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ScienceDirect
Discrete Applied Mathematics 155 (2007) 2371–2376
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

A factorization of the symmetric Pascal matrix involving the Fibonacci matrix

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Received 18 August 2004; received in revised form 23 November 2005; accepted 30 June 2007
Available online 20 August 2007

Abstract

In this short note, we give a factorization of the Pascal matrix. This result was apparently missed by Lee et al. [Some combinatorial identities via Fibonacci numbers, Discrete Appl. Math. 130 (2003) 527–534].

MSC: 05A10; 11B39; 15A23

Keywords: Pascal matrix; Fibonacci matrix; Factorization of matrix

1. Introduction

For a fixed n, the $n \times n$ lower triangular Pascal matrix, $P_n = [p_{i,j}]_{i,j=1,2,\ldots,n}$, (see [1,6]), is defined by

$$p_{i,j} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \geqslant j, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let F_n be the nth Fibonacci number with the generating series $\sum_{n=0}^{\infty} F_n x^n = \frac{x}{1-x-x^2}$. The $n \times n$ Fibonacci matrix $\mathscr{F}_n = [f_{i,j}]_{i,j=1,2,\ldots,n}$ is the unipotent lower triangular Toeplitz matrix defined by

$$f_{i,j} = \begin{cases} F_{i-j+1} & \text{if } i-j+1 \geqslant 0, \\ 0 & \text{if } i-j+1 < 0. \end{cases} \tag{2}$$

In [4], Lee et al. discussed the factorizations of Fibonacci matrix $\mathscr{F}_n$ and the eigenvalues of symmetric Fibonacci matrices $\mathscr{F}_n \mathscr{F}_n^{\mathrm{T}}$. The inverse of $\mathscr{F}_n$ was also given as follows:

$$\mathscr{F}_n^{-1} = [f'_{i,j}]_{i,j=1,2,\ldots,n} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i-2 \leqslant j \leqslant i-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

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0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.06.024

In fact, formula (3) is an immediate consequence of the isomorphism between lower formal power series and lower triangular Toeplitz matrices.

In [5], Lee et al. obtained the following result:

$$P_n = \mathscr{F}_n \mathscr{L}_n, \tag{4}$$

where $\mathscr{L}_n = [l_{i,j}]_{i,j=1,2,\ldots,n}$ is defined by

$$l_{i,j} = \binom{i-1}{j-1} - \binom{i-2}{j-1} - \binom{i-3}{j-1}.$$

In this short note, we give a second factorization of the Pascal matrix which was apparently missed by the authors in [5].

2. The main results

First, we define an $n \times n$ matrix $\mathscr{R}_n = [r_{i,j}]_{i,j=1,2,\ldots,n}$ as follows:

$$r_{i,j} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1}. \tag{5}$$

From the definition of $\mathscr{R}_n$, it is easy to see that $\mathscr{R}_n$ is unipotent lower triangular. It satisfies $r_{i,1} = -\frac{1}{2}(i+1)(i-2)$ for $i \geqslant 2$ and $r_{i,j} = r_{i-1,j} + r_{i-1,j-1}$ for $i, j \geqslant 2$.

Next we give the following factorization of the Pascal matrix.

Theorem 2.1. *We have*

$$P_n = \mathscr{R}_n \mathscr{F}_n. \tag{6}$$

Proof. It suffices to prove $P_n \mathscr{F}_n^{-1} = \mathscr{R}_n$. For $i \geqslant 1$ we have $\sum_{k=1}^{i} p_{i,k} f'_{k,1} = p_{i,1} f'_{1,1} + p_{i,2} f'_{2,1} + p_{i,3} f'_{3,1} = 1 + \binom{i-1}{1}(-1) + \binom{i-1}{2}(-1) = -\frac{1}{2}(i+1)(i-2) = r_{i,1}$, and for $i \geqslant 1,\ j \geqslant 2$, we have $\sum_{k=1}^{n} p_{i,k} f'_{k,j} = p_{i,j} f'_{j,j} + p_{i,j+1} f'_{j+1,j} + p_{i,j+2} f'_{j+2,j} = \binom{i-1}{j-1} - \binom{i-1}{j} - \binom{i-1}{j+1} = r_{i,j}$, which implies that $P_n \mathscr{F}_n^{-1} = \mathscr{R}_n$, as desired. □

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -5 & -1 & 2 & 1 & 0 \\ -9 & -6 & 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 5 & 3 & 2 & 1 & 1 \end{pmatrix}.$$

From the theorem, we have the following combinatorial identity involving the Fibonacci numbers.

Corollary 2.2.

$$\binom{n-1}{r-1} = F_{n-r+1} + (n-2)F_{n-r} + \frac{1}{2}(n^2 - 5n + 2)F_{n-r-1} + \sum_{k=r}^{n-3} \binom{n-1}{k-1} \left[2 - \frac{n}{k} - \frac{(n-k)(n-k-1)}{k(k+1)} \right] F_{k-r+1}. \tag{7}$$

In particular,

$$\sum_{k=1}^{n}\left(\binom{n-1}{k-1}-\binom{n-1}{k}-\binom{n-1}{k+1}\right)F_k=1. \tag{8}$$

Lemma 2.3.

$$\sum_{k=3}^{i}\left\{\binom{i-2}{k-2}-\binom{i-2}{k-1}-\binom{i-2}{k}\right\}F_k=\frac{1}{2}(i+1)(i-2). \tag{9}$$

Proof. We argue by induction on i. If $i=3,4$, then lemma is true, respectively. Suppose the lemma is true for $i\geqslant 4$. Then

$$\begin{aligned}
&\sum_{k=3}^{i+1}\left\{\binom{i-1}{k-2}-\binom{i-1}{k-1}-\binom{i-1}{k}\right\}F_k\\
&=\sum_{k=3}^{i}\left\{\binom{i-2}{k-2}-\binom{i-2}{k-1}-\binom{i-2}{k}\right\}F_k+\sum_{k=3}^{i+1}\left\{\binom{i-2}{k-3}-\binom{i-2}{k-2}-\binom{i-2}{k-1}\right\}F_k\\
&=\frac{1}{2}(i+1)(i-2)+\sum_{k=2}^{i}\left\{\binom{i-2}{k-2}-\binom{i-2}{k-1}-\binom{i-2}{k}\right\}F_{k+1}\\
&=\frac{1}{2}(i+1)(i-2)+\sum_{k=2}^{i}\left\{\binom{i-2}{k-2}-\binom{i-2}{k-1}-\binom{i-2}{k}\right\}\{F_k+F_{k-1}\}\\
&=\frac{1}{2}(i+1)(i-2)+1-(i-2)-\binom{i-2}{2}+\frac{1}{2}(i+1)(i-2)\\
&\quad+\sum_{k=2}^{i}\left\{\binom{i-2}{k-2}-\binom{i-2}{k-1}-\binom{i-2}{k}\right\}F_{k-1}\\
&=(i+1)(i-2)+1-(i-2)-\binom{i-2}{2}+\sum_{k=1}^{i-1}\left\{\binom{i-2}{k-1}-\binom{i-2}{k}-\binom{i-2}{k+1}\right\}F_k\\
&\quad\text{(by applying (8))}\\
&=(i+1)(i-2)+1-(i-2)-\binom{i-2}{2}+1\\
&=\frac{1}{2}(i+2)(i-1).
\end{aligned}$$

Hence the lemma is also true for $i+1$. By induction, we complete the proof. □

Note. Since $\frac{1}{2}(i+1)(i-2)$ is a linear combination of $\binom{i}{k}$ for $k=0,1,2$ $\left(\text{or } \binom{i+1}{k}\right)$, the referee pointed out that Lemma 2.3 follows also from Theorem 2.1.

We define the $n \times n$ matrices $\mathscr{U}_n$, $\overline{\mathscr{U}}_n$ and $\overline{\mathscr{R}}_n$ by

$$\mathscr{U}_n = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -F_3 & 1 & 1 & 0 & \dots & 0 & 0 \\ -F_4 & 0 & 1 & 1 & \dots & 0 & 0 \\ -F_5 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ -F_n & 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}, \tag{10}$$

$\overline{\mathscr{U}}_k = I_{n-k} \oplus \mathscr{U}_k$ and $\overline{\mathscr{R}}_n = [1] \oplus \mathscr{R}_{n-1}$, i.e., $\overline{A}$ is the matrix A shifted one row down and one column to the right with first column given by $(1, 0, 0, \dots)$. From the definition of $\overline{\mathscr{U}}_k$, we have $\overline{\mathscr{U}}_1 = \overline{\mathscr{U}}_2 = I_n$ and $\overline{\mathscr{U}}_n = \mathscr{U}_n$. Hence

Lemma 2.4.

$$\mathscr{R}_n = \overline{\mathscr{R}}_n \mathscr{U}_n. \tag{11}$$

Proof. The (i, j) element of $\overline{\mathscr{R}}_n$ is $r_{i-1,j-1}$, $(i, j = 2, 3, \dots, n)$, or 1 $(i = 1, j = 1)$, or 0 $(i \neq 1, j = 1$ or $i = 1, j \neq 1)$.

Let $\overline{\mathscr{R}}_n \mathscr{U}_n = (D_{i,j})$ and $\mathscr{U}_n = (u_{i,j})$. Obviously, $D_{1,1} = 1 = r_{1,1}$, $D_{2,1} = 0 = r_{2,1}$ and $D_{i,j} = 0$ $(i < j)$. For $i \geqslant 3$, by Lemma 2.3, we have

$$\begin{aligned} D_{i,1} &= \sum_{k=1}^{i} r_{i-1,k-1} u_{k,1} \\ &= -\sum_{k=3}^{i} \left\{ \binom{i-2}{k-2} - \binom{i-2}{k-1} - \binom{i-2}{k} \right\} F_k \\ &= -\frac{1}{2}(i+1)(i-2) \\ &= r_{i,1}. \end{aligned}$$

When $i \geqslant j \geqslant 2$, we have

$$D_{i,j} = \sum_{k=1}^{i} r_{i-1,k-1} u_{k,j} = r_{i-1,j-1} + r_{i-1,j} = r_{i,j}.$$

Thus, $\mathscr{R}_n = \overline{\mathscr{R}}_n \mathscr{U}_n$. $\square$

Example.

$$\mathscr{R}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -5 & -1 & 2 & 1 & 0 \\ -9 & -6 & 1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -5 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{pmatrix} = \overline{\mathscr{R}}_5 \mathscr{U}_5.$$

An immediate consequence of Lemma 2.4 and the definition of the $\overline{\mathscr{U}}_k$ is

Theorem 2.5.

$$\mathscr{R}_n = \overline{\mathscr{U}}_1\overline{\mathscr{U}}_2 \ldots \overline{\mathscr{U}}_{n-1}\overline{\mathscr{U}}_n. \tag{12}$$

Example.

$$\mathscr{R}_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -5 & -1 & 2 & 1 & 0 \\ -9 & -6 & 1 & 3 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let

$$S_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad S_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$S_k = S_0 \oplus I_k$, for $k \in \mathbb{N}$, $\overline{\mathscr{F}}_n = [1] \oplus \mathscr{F}_{n-1}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and $G_k = I_{n-k} \oplus S_{k-3}$ for $k \geqslant 3$. In [4], the authors gave the following result:

$$\mathscr{F}_n = G_1 G_2 \ldots G_n.$$

Hence we have:

Theorem 2.6.

$$P_n = \overline{\mathscr{U}}_1\overline{\mathscr{U}}_2 \ldots \overline{\mathscr{U}}_{n-1}\overline{\mathscr{U}}_n G_1 G_2 \ldots G_n. \tag{13}$$

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 1 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. A remark

In this note, all matrix-identities are expressed using finite matrices. Since all matrix-identities involve lower-triangular matrices, they have an analogue for infinite matrices. We state them briefly as follows.

Let P, $\mathscr{F}$, $\mathscr{L}$, $\mathscr{U}$ and $\mathscr{R}$ are the infinite cases of the matrices P_n, $\mathscr{F}_n$, $\mathscr{L}_n$, $\mathscr{U}_n$ and $\mathscr{R}_n$, respectively. Furthermore, define

$$\mathscr{U}^{(k)} = I_k \oplus \mathscr{U}$$

and

$$\mathscr{R}^{(k)} = I_k \oplus \mathscr{R}.$$

Then $P = \mathscr{F}\mathscr{L} = \mathscr{R}\mathscr{F}$ (cf. (4) and Theorem 2.1), $\mathscr{R} = \mathscr{R}^{(1)}\mathscr{U}$ (cf. Lemma 2.4) and $\mathscr{R} = \mathscr{R}^{(t+1)}\mathscr{U}^{(t)} \ldots \mathscr{U}^{(2)}\mathscr{U}^{(1)}\mathscr{U}$, where t is an arbitrary nonnegative integer (cf. Theorem 2.5).

Acknowledgements

We would like to thank the two anonymous referees for their many valuable suggestions. This research is supported by the National Natural Science Foundation of China (Grant No. 10471016), the Natural Science Foundation of Henan Province (Grant No. 0511010300) and the Natural Science Foundation of the Education Department of Henan Province (Grant No. 2003110009).

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