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The Generalized Pascal-Like Triangle and Applications

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Abstract

We construct the generalized Pascal-like triangle and derive the explicit formulas for the second order linear recurrences by using some properties of this triangle. Applications to earlier results about generalized Fibonacci and Lucas numbers.

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1 Introduction

The second order linear recurrence sequence $\{W_n\}_{n\geq 0}$ of real numbers is defined by

$$W_{n+2} = aW_{n+1} + bW_n \tag{1}$$

where $W_0 = p$ and $W_1 = q$.

If $p = 0, q = 1$, then $W_n = U_n$ is the generalized Fibonacci numbers. If $p = 2, q = a$, then $W_n = V_n$ is the generalized Lucas numbers. For $a = b = 1$, U_n and V_n are the well-known Fibonacci numbers F_n and Lucas numbers L_n, respectively.

It is well-known that the explicit formulas for the generalized Fibonacci and Lucas numbers are

$$U_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} a^{n-2i}b^i \ , \ V_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} a^{n-2i}b^i,$$

respectively, see the equations (2.7) and (2.8) in [2], also [1].

In this paper we consider the second order linear recurrent sequence and derive the explicit formula for this sequence by using the Pascal-like triangle which is defined.

2 The generalized Pascal-like triangle

Definition 2.1. *Let n be a positive integer. For $i \in \mathbb{Z}$, the $A_{n,i}$ is defined as*

$$A_{n,i} = \begin{cases} a^{n-1}q & ; i = 0 \\ b^n p & ; n = i \\ aA_{n-1,k} + bA_{n-1,k-1} & ; 0 < i < n \\ 0 & ; \textit{otherwise} \end{cases}. \tag{2}$$

We see that $aA_{n,0} = A_{n+1,0}$ and $bA_{n,n} = A_{n+1,n+1}$.

Definition 2.2. *The generalized Pascal-like triangle is defined as follows*

	0	1	2	3	4	...	n	...
1	$A_{1,0}$	$A_{1,1}$						
2	$A_{2,0}$	$A_{2,1}$	$A_{2,2}$					
3	$A_{3,0}$	$A_{3,1}$	$A_{3,2}$	$A_{3,3}$				
4	$A_{4,0}$	$A_{4,1}$	$A_{4,2}$	$A_{4,3}$	$A_{4,4}$			
$\vdots$			$\vdots$					
n	$A_{n,0}$	$A_{n,1}$	$A_{n,2}$		$\dots$		$A_{n,n}$	
$\vdots$			$\vdots$					

The following triangle will be shown any elements of the Pascal-like triangle in the variables a, b, p and q by using Definition 2.1.

	0	1	2	3	4	...
1	q	bp				
2	aq	$abp + bq$	b^2p			
3	a^2q	$a^2bp + 2abq$	$2ab^2p + b^2q$	b^3p		
4	a^3q	$a^3bp + 3a^2bq$	$3a^2b^2p + 3ab^2q$	$3ab^3p + b^3q$	b^4p	
5	a^4q	$a^4bp + 4a^3bq$	$4a^3b^2p + 6a^2b^2q$	$\dots$		
6	a^5q	$a^5bp + 5a^4bq$	$\dots$			
7	a^6q	$\dots$				
8	$\dots$					

We see that the sums of elements in each row of the Pascal-like triangle is $(a+b)^{n-1}(q+bp)$.

The following theorem gives an alternative definition of $A_{n,i}$ as the binomial sums.

Theorem 2.3. *Let $n \in \mathbb{N}$ and $0 \le i \le n$. Then*

$$A_{n,i} = a^{n-i}b^i p\binom{n-1}{i-1} + a^{n-i-1}b^i q\binom{n-1}{i}. \tag{3}$$

Proof. For $n = 1$, we see that the equation (3) holds for $i = 0, 1$. By induction on n, assume that (3) is true for $k \in \mathbb{N}$ and $0 \le i \le k$. By (2) and the inductive hypothesis, we get

$$\begin{aligned}
A_{k+1,i} &= aA_{k,i} + bA_{k,i-1} \\
&= a\left[a^{k-i}b^i p\binom{k-1}{i-1} + a^{k-i-1}b^i q\binom{k-1}{i}\right] \\
&\quad + b\left[a^{k-i+1}b^{i-1} p\binom{k-1}{i-2} + a^{k-i}b^{i-1} q\binom{k-1}{i-1}\right] \\
&= a^{k-i+1}b^i p\left[\binom{k-1}{i-1} + \binom{k-1}{i-2}\right] + a^{k-i}b^i q\left[\binom{k-1}{i} + \binom{k-1}{i-1}\right] \\
&= a^{k-i+1}b^i p\binom{k}{i-1} + a^{k-i}b^i q\binom{k}{i},
\end{aligned}$$

showing that (3) works for $n = k + 1$. □

3 Explicit formulas

In this section we derive the explicit formulars for the second order recurrence sequence.

Theorem 3.1. *Let $n \in \mathbb{N}$. We have*

$$W_n = \sum_{i=0}^{\lfloor n/2 \rfloor} A_{n-i,i}. \tag{4}$$

Proof. For $n = 1$, we see that the equation (4) holds. By induction on n, assume that (4) is true for $k \in \mathbb{N}$ and $0 \le i \le k$. By (1),(2) and the inductive hypothesis, we get

$$\begin{aligned}
W_{k+1} &= aW_k + bW_{k-1} \\
&= a\sum_{i=0}^{\lfloor k/2 \rfloor} A_{k-i,i} + b\sum_{i=0}^{\lfloor (k-1)/2 \rfloor} A_{k-i-1,i} \\
&= aA_{k,0} + a\sum_{i=1}^{\lfloor k/2 \rfloor} A_{k-i,i} + b\sum_{i=1}^{\lfloor (k+1)/2 \rfloor} A_{k-i,i-1} \\
&= \begin{cases} A_{k+1,0} + \displaystyle\sum_{i=1}^{\lfloor k/2 \rfloor} A_{k-i+1,i} & ; k \text{ is even} \\ A_{k+1,0} + \displaystyle\sum_{i=1}^{\lfloor (k-1)/2 \rfloor} A_{k-i+1,i} + bA_{\frac{k-1}{2},\frac{k-1}{2}} & ; k \text{ is odd} \end{cases}
\end{aligned}$$

$$W_{k+1} = \sum_{i=0}^{\lfloor (k+1)/2 \rfloor} A_{k-i+1,i}$$

showing that (4) works for $n = k+1$. □

By using Theorem 2.3, we can write W_n in the binomial sum.

Corollary 3.2. *Let $n \in \mathbb{N}$. We have*

$$W_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \left[a^{n-2i} b^i p \binom{n-i-1}{i-1} + a^{n-2i-1} b^i q \binom{n-i-1}{i} \right].$$

Particular cases for the Corollary 3.2:

- If we take $p = 0$ and $q = 1$, then $W_n = U_n$ and

$$U_n = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i-1}{i} a^{n-2i-1} b^i.$$

- If we take $p = 2$ and $q = a$, then $W_n = V_n$ and

$$\begin{aligned} V_n &= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[2a^{n-2i} b^i \binom{n-i-1}{i-1} + a^{n-2i} b^i \binom{n-i-1}{i} \right] \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[\binom{n-i}{i} + \binom{n-i-1}{i-1} \right] a^{n-2i} b^i \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \left[\binom{n-i}{i} + \frac{i}{n-i} \binom{n-i}{i} \right] a^{n-2i} b^i \\ &= \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n}{n-i} \binom{n-i}{i} a^{n-2i} b^i. \end{aligned}$$

Two above identities are two identities in section 1.

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