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CATALAN AND RELATED SEQUENCES ARISING FROM INVERSES OF PASCAL'S TRIANGLE MATRICES

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Here is recorded a fascinating sequence of sequences which arise in the first column of matrix inverses of matrices containing certain columns of Pascal's triangle. The convolution arrays of these sequences are computed, leading to determinant relationships, a general formula for any element in the convolution array for any of these sequences, and a class of combinatorial identities.

The sequence $S_1 = \{1, 1, 2, 5, 14, 42, \cdots\}$ is the sequence of Catalan numbers [1], and the sequence $S_2 = \{1, 1, 3, 12, 55, \cdots\}$ appeared in an enumeration problem given by Carlitz [2, p. 125].

1. SEQUENCES ARISING FROM INVERSES OF PASCAL'S TRIANGLE MATRICES

We form a series of $n \times n$ matrices P_i, $i = 0, 1, 2, 3, \cdots$, by placing every $(i+1)^{st}$ column of Pascal's triangle on and below the main diagonal, and zeroes elsewhere. Then, P_0 contains Pascal's triangle itself, while P_1 contains every other column of Pascal's triangle and P_2 every third column. We call the inverse of P_i the matrix P_i^{-1} and record the convolution arrays for the sequences S_i which arise as the absolute values of the elements in the first column of P_i^{-1} in the tables which follow.

Table 1.1 Non-Zero Elements of the Matrices P_i^{-1} and P_i

	P_i^{-1}						P_i					
$i = 0$	1						1					
	−1	1					1	1				
	1	−2	1				1	2	1			
	−1	3	−3	1			1	3	3	1		
	1	−4	6	−4	1		1	4	6	4	1	
	−1	5	−10	10	−5	1	1	5	10	10	5	1
	...						...					
$i = 1$	1						1					
	−1	1					1	1				
	2	−3	1				1	3	1			
	−5	9	−5	1			1	6	5	1		
	14	−28	20	−7	1		1	10	15	7	1	
	...						...					
$i = 2$	1						1					
	−1	1					1	1				
	3	−4	1				1	4	1			
	−12	18	−7	1			1	10	7	1		
	55	−88	42	−10	1		1	20	28	10	1	
	...						...					
$i = 3$	1						1					
	−1	1					1	1				
	4	−5	1				1	5	1			
	−22	30	−9	1			1	15	9	1		
	140	−200	72	−13	1		1	35	45	13	1	
	...						...					

Next, we will compute the convolution arrays for the sequences S_i which are tabulated below as well as establish the form of the n^{th} term.

Table 1.2 The Sequences S_i Arising from Matrices P_i^{-1}

i	S_i	n^{th} term
0	1, 1, 1, 1, 1, ···	$\binom{n}{n}$
1	1, 1, 2, 5, 14, ···	$\frac{1}{n+1}\binom{2n}{n}$
2	1, 1, 3, 12, 55, ···	$\frac{1}{2n+1}\binom{3n}{n}$
3	1, 1, 4, 22, 140, ···	$\frac{1}{3n+1}\binom{4n}{n}$
4	1, 1, 5, 35, 285, ···	$\frac{1}{4n+1}\binom{5n}{n}$
···	···	···
k	1, 1, $k+1$, ···	$\frac{1}{kn+1}\binom{(k+1)n}{n} = \frac{1}{n}\binom{(k+1)n}{n-1}$

It is important to note that convolutions of the sequences S_i arising from P_i^{-1} have as their i^{th} convolution that same sequence, less its first element. Let $S_i(x)$ be the generating function for the sequence S_i, and let * denote a convolution. We easily calculate:

$$i = 1: \quad (1, 1, 2, 5, 14, \cdots)*(1, 1, 2, 5, 14, \cdots) = (1, 2, 5, 14, \cdots)$$

(1.1) $$xS_1^2(x) = S_1(x) - 1$$

$$i = 2: \quad (1, 1, 3, 12, 55, \cdots)*(1, 1, 3, 12, 55, \cdots)*(1, 1, 3, 12, 55, \cdots) = (1, 3, 12, 55, \cdots)$$

(1.2) $$xS_2^3(x) = S_2(x) - 1$$

$$i = 3: \quad (1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots)*(1, 1, 4, 22, \cdots) = (1, 4, 22, \cdots)$$

(1.3) $$xS_3^4(x) = S_3(x) - 1.$$

In fact, it will be shown by the Lemma [3] following, that

(1.4) $$xS_i^{i+1} = S_i(x) - 1,$$

which will allow an easy construction of the convolution array for S_i.

Lemma: Two infinite matrices (denoted by giving successive column generators),

$$(f^m(x), xf^{m+k}(x), x^2f^{m+2k}(x), \cdots) \quad \text{and} \quad (A^m(x), xA^{m+k}(x), x^2A^{m+2k}(x), \cdots)$$

are inverses if

$$A(x)f(xA^k(x)) = 1.$$

Here, we take $f(x) = 1/(1-x)$, the generating function for the first column of the Pascal matrix, and let $A(x) = S_i(-x)$, where $S_i(x)$ is the generating function for the sequence S_i, and take $k = i + 1$. Then

$$1 = A(x)f(xA^k(x)) = S_i(-x)[1 - xS_i^{i+1}(-x)]^{-1},$$

or

$$1 - xS_i^{i+1}(-x) = S_i(-x)$$

which establishes (1.4) upon replacing $(-x)$ by x and rearranging terms.

Also notice that, in a convolution triangle, the generating function for the i^{th} column is the i^{th} power of the generating function for the first column. Putting this together with (1.4) gives us a neat way to generate the convolution triangle for any one of the sequences S_i. For example, for $i = 1$,

$$xS_1^2(x) = S_1(x) - 1$$

(1.5) $$xS_1^{k+1}(x) = S_1^k(x) - S_1^{k-1}(x)$$

$$S_1^k(x) = S_1^{k-1}(x) + xS_1^{k+1}(x)$$

which means that we have a Pascal-like rule of formation for the elements of the convolution triangle. An element in the k^{th} column in the sum of elements in the $(k-1)^{st}$ and $(k+1)^{st}$ columns as shown in the convolution triangle for S_1 (the Catalan numbers) given below:

Table 1.3 Convolution Triangle for S_1 : 1, 1, 2, 5, 14, 42, ⋯

1	1	1	1	1	1	⋯
1	2	3	4	5	6	⋯
2	5	9	14	20	27	⋯
5	14	28	48	75	110	⋯
14	42	90	165	275	429	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯

Scheme: $z = x + y$

		y
x	z	

Notice that, except for spacing, the rule of formation is the same as that for Pascal's triangle. For Pascal's triangle in rectangular form, the scheme would be a diagram like below, where $z = x + y$:

	y
x	z

Similarly, for $i = 2$, we obtain

(1.6) $$S_2^k(x) = S_2^{k-1}(x) + xS_2^{k+2}(x)$$

which leads to the generation of the convolution triange for S_2 below.

Table 1.4 Convolution Triange for S_2 : 1, 1, 3, 12, 55, ⋯

1	1	1	1	1	1	⋯
1	2	3	4	5	6	⋯
3	7	12	18	25	33	⋯
12	30	55	88	130	182	⋯
55	143	273	455	700	1020	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯

Scheme: $z = x + y$

			y
x	z		

For $i = 3$, we have

(1.7) $$S_3^k(x) = S_3^{k-1}(x) + xS_3^{k+3}(x)$$

which gives a scheme similar to those preceding, using a grid in which the column entries to be added are separated by three spaces, as computed below:

Table 1.5 Convolution Triangle for S_3 : 1, 1, 4, 22, 140, ⋯

1	1	1	1	1	1	⋯
1	2	3	4	5	6	⋯
4	9	15	22	30	39	⋯
22	52	91	140	200	272	⋯
140	340	612	969	1425	1995	⋯
⋯	⋯	⋯		⋯	⋯	⋯

Scheme: $z = x + y$

				y
x	z			

Returning for a moment to the matrices P_i^{-1} and comparing them to the convolution arrays for the sequences just given, notice that, ignoring signs, P_1^{-1} contains the alternate columns of the Catalan convolution array, and that P_i^{-1} is always composed of columns of a convolution array for the sequence in the first column. In fact, except for signs, the matrix P_i^{-1} always contains the zeroth column, the $(i+1)^{st}$ column, the $2(i+1)^{nd}$ column,

$\cdots$, and has its $(k+1)^{st}$ column given by the $k(i+1)^{st}$ column of the convolution array for the sequence S_i. (Notice that the count of the columns for matrices begins with one, but for convolution arrays begins with zero.) We have proved this already in applying the Lemma.

Now, to generalize, the formulation of the convolution triangle for S_i would require a grid in which column entries to be added were separated by i spaces, so that the generating function $S_i(x)$ for the zero th column of the convolution array for S_i satisfies

$$S_i^k(x) = S_i^{k-1}(x) + xS_i^{k+1}(x), \tag{1.8}$$

where, of course, $S_i^k(x)$ is the generating function for the $(k-1)^{st}$ column, $k = 1, 2, 3, \cdots$.

Then, notice that this means that each row in the convolution array for any of the sequences S_i is the partial sum of the previous row from some point on. Thus, each convolution array written in rectangular form has its i^{th} row an arithmetic progression of order i, $i = 0, 1, 2, 3, \cdots$, and the constant of each of these progressions is 1. By previous results [4], we have

Theorem 1.1. The determinant of any $n \times n$ array taken to include the row of 1's in the convolution array written in rectangular form for any of the sequences S_i has value one.

It will also be shown in a later paper that the determinant of any $n \times n$ array taken to include the row of integers (1, 2, 3, 4, $\cdots$) and its first column the $(j-1)^{st}$ column of the convolution array has value

$$\binom{n+j-1}{n}, \qquad j = 1, 2, 3, \cdots.$$

2. GENERATION OF CONVOLUTION TRIANGLES FOR SEQUENCES S_i FROM PASCAL'S TRIANGLE

The convolution triangles for these sequences S_i are also available from Pascal's triangle in a reasonable way. If one looks at Pascal's triangle as given in Table 2.1,

Table 2.1 Pascal's Triangle

						1						
					1		1					
				1		2		1				
			1		3		3		1			
		1		4		6		4		1		
	1		5		10		10		5		1	
1		6		15		20		15		6		1

. . . .

and takes diagonals parallel to the central diagonal

$$1, 2, 6, 20, 20, 70, 252, \cdots, \binom{2n}{n}, \cdots,$$

one sees that

$$1/1,\ 2/2,\ 6/3,\ 20/4,\ 70/5,\ 252/6, \cdots = 1, 1, 2, 5, 14, 42, \cdots$$

$$2(1/2,\ 3/3,\ 10/4,\ 35/5,\ 126/6, \cdots) = 1, 2, 5, 14, 42, \cdots$$

$$3(1/3,\ 4/4,\ 15/5,\ 56/6,\ 210/7, \cdots) = 1, 3, 9, 28, 90, \cdots$$

$$4(1/4,\ 5/5,\ 21/6,\ 84/7,\ 330/8, \cdots) = 1, 4, 14, 48, 165, \cdots,$$

where successive parallel diagonals of Pascal's triangle produce successive columns of the Catalan convolution triangle.

To write the convolution triangle for the sequence S_2, one uses the diagonal

$$1, 3, 15, 84, 495, \cdots, \binom{3n}{n}, \cdots,$$

and diagonals parallel to it:

$$1/1,\ 3/3,\ 15/5,\ 84/7,\ 495/9, \cdots = 1, 1, 3, 12, 55, \cdots$$

$$2(1/2,\ 4/4,\ 21/6,\ 120/8, \cdots) = 1, 2, 7, 30, \cdots$$

$$3(1/3,\ 5/5,\ 28/7,\ 165/9,\ \cdots) = 1, 3, 12, 55, \cdots$$
$$4(1/4,\ 6/6,\ 36/8,\ 220/10,\ \cdots) = 1, 4, 18, 88, \cdots$$
$$5(1/5,\ 7/7,\ 45/9,\ 286/11,\ \cdots) = 1, 5, 25, 130, \cdots.$$

Notice that we again produce successive columns of the convolution triangle from successive diagonals of Pascal's triangle.

As a final example, we write the convolution triangle for S_3 from the diagonal

$$1, 4, 28, 220, 1820, \cdots, \binom{4n}{n}, \cdots$$

and diagonals parallel to it:

$$1/1,\ 4/4,\ 28/7,\ 220/10,\ 1820/13,\ \cdots = 1, 1, 4, 22, 140, \cdots$$
$$2(1/2,\ 5/5,\ 36/8,\ 286/11,\ 2380/14,\ \cdots) = 1, 2, 9, 52, 340, \cdots$$
$$3(1/3,\ 6/6,\ 45/9,\ 364/12,\ \cdots) = 1, 3, 15, 91, \cdots$$
$$4(1/4,\ 7/7,\ 55/10,\ 455/13,\ \cdots) = 1, 4, 22, 140, \cdots$$
$$5(1/5,\ 8/8,\ 66/11,\ 560/14,\ \cdots) = 1, 5, 25, 200, \cdots.$$

Before we continue to the general case, observe the arithmetic progressions appearing in the denominators. For the Catalan numbers, the sequence S_1, the common difference is one; for S_2, two; and for S_3, three. For S_3, for example, we find the parallel diagonals from Pascal's rectangular array by beginning in the leftmost column and counting to the right one and down 4 throughout the array. To get the sequence S_3 itself, we multiply the Pascal diagonal 1, 4, 28, 220, $\cdots$ by 1 and divide by 1, 4, 7, 10, 13, $\cdots$; to get the first convolution or S_3^2, we multiply the first diagonal parallel to 1, 4, 28, 220, $\cdots$ by 2 and divide by 2, 5, 8, 11, $\cdots$; for the second convolution or S_3^3, we take the next parallel diagonal, multiply by 3, and divide by 3, 6, 9, 12, $\cdots$; and for S_3^k, we multiply the k^{th} diagonal by k and divide by $k,\ k+3,\ k+6,\ k+9,\ \cdots$.

To find the diagonals easily, write Pascal's triangle in rectangular form:

Table 2.2 Pascal's Triangle in Rectangular Form

1	1	1	1	1	1	1	1	···
1	2	3	4	5	6	7	8	···
1	3	6	10	15	21	28	36	···
1	4	10	20	35	56	84	120	···
1	5	15	35	70	126	210	330	···
1	6	21	56	126	252	462	792	···
1	7	28	84	210	462	924	1716	···
···	···	···	···	···	···	···	···	···

Then the sequence S_i is given by

$$\frac{1}{ni+1}\binom{(i+1)n}{n},$$

which diagonal is found by beginning in the leftmost column and counting to the right one and down $(i+1)$ throughout the rectangular Pascal array. The diagonals which lead to the convolution array for S_i are parallel and below this first diagonal. To find the $(k-1)^{st}$ convolution S_i^k, we multiply the k^{th} diagonal by k and divide by $k,\ k+i,\ k+2i,\ k+3i,\ \cdots$. The diagonals used to find the convolution triangle for S_2 are marked in the array above.

Now, we can find all the positive integral powers of the Catalan sequence in the convolution sense. However, let us not neglect the zero or negative powers. Here, we must adopt a convention, and call 0/0 = 1 and −0/0 = −1. We find S_1^0, S_1^{-1}, and S_1^{-2} by following the same process as given above but using an extended Pascal's triangle which includes coefficients for the binomial expansion of $(1+x)^{-k}$.

To see this in perspective, let us write down the parallel diagonals and the numbers to multiply and divide by.

$\cdots$

S_1^3 : 3(1/3, 4/4, 15/5, 56/6, 210/7, ⋯) = 1, 3, 9, 28, 90, ⋯

S_1^2 : 2(1/2, 3/3, 10/4, 35/5, 126/6, ⋯) = 1, 2, 5, 14, 42, ⋯

S_1 : 1(1/1, 2/2, 6/3, 20/4, 70/5, ⋯) = 1, 1, 2, 5, 14, ⋯

S_1^0 : 0(1/0, 1/1, 3/2, 10/3, 35/4, ⋯) = 1, 0, 0, 0, 0, ⋯

S_1^{-1} : −1(1/−1, 0/0, 1/1, 4/2, 15/3, 56/4, ⋯) = 1, −1, −1, −2, −5, −14, ⋯

S_1^{-2} : −2(1/−2, −1/−1, 0/0, 1/1, 5/2, ⋯) = 1, −2, −1, −2, −5, ⋯

$\cdots$

Thus, you see that if we write down the extra terms from "Pascal's attic," the process works in reverse to obtain all columns of the Catalan convolution triangle. This process will provide the zero and negative powers for any of the sequences S_i. One can also complete the Catalan convolution array to the left to provide negative integral powers by using its rule of formation in reverse, which is the following scheme:

		y
x	z	

$x = z - y$

The rule of formation can be rewritten to work to the left for the convolution array for any of the sequences S_i.

Now, write the complete Pascal array down in rectangular form as

⋯	1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	⋯
⋯	21	15	10	6	3	1	0	0	1	3	6	10	15	21	⋯
⋯	−35	−20	−10	−4	−1	0	0	0	1	4	10	20	35	56	⋯
⋯	35	15	5	1	0	0	0	0	1	5	15	35	70	126	⋯
⋯	−21	−6	−1	0	0	0	0	0	1	6	21	56	126	252	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

This is the regular arrangement. Now, if we move the i^{th} row i places to the left, i = 0, 1, 2, ⋯, we form

⋯	1	1	1	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	⋯
⋯	10	6	3	1	0	0	1	3	6	10	15	21	28	36	⋯
⋯	−4	−1	0	0	0	1	4	10	20	35	56	84	120	165	⋯
⋯	0	0	0	0	1	5	15	35	70	126	210	330	495	715	⋯
⋯	0	0	0	1	6	21	56	126	252	462	792	⋯	⋯	⋯	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

Now, all diagonals which are parallel to 1, 2, 6, 20, 70, ⋯ are all vertical. By proper processing, as just described, we can obtain all columns of the Catalan convolution triangle. To obtain the column which gives us S_1^k, we multiply the column above which starts with $1, k+1, \cdots$, by k and divide successive terms by $k, k+1, k+2, k+3, k+4, \cdots$, for $k = 0, \pm 1, \pm 2, \pm 3, \cdots$, where we adopt the convention that 0/0 = 1 and −0/0 = −1. If we begin again with the regular arrangement, but this time move the i^{th} row $2i$ spaces to the right, we obtain an arrangement which has column 1, −2, 6, −20, 70, ⋯, as

⋯	1	1	1	1	1	1	1	1	1	1	1	⋯
⋯	−5	−4	−3	−2	−1	0	1	2	3	4	5	⋯
⋯	21	15	10	6	3	1	0	0	1	3	6	⋯
⋯	−84	−56	−35	−20	−10	−4	−1	0	0	0	1	⋯
⋯	330	210	126	70	35	15	5	1	0	0	0	⋯
⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯	⋯

With the same processing as above, we obtain the Catalan convolution array with alternating signs. This shows that Pascal's triangle itself contains all that the inverses of the Pascal matrices gives from properly processed columns in the Pascal convolution array. By similar movement of the rows of Pascal's triangle and proper processing, we can obtain S_i^k, $i = 0, 1, 2, 3, \cdots; k = 0, \pm 1, \pm 2, \cdots$.

As we already know, the Catalan sequence S_1 and its convolution triangle are obtained by processing properly the diagonal 1, 2, 6, 20, 70, ⋯, and those diagonals parallel to it. Since Pascal's triangle has symmetry, we can use the parallel diagonals either above or below the central diagonal, when Pascal's triangle is written in rectangular form as in Table 4.4. Then, S_1^k is obtained by multiplying the parallel diagonal which begins with $1, k+1, \cdots$ by k and dividing successive entries by $k, k+1, k+2, \cdots$. Now, suppose that we try the same process for the Catalan convolution array, using diagonals parallel to $1, 2, 9, 48, 275, \cdots$, the central diagonal of the array, as given in Table 1.3.

$$1/1, 2/2, 9/3, 48/4, 275/5, \cdots = 1, 1, 3, 12, 55, \cdots = S_2$$

$$2(1/2, 3/3, 14/4, 75/5, 429/6, \cdots) = 1, 2, 7, 30, 143, \cdots = S_2^2$$

$$3(1/3, 4/4, 20/5, 110/6, 637/7, \cdots) = 1, 3, 12, 55, 273, \cdots = S_2^3$$

$$4(1/4, 5/5, 27/6, 154/7, \cdots) = 1, 4, 18, 88, \cdots = S_2^4$$

Surely you recognize the convolution array for the next of our sequences, S_2! If this same process is used on the convolution array for S_i, one obtains the convolution array for S_{i+1}. See [8], [9], [10].

3. A SECOND GENERATION OF THE SEQUENCES S_i FROM PASCAL'S TRIANGLE

These arrays can be obtained in yet another way from the diagonals of Pascal's triangle written in rectangular form. To obtain the convolution array for $S_2 = \{1, 1, 3, 12, 55, 273, \cdots\}$, we multiply successive diagonals and divide by successive members of an arithmetic progression with constant difference 3 as follows:

$$1(1/1, 4/4, 21/7, 120/10, \cdots) = 1, 1, 3, 12, \cdots = S_2$$

$$2(1/2, 5/5, 28/8, 165/11, \cdots) = 1, 2, 7, 30, \cdots = S_2^2$$

$$3(1/3, 6/6, 36/9, 220/12, \cdots) = 1, 3, 12, 55, \cdots = S_2^3$$

$$4(1/4, 7/7, 45/10, 286/13, \cdots) = 1, 4, 18, 88, \cdots = S_2^4$$

The diagonals are obtained by beginning in the row of ones in the Pascal rectangular array and counting down one and right two, or by beginning in the column of ones and counting to the right one and down two. The multiplier is the same as the exponent of S_2^k, and the arithmetic progression used is $k, k+3, k+6, \cdots, k+3n, n = 0, 1, 2, \cdots$.

To obtain the Catalan sequence, and its convolution triangle, we can use the diagonals obtained by counting down one and right one beginning in the column of ones (or in the row of ones) so that the beginning diagonal is 1, 3, 10, 35, ⋯, and dividing by successive terms of arithmetic progressions with constant difference two as follows:

$$1(1/1, 3/3, 10/5, 35/7, 126/9, \cdots) = 1, 1, 2, 5, 14, \cdots = S_1$$

$$2(1/2, 4/4, 15/6, 56/8, 210/10, \cdots) = 1, 2, 5, 14, 42, \cdots = S_1^2$$

$$3(1/3, 5/5, 21/7, 84/9, 330/11, \cdots) = 1, 3, 9, 28, 90, \cdots = S_1^3$$

Again the multiplier is the same as the exponent for S_1^k, and the arithmetic progression used for the divisors is $k+2n, n = 0, 1, 2, \cdots$.

Then, we have a dual system working here for extracting the convolution array of the sequence S_i from Pascal's triangle written in rectangular form. To obtain the convolution array for S_i, we find successive diagonals from Pascal's array by beginning in the column of ones and counting right one and down i, taking the first diagonal as $1, i+2, \cdots$. (Or, we can work to the right, taking the diagonals successively that are parallel to the diagonal beginning with $1, i+2, \cdots$, obtained by counting down one and right i throughout the array.)

To write S_i^k, we take the k^{th} diagonal which begins $1, k+i+1, \cdots$, multiply by k, and divide successively by the successive terms of the arithmetic progression $k + in$, $n = 0, 1, 2, \cdots$. Explicitly, we write the m^{th} element of S_i^k as

$$\frac{k}{k+im}\binom{(i+1)m+k-1}{m}$$

for $i = 0, 1, 2, \cdots;\ k = 1, 2, 3, \cdots;\ m = 0, 1, 2, \cdots$.

Many cases were shown which verify that the m^{th} term of the $(k-1)^{st}$ convolution of the sequence S_i, denoted by $s_i(m,k)$, is given by

$$s_i(m,k) = \frac{k}{k+im}\binom{(i+1)m+k-1}{m}, \tag{3.1}$$

$m = 0, 1, 2, \cdots;\ k = 1, 2, 3, \cdots;\ i = 0, 1, 2, \cdots$. Applying (1.8) leads to a rule of formation for the convolution array for any sequence S_i,

$$s_i(m,k) = s_i(m, k-1) + s_i(m-1, k+i). \tag{3.2}$$

Assume that (3.1) holds for all convolutions for the first $(m-1)$ terms, and holds for the first $(k-2)$ convolutions for the first m terms. Then $s_i(m,k)$ again will have the desired form of (3.1) as shown by

$$s_i(m,k) = s_i(m,k-1) + s_i(m-1,k+i) = \frac{k-1}{k-1+im}\binom{(i+1)m+k-2}{m} + \frac{k+i}{k+im}\binom{(i+1)m+k-2}{m-1}$$

$$= \binom{(i+1)m+k-1}{m}\left[\frac{k-1}{k-1+im}\cdot\frac{im+k-1}{im+m+k-1} + \frac{k}{k+im}\cdot\frac{m}{im+m+k-1}\right]$$

$$= \binom{(i+1)m+k-1}{m}\cdot\frac{k(im+m+k-1)}{(k+im)(im+m+k-1)}.$$

4. THE SEQUENCE OF SEQUENCES S_i TAKEN AS A RECTANGULAR ARRAY

Next, suppose one simply considers the sequence of sequences S_i as the rows of a rectangular array, and considers the progressions appearing in the columns. We omit the first term for each sequence S_i

Table 4.1 The Sequences S_i

S_0:	1	1	1	1	1	1	1	1	···
S_1:	1	2	5	14	42	132	429	1,430	···
S_2:	1	3	12	55	273	1,428	7,752	43,263	···
S_3:	1	4	22	140	969	7,084	53,820	420,732	···
S_4:	1	5	35	285	2,530	23,751	231,880	2,330,445	···
S_5:	1	6	51	506	5,481	62,832	749,398	9,203,634	···
S_6:	1	7	70	819	10,472	141,776	1,997,688	28,989,675	···
S_7:	1	8	92	1,240	18,278	285,384	4,638,348	77,652,024	···
	···	···	···	···	···	···	···	···	···
Order of AP:	0	1	2	3	4	5	6	7	
Constant:	1	1	3	16	125	1296	16807	262,144	···
Form:	1^{-1}	2^0	3^1	4^2	5^3	6^4	7^5	8^6	n^{n-2}

Notice that the k^{th} column is an arithmetic progression of order $(k-1)$, with common difference k^{k-2}. This means, using Eves' Theorem [4], [5],

Eves' Theorem: Consider a determinant of order n whose i^{th} column $(i = 1, 2, \cdots, n)$ is composed of any n successive terms of an arithmetic progression of order $(i-1)$ with constant a_i. The value of the determinant is the product $a_1 a_2 \cdots a_n$.

that we can write Theorems 4.1 and 4.2.

Theorem 4.1: The determinant of any $n \times n$ array taken to include the column of 1's in the sequence of sequences S_i rectangular array has value

$$\prod_{j=1}^{n} j^{j-2}.$$

Theorem 4.2: Take a determinant of order n with its first column in the column of integers, and its first row along the row of ones of the rectangular sequence of sequences S_i array. The value of the determinant is

$$\prod_{j=1}^{n+1} j^{j-2}.$$

Proof: Subtract the $(i-1)^{st}$ row from the i^{th} row, $i = n, n-1, \cdots, 2$, to obtain a determinant whose k^{th} column is an arithmetic progression of order $k-1$ with constant $(k+1)^{(k+1)-2}$ and apply Eve's Theorem.

Further, the following result seems to be true.

Conjecture: Take an $n \times n$ determinant such that its first column is the column of integers in the sequence of sequences S_i rectangular array and its first row is the k^{th} row, $k = 1, 2, 3, \cdots$. Then its determinant is given by

$$\left(\prod_{j=1}^{n+1} j^{j-2}\right) \cdot \binom{n+k-1}{n} .$$

To prove that the constants of the arithmetic progressions have the form given, we quote Hsu [6, p. 480]:

$$\sum_{r=0}^{n'} (-1)^r \binom{n'}{r} \binom{sr+t}{m} = \begin{cases} 0, & m < n \\ (-s)^{n'}, & m = n \end{cases}$$

and substitute $n' = n-1,\ t = n^2,\ s = -n,\ m = n-1$, to obtain

$$\frac{1}{n}\sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r}\binom{n^2-rn}{n-1} = \frac{1}{n}(n^{n-1}) = n^{n-2}, \tag{4.1}$$

where we also make use of the known general form for the m^{th} term of S_i.

5. A CLASS OF COMBINATORIAL IDENTITIES

Returning to the first section, in Table 1.1 we computed matrices P_i^{-1}. Now, since $P_iP_i^{-1} = I$, we can write an entire class of combinatorial identities. Notice that, since we are dealing with infinite matrices such that all non-zero elements appear on and below the main diagonals, $P_iP_i^{-1} = I$ for any $n \times n$ matrices P_i, P_i^{-1}, and I formed from the $n \times n$ blocks in the upper left of the original infinite matrices. Since P_i contains elements taken from Pascal's triangle, it is a simple matter to write the element in its $(n+1)^{st}$ row and $(j+1)^{st}$ column as

$$p_i(n,j) = \binom{n+ij}{j+ij}, \qquad n = 0, 1, 2, \cdots;\ j = 0, 1, 2, \cdots. \tag{5.1}$$

Now, the elements in P_i^{-1} are the same as those in the convolution array for S_i, except for sign. When $i = 1$, we have the Catalan convolution array, and the element of P_1^{-1} in its $(r+1)^{st}$ row and $(p+1)^{st}$ column is given by the $(r-p)^{th}$ element of the $(2p)^{th}$ convolution of S_1, or the $(r-p)^{th}$ element in the sequence S_1^{2p+1}, which is, by (3.1),

$$p_1^*(r,p) = (-1)^{r-p}s_1(r-p, 2p+1) = \frac{(-1)^{r-p}(2p+1)}{p+1+r}\binom{2r}{r-p}$$

while

$$p_1(n,j) = \binom{n+j}{2j} .$$

Since $P_1P_1^{-1} = I$, the element in the $(n+1)^{st}$ row and $(p+1)^{st}$ column of I is given by

$$0 = \sum_{j=0}^{n} p_1(n, j)p_1^*(j, p), \qquad n \neq p.$$

Now, when $p = 0$, we have the first column of P_1^{-1}, of the sequence S_1 of Catalan numbers, and

(5.2) $$0 = \sum_{j=0}^{n} \frac{(-1)^j}{j+1} \binom{2j}{j}\binom{n+j}{2j}$$

which was given as (3.100) by Gould [7].

Since $n \geq p + 1$ gives non-diagonal elements of I, we also have the more general

(5.3) $$0 = \sum_{j=p}^{n} \frac{(-1)^{j-p}(2p+1)}{p+1+j} \binom{2j}{j-p}\binom{n+j}{2j} .$$

We can further generalize by not restricting i. Let the element in the $(r+1)^{st}$ row and $(p+1)^{st}$ column of P_i^{-1} be

(5.4) $$p_i^*(r, p) = (-1)^{r-p}s_i(r-p, ip+1) = \frac{(-1)^{r-p}[(i+1)p+1]}{p+1+ir} \binom{r+ir}{r-p} .$$

Since $P_iP_i^{-1} = I$, for $n \geq p + 1$ we obtain a non-diagonal element, giving the very general identity

(5.5) $$0 = \sum_{j=p}^{n} \frac{(-1)^{j-p}[(1+i)p+1]}{p+1+ij} \binom{j+ij}{j-p}\binom{n+ij}{j+ij} ,$$

for $i = 0, 1, 2, 3, \cdots; p = 0, 1, 2, \cdots$; and $n \geq p + 1$.

Notice that, for $i = 0$, we have Pascal's triangle in both P_i and P_i^{-1}, leading to

(5.6) $$0 = \sum_{j=p}^{n} (-1)^{j-p} \binom{j}{j-p}\binom{n}{j} ,$$

and, when $i = 0$ and $p = 0$, to the familiar identity,

(5.7) $$0 = \sum_{j=0}^{n} (-1)^j \binom{n}{j} .$$

For $p = 0$ in (5.5), we are in the first column, and

(5.8) $$0 = \sum_{j=0}^{n} \frac{(-1)^j}{1+ij} \binom{j+ij}{j}\binom{n+ij}{j+ij} = \sum_{j=0}^{n} \frac{(-1)^j}{1+ij} \binom{n+ij}{ij}\binom{n}{j}$$

gives a recursion relation for the terms of S_i, as

(5.9) $$0 = \sum_{j=0}^{n} (-1)^j s_i(j, 1) \binom{n+ij}{j+ij} ,$$

where $s_i(j, 1)$ is the j^{th} term of the sequence S_i^1.

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EXPONENTIALS AND BESSEL FUNCTIONS

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A Bessel function of order n may be defined as follows:

(1)
$$J_n(x) = \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda}{\Gamma(\lambda+1)\Gamma(\lambda+n+1)}\left(\frac{x}{2}\right)^{n+2\lambda}$$

It may be easily shown that for integral $n, J_n(x)$ is the coefficient of U^n in the expansion of

$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)\right]$$

i.e.,

(2)
$$\exp\left[\frac{x}{2}\left(u-\frac{1}{u}\right)\right] = \sum_{n=-\infty}^{\infty} U^n J_n(x)$$

Now let

(3)
$$u - \frac{1}{u} = L_{2k+1},$$

where L_{2k+1} is a Lucas number defined by

(4)
$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2},$$

where n is any integer.

Equation (3) becomes $u^2 - uL_{2k+1} - 1 = 0$ with roots

$$\left(\frac{1+\sqrt{5}}{2}\right)^{2k+1} = \alpha^{2k+1} \quad \text{and} \quad \left(\frac{1-\sqrt{5}}{2}\right)^{2k+1} = \beta^{2k+1},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

are the roots of the well known quadratic

(5)
$$\phi^2 = \phi + 1.$$

[Continued on page 418.]