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We first encountered Pascal's matrices while working on a probability problem involving repeated flips of an unfair coin. Since they are derived naturally from Pascal's triangle, Pascal's matrices immediately caught our attention. However, it was the striking simplicity of the powers of these matrices that intrigued us and prompted us to write this paper. Before defining Pascal's matrices, we'll describe the probability question which led us to them.

If the probability of heads on each flip is p, and thus the probability of tails is $1 - p$, then in a sequence of n flips the probability of any event is given by an expression of the form

$$\sum_{i=0}^{n} a_i p^{n-i}(1-p)^i,$$

where the a_i's are integers which depend on the event in question, with $0 \le a_i \le \binom{n}{i}$. Of course, this can be multiplied out to yield a polynomial in p of degree n with integer coefficients. While trying to resolve an open problem in [1], we found that we needed to reverse this process: Given a polynomial in p with integer coefficients, is it the formula for the probability of some event? As we will show later, the answer to this question involves a Pascal's matrix.

The $n \times n$ Pascal's matrix is obtained by taking the first n rows of Pascal's triangle and filling in with 0's on the right. Specifically, we define the $n \times n$ Pascal's matrix P by

$$P_{ij} = \begin{cases} \binom{i-1}{j-1} & \text{if } i \ge j, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the first four Pascal's matrices are

$$(1), \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

Computing the inverses of these matrices reveals a rather suggestive pattern. In particular, for $n = 4$ we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}.$$