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Fibonacci, Lucas, and Pell Numbers, and Pascal's Triangle

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Fibonacci, Lucas, Pell, and Pell–Lucas numbers belong to a large family of positive integers. Using Lockwood's identity, developed from the binomial theorem, we show how they can be computed from Pascal's triangle, the well-known triangular array of the binomial coefficients $\binom{n}{k}$, where $0 \leq k \leq n$. This close link with Pascal's triangle brings these numbers within reach of many mathematical amateurs.

Introduction

Fibonacci numbers F_n and Lucas numbers L_n continue to provide invaluable opportunities for exploration, and contribute handsomely to the beauty of mathematics, especially number theory. They are often defined recursively as follows:

$$\begin{aligned} &\mathrm{F}_1 = 1 = \mathrm{F}_2, && \mathrm{L}_1 = 1,\ \mathrm{L}_2 = 3, \\ &\mathrm{F}_n = \mathrm{F}_{n-1} + \mathrm{F}_{n-2}, \quad n \geq 3; && \mathrm{L}_n = \mathrm{L}_{n-1} + \mathrm{L}_{n-2}, \quad n \geq 3. \end{aligned}$$

(Both Fibonacci and Lucas numbers satisfy the same recurrence relation.) These recursive definitions can be used to develop Binet's explicit formulas (see reference 1)

$$\mathrm{F}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad \mathrm{L}_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are solutions of the quadratic equation $x^2 = x + 1$ and $n \geq 1$. Notice that $\alpha + \beta = 1$, $\alpha - \beta = \sqrt{5}$, and $\alpha\beta = -1$. (These facts will come in handy a bit later.) They can be confirmed using induction, or generating functions as the French mathematician Abraham De Moivre (1667–1754) did in 1718. The first five Fibonacci numbers are 1, 1, 2, 3, and 5; the first five Lucas numbers are 1, 3, 4, 7, and 11.

Lucas' formula

In 1876, the French mathematician François Edouard Anatole Lucas (1842–1891) discovered another explicit formula for F_n:

$$\mathrm{F}_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k}, \tag{1}$$

where $\lfloor x \rfloor$ denotes the *floor* of the real number x, that is, the greatest integer less than or equal to x.

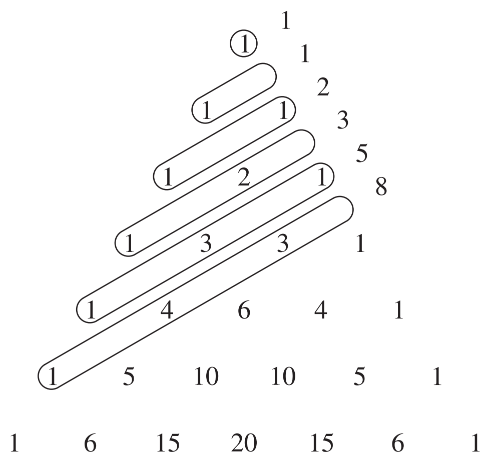

Figure 1 Pascal's triangle.

This formula can be established using the strong version of induction. Since

$$\sum_{k=0}^{0} \binom{1-k-1}{k} = \binom{0}{0} = 1 = \mathrm{F}_1$$

and

$$\sum_{k=0}^{0} \binom{2-k-1}{k} = \binom{0}{0} = 1 = \mathrm{F}_2,$$

(1) holds when $n = 1$ and $n = 2$.

Now assume that (1) holds for all positive integers less than or equal to m, where m is an arbitrary positive integer, i.e.

$$\mathrm{F}_m = \sum_{k=0}^{\lfloor (m-1)/2 \rfloor} \binom{m-k-1}{k}.$$

There are two cases we need to consider: m is odd and m is even. In each case, using Pascal's identity and a lot of algebra, it can be shown that the formula works for $n = m + 1$ (see reference 1). Thus, by the strong version of induction, Lucas' formula holds for every positive integer n.

It follows by Lucas' formula that Fibonacci numbers can be obtained as sums of the binomial coefficients along the rising-diagonals in Pascal's triangle; see figure 1. We will develop another method for computing them from Pascal's triangle. But before we do, we will show how Lucas, Pell, and Pell–Lucas numbers (defined later) can be extracted from the array. To this end, we need an identity which can be obtained from the binomial theorem.

Lockwood's identity

Let x and y be arbitrary real numbers. Then, by the binomial theorem, we have

$$\begin{aligned} x + y &= (x + y), \\ x^2 + y^2 &= (x + y)^2 - 2xy, \\ x^3 + y^3 &= (x + y)^3 - 3(xy)(x + y), \\ x^4 + y^4 &= (x + y)^4 - 4(xy)(x + y)^2 + 2(xy)^2, \\ x^5 + y^5 &= (x + y)^5 - 5(xy)(x + y)^3 + 5(xy)^2(x + y). \end{aligned}$$

In each case, the expression $x^n + y^n$ is expressed as a sum of $\lfloor n/2 \rfloor + 1$ terms in xy and $x + y$.

More generally, we have the following identity, developed by E. H. Lockwood in 1967 (see reference 2):

$$x^n + y^n = (x + y)^n + \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k},$$

where $n \geq 1$. This identity also can be confirmed using the strong version of induction, Pascal's identity, and a lot of messy algebra. It can be rewritten as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x + y)^{n-2k}, \tag{2}$$

where

$$\binom{r}{-1} = 0.$$

It follows from (2), for example, that

$$x^7 + y^7 = (x + y)^7 - 7(xy)(x + y)^5 + 14(xy)^2(x + y)^3 - 7(xy)^3(x + y).$$

Lucas numbers and Pascal's triangle

Lockwood's identity yields several interesting dividends. To begin with, we now can extract Lucas numbers from Pascal's triangle. To see this, we let $x = \alpha$ and $y = \beta$ in (2). This yields

$$\begin{aligned} \mathrm{L}_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (-1)^k \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right]. \end{aligned} \tag{3}$$

1

1 1

1 2 1

1 3 **3** 1

1 **4** 6 **4** 1

1 5 **10** 10 5 1

1 **6** 15 20 15 6 1

1 7 21 35 35 21 7 1

Figure 2

Consequently, L_n can be computed by adding up the elements along two alternate rising-diagonals. For example,

$$
\begin{aligned}
L_7 &= \sum_{k=0}^{3}\left[\binom{7-k}{k} + \binom{6-k}{k-1}\right] \\
&= \left[\binom{7}{0} + \binom{6}{-1}\right] + \left[\binom{6}{1} + \binom{5}{0}\right] + \left[\binom{5}{2} + \binom{4}{1}\right] + \left[\binom{4}{3} + \binom{3}{2}\right] \\
&= (1 + 0) + (6 + 1) + (10 + 4) + (4 + 3) \\
&= (0 + 1 + 4 + 3) + (1 + 6 + 10 + 4) \\
&= 29
\end{aligned}
$$

(see the bold-faced numbers in figure 2).

Notice that (2) can also be written as follows:

$$x^n + y^n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n}{n-k}\binom{n-k}{k}(xy)^k(x+y)^{n-2k}.$$

Consequently,

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k}.$$

Thus L_n can be computed using the elements on the rising-diagonal beginning at $\binom{n}{0}$ with weights $n/(n-k)$. For example,

$$\begin{aligned} L_7 &= \sum_{k=0}^{3} \frac{7}{7-k}\binom{7-k}{k} \\ &= \frac{7}{7}\binom{7}{0} + \frac{7}{6}\binom{6}{1} + \frac{7}{5}\binom{5}{2} + \frac{7}{4}\binom{4}{3} \\ &= 1 + 7 + 14 + 7 \\ &= 29, \end{aligned}$$

as expected.

Equation (3), coupled with (1), yields the following well-known formula connecting Fibonacci and Lucas numbers:

$$\begin{aligned} L_n &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k-1}{k-1} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\ &= \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} \binom{n-i-2}{i} + \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \\ &= F_{n-1} + F_{n+1}. \end{aligned}$$

This can also be established using Binet's formulas, which is a lot simpler (see reference 1).

Next we turn to Pell and Pell–Lucas numbers.

Pell and Pell–Lucas families

Pell numbers P_n and Pell–Lucas numbers Q_n are also often defined recursively as follows:

$$\begin{aligned} &P_1 = 1,\ P_2 = 2, && Q_1 = 1,\ Q_2 = 3, \\ &P_n = 2P_{n-1} + P_{n-2}, \quad n \geq 3; && Q_n = 2Q_{n-1} + Q_{n-2}, \quad n \geq 3. \end{aligned}$$

They can also be defined by Binet-like formulas as follows:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \qquad Q_n = \frac{\gamma^n + \delta^n}{2},$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$, and $n \geq 1$. Notice that $\gamma + \delta = 2$, $\gamma - \delta = 2\sqrt{2}$, and $\gamma\delta = -1$. The first five Pell numbers are 1, 2, 5, 12, and 29; the first five Pell–Lucas numbers are 1, 3, 7, 17, and 41.

Pell–Lucas numbers and Pascal's triangle

We can also extract Pell–Lucas numbers from Pascal's triangle with proper weights. To see this, we let $x = \gamma$ and $y = \delta$ in (2). Then

$$Q_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 2^{n-2k-1}.$$

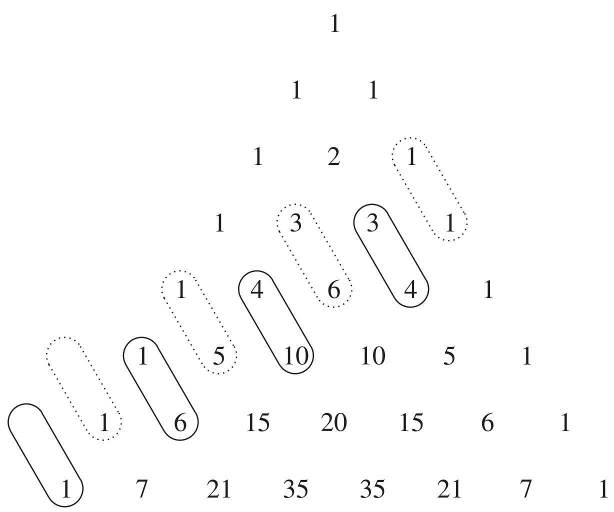

Figure 3

For example,

$$
\begin{aligned}
Q_7 &= \sum_{k=0}^{3}\left[\binom{7-k}{k}+\binom{6-k}{k-1}\right]2^{6-2k}\\
&= \left[\binom{7}{0}+\binom{6}{-1}\right]2^6+\left[\binom{6}{1}+\binom{5}{0}\right]2^4+\left[\binom{5}{2}+\binom{4}{1}\right]2^2+\left[\binom{4}{3}+\binom{3}{2}\right]2^0\\
&= (1+0)\cdot 2^6+(6+1)\cdot 2^4+(10+4)\cdot 2^2+(4+3)\cdot 2^0\\
&= 239.
\end{aligned}
$$

Consequently, Q_7 can be computed by multiplying the sums of the entries inside the solid loops beginning at $\binom{7}{0}$ in figure 3 by the weights 2^6, 2^4, 2^2, and 2^0 respectively, and then adding up the products.

Likewise, $Q_6 = (1+0)\cdot 2^5 + (5+1)\cdot 2^3 + (6+3)\cdot 2^1 + (1+1)\cdot 2^{-1} = 99$; see the dotted loops in figure 3.

Odd-numbered Fibonacci numbers and Pascal's triangle

Next we will show that odd-numbered Fibonacci numbers can be computed from Pascal's triangle in a different way. To this end, we let n be odd and change y to $-y$ in (2). Then we obtain

$$x^n - y^n = \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](-xy)^k(x-y)^{n-2k}. \tag{4}$$

Letting $x=\alpha$ and $y=\beta$, this yields

$$(\alpha-\beta)F_n = \sum_{k=0}^{(n-1)/2}(-1)^k\left[\binom{n-k}{k}+\binom{n-k-1}{k-1}\right](\alpha-\beta)^{n-2k}.$$

So

$$\mathrm{F}_n = \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 5^{(n-2k-1)/2},$$

where n is odd.

For example,

$$\begin{aligned}
\mathrm{F}_7 &= \sum_{k=0}^{3} (-1)^k \left[\binom{7-k}{k} + \binom{6-k}{k-1} \right] 5^{3-k} \\
&= \left[\binom{7}{0} + \binom{6}{-1} \right] 5^3 - \left[\binom{6}{1} + \binom{5}{0} \right] 5^2 + \left[\binom{5}{2} + \binom{4}{1} \right] 5^1 - \left[\binom{4}{3} + \binom{3}{2} \right] 5^0 \\
&= (1+0) \cdot 5^3 - (6+1) \cdot 5^2 + (10+4) \cdot 5^1 - (4+3) \cdot 5^0 \\
&= 13
\end{aligned}$$

(see the solid loops in figure 3).

Odd-numbered Pell numbers and Pascal's triangle

Using (4), odd-numbered Pell numbers also can be computed from Pascal's triangle. To see this, letting $x = \gamma$ and $y = \delta$, (4) yields

$$\begin{aligned}
(\gamma - \delta)\mathrm{P}_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (-\gamma\delta)^k (\gamma - \delta)^{n-2k}, \\
\mathrm{P}_n &= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (2\sqrt{2})^{n-2k-1} \\
&= \sum_{k=0}^{(n-1)/2} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] 8^{(n-2k-1)/2},
\end{aligned}$$

where n is odd. Thus, P_n can be computed using the same loops for F_n, but with different weights, where n is odd.

For example,

$$\begin{aligned}
\mathrm{P}_7 &= \sum_{k=0}^{3} (-1)^k \left[\binom{7-k}{k} + \binom{6-k}{k-1} \right] 8^{3-k} \\
&= (1+0) \cdot 8^3 - (6+1) \cdot 8^2 + (10+4) \cdot 8^1 - (4+3) \cdot 8^0 \\
&= 169
\end{aligned}$$

(see the solid loops in figure 3).

Acknowledgments

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