$x^{[-m,b]} = x^2/x^{[m+2,b]},\ m \text{ a positive integer.}$$

The usual central factorial ($b = 1$) will be denoted by $x^{[m]}$. Note that these factorials are called "Stephensen polynomials" by some authors.

Carlitz and Riordan [1] and Riordan [5, p. 213] studied the connection constants of the sequences $x^{[m]}$ and x^n, that is, the central factorial numbers $t(m, n)$ and $T(m, n)$:

$$x^{[m]} = \sum_{n=0}^{m} t(m, n)x^n,\quad x^m = \sum_{n=0}^{m} T(m, n)x^{[n]};$$

these numbers also appeared in the paper of Comtet [2]. In this paper we discuss some properties of the connection constants of the sequences $x^{[m,g]}$ and $x^{[n,h]}$, $h \neq g$, of generalized central factorials, that is, the numbers $K(m, n, s)$:

$$x^{[m,g]} = \sum_{n=0}^{m} g^m h^{-n} K(m, n, s)x^{[n,h]},\quad s = h/g.$$

2. EXPANSIONS OF CENTRAL FACTORIALS

The central difference operator with increment a, denoted by δ_a, is defined by

$$\delta_a f(x) = f(x + a/2) - f(x - a/2)$$

Note that

$$\delta_a = E_a^{\frac{1}{2}} - E_a^{-\frac{1}{2}} = E_a^{-\frac{1}{2}}\Delta_a, \tag{2.1}$$