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SOME SUMS RELATED TO SUMS OF ORESME NUMBERS

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1. INTRODUCTION

In [1] A. F. Horadam presented a history of number attributed to Nicole Oresme, namely the sequence

$$\{O_n\} = \left\{\frac{n}{2^n}\right\} = \left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \ldots, \frac{n}{2^n}, \ldots\right\}. \tag{1.1}$$

As with all of Horadam's papers, an abundance of properties of these numbers was obtained. Those generalized in this paper are referenced here.

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, O_0 = 0, O_1 = \frac{1}{2} \tag{1.2}$$

$$O_{n+2} - \frac{3}{4}O_n + \frac{1}{4}O_{n-1} = 0 \tag{1.3}$$

$$O_{n+2} - \frac{3}{4}O_{n+1} + \frac{1}{16}O_{n-1} = 0 \tag{1.4}$$

$$\sum_{j=0}^{n-1} O_j = 4\left(\frac{1}{2} - O_{n+1}\right) \tag{1.5}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

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$$\sum_{j+0}^{\infty} O_j = 2 \tag{1.6}$$

$$\sum_{j=0}^{n-1} (-1)^j O_j = \frac{4}{9}\left[-\frac{1}{2} + (-1)^n (O_{n+1} - 2O_n)\right] \tag{1.7}$$

$$\sum_{j=0}^{n-1} O_{2j} = \frac{4}{9}\left[2 + O_{2n-1} - 5O_{2n}\right] \tag{1.8}$$

$$\sum_{j=0}^{n-1} O_{2j+1} = \frac{1}{9}\left[10 + 5O_{2n-1} - 16O_{2n}\right] \tag{1.9}$$

$$O_{n+1}O_{n-1} - O_n^2 = -\left(\frac{1}{4}\right)^n \tag{1.10}$$

and the generating function

$$\sum_{n=0}^{\infty} O_n x^n = \sum_{n=0}^{\infty} \frac{n}{2^n} x^n = \frac{2x}{(2-x)^2}. \tag{1.11}$$

2. ORESME, FIBONACCI, LUCAS AND PELL SUMS

Variations of (1.11) can be obtained from derivatives of the generating function for the sum of the geometric series. The following converge for $|x| < k$.

$$\sum_{n=0}^{\infty} \frac{x^n}{k^n} = \frac{k}{k-x} \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{nx^n}{k^n} = \frac{kx}{(k-x)^2} \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{n^2 x^n}{k^n} = \frac{kx(k+x)}{(k-x)^3} \tag{2.3}$$

$$\sum_{n=0}^{\infty} \frac{n^3 x^n}{k^n} = \frac{kx(k^2+4kx+x^2)}{(k-x)^4} \tag{2.4}$$

$$\sum_{n=0}^{\infty} \frac{n^4 x^n}{k^n} = \frac{kx(k^3+11k^2x+11kx^2+x^3)}{(k-x)^5} \tag{2.5}$$

...

Proof of (2.4):

Let $f_0(x) = \frac{1}{1-x} = \sum x^n$. Then $f_0'(x) = \frac{1}{(1-x)^2}$.

Let $f_1(x) = \frac{x}{(1-x)^2} = \sum nx^n$. Then $f_1'(x) = \frac{1+x}{(1-x)^3}$.

Let $f_2(x) = \frac{x(1+x)}{(1-x)^3} = \sum n^2x^n$. Then $f_2'(x) = \frac{1+4x+x^2}{(1-x)^4}$.

Hence $\sum n^3x^n = \frac{x(1+4x+x^2)}{(1-x)^4}$ so that $\sum n^3\left(\frac{x^n}{k^n}\right) = \frac{kx(k^2+4kx+x^2)}{(k-x)^4}$. □

(2.1) to (2.5) form sequences of sums $S_m(n^m, k, x)$. The special case with $k = 2$ and $x = 1$ yields a sequence of Oresme sums

$$\{S_m(m,2,1)\} = \{s_o, s_1, s_2, s_3, s_4, \dots\} = \left\{\sum_{n=0}^{\infty} \frac{n^m}{2^n}\right\} = \left\{\sum_{n=0}^{\infty} n^{m-1}0_n\right\}$$

$$= \{2, 2, 6, 26, 150, \dots\}. \tag{2.6}$$

By using the generating functions (2.1)-(2.5) analogous sums and sequences of sums can be found for the Fibonacci, Lucas, and Pell numbers.

Thus for the Fibonacci numbers, F_n, since $|x| = |\frac{1\pm\sqrt{5}}{2}| < 2$ the following converge if $k \geq 2$.

$$\sum_{n=0}^{\infty} \frac{F_n}{k^n} = \frac{k}{k^2-k-1} \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{nF_n}{k^n} = \frac{k(k^2+1)}{(k^2-k-1)^2} \tag{2.8}$$

$$\sum_{n=0}^{\infty} \frac{n^2F_n}{k^n} = \frac{k(k^4+k^3+6k^2-k+1)}{(k^2-k-1)^3} \tag{2.9}$$

$$\sum_{n=0}^{\infty} \frac{n^3 F_n}{k^n} = \frac{k(k^6+4k^5+24k^4+24k^2-4k+1)}{(k^2-k-1)^4} \tag{2.10}$$

$$\sum_{n=0}^{\infty} \frac{n^4 F_n}{k^n} = \frac{k(k^8+11k^7+87k^6+48k^5+240k^4-48k^3+87k^2-11k+1)}{(k^2-k-1)^5} \tag{2.11}$$

$$\cdots$$

Proof of (2.10):

$$\sum \frac{n^3 F_n}{k^n} = \frac{1}{\sqrt{5}} \sum \left[\left(\frac{\alpha}{k}\right)^n - \left(\frac{\beta}{k}\right)^n \right] = \frac{1}{\sqrt{5}} \left[\frac{k\alpha(k^2+4\alpha k+\alpha^2)}{(k-\alpha)^4} - \frac{k\beta(k^2+4\beta k+\beta^2)}{(k-\beta)^4} \right]$$

$$= \frac{k}{\sqrt{5}(k^2-k-1)^4} \left[\alpha(k^2+4\alpha k+\alpha^2)(k-\beta)^4 - \beta(k^2+4\beta k+\beta^2)(k-\alpha)^4 \right]$$

The bracketed expression in the numerator becomes

$$(\alpha-\beta)k^6 + 4(\alpha^2-\beta^2)k^5 + [6(\alpha-\beta)+16(\alpha-\beta)+\alpha^3-\beta^3]k^4 +$$
$$[\alpha^3-\beta^3+16(\alpha-\beta)+6(\alpha-\beta]k^2 + 4(\beta^2-\alpha^2)k + \alpha - \beta.$$

Thus the sum simplifies to

$$\sum \frac{n^3 F_n}{k^n} = \frac{k(k^6+4k^5+24k^4+24k^2-4k+1)}{(k^2-k-1)^4}. \quad \square$$

(2.7) to (2.11) form sequences $\{F_k^{(m)}(n^m, k, F_n)\}$, $m \ge 0$, $k \ge 2$ from which the following sequences of sums are obtained.

$$F_k^{(0)} = \{fs_2^{(0)}, fs_3^{(0)}, fs_4^{(0)}, \ldots\}$$
$$= \left\{ \frac{2}{1}, \frac{3}{5}, \frac{4}{11}, \frac{5}{19}, \frac{6}{29}, \frac{7}{41}, \frac{8}{55}, \frac{9}{71}, \frac{10}{89}, \cdots \right\} \tag{2.12}$$

$$F_k^{(1)} = \{fs_2^{(1)}, fs_3^{(1)}, fs_4^{(1)}, \ldots\}$$
$$= \left\{ \frac{10}{1}, \frac{30}{5^2}, \frac{68}{11^2}, \frac{130}{19^2}, \frac{222}{29^2}, \frac{350}{41^2}, \frac{520}{55^2}, \frac{738}{71^2}, \frac{1010}{89^2}, \cdots \right\} \tag{2.13}$$

$$F_k^{(2)} = \{fs_2^{(2)}, fs_3^{(2)}, fs_4^{(2)}, \ldots\}$$

$$= \left\{\frac{94}{1}, \frac{480}{5^3}, \frac{1652}{11^3}, \frac{4480}{19^3}, \frac{10338}{29^3}, \frac{21224}{41^3}, \frac{39880}{55^3}, \frac{69912}{71^3}, \frac{115910}{89^3}, \ldots\right\} \tag{2.14}$$

$$F_k^{(3)} = \{fs_2^{(3)}, fs_3^{(3)}, fs_4^{(3)}, \ldots\}$$

$$= \left\{\frac{1330}{1}, \frac{11550}{5^4}, \frac{58820}{11^4}, \frac{218530}{19^4}, \frac{658230}{29^4}, \frac{1705550}{41^4}, \frac{3944200}{55^4}, \frac{8343090}{71^4}, \frac{16423610}{89^4}, \ldots\right\} \tag{2.15}$$

$$F_k^{(4)} = \{fs_2^{(4)}, fs_3^{(4)}, fs_4^{(4)}, \ldots\}$$

$$= \left\{\frac{25102}{1}, \frac{373800}{5^5}, \frac{2843924}{11^5}, \frac{14527480}{19^5}, \frac{56969826}{29^5}, \frac{185009552}{41^5}, \frac{521513800}{55^5}, \frac{1316481264}{71^5}, \frac{3041605910}{89^5}, \ldots\right\}. \tag{2.16}$$

...

Similarly for the Lucas numbers, L_n, since again, $|x| = |\frac{1\pm\sqrt{5}}{2}| < 2$ the following converge if $k \geq 2$.

$$\sum_{n=0}^{\infty} \frac{L_n}{k^n} = \frac{k(2k-1)}{k^2-k-1} \tag{2.17}$$

$$\sum_{n=0}^{\infty} \frac{nL_n}{k^n} = \frac{k(k^2+4k-1)}{(k^2-k-1)^2} \tag{2.18}$$

$$\sum_{n=0}^{\infty} \frac{n^2L_n}{k^n} = \frac{k(k^4+9k^3+9k-1)}{(k^2-k-1)^3} \tag{2.19}$$

$$\sum_{n=0}^{\infty} \frac{n^3L_n}{k^n} = \frac{k(k^6+20k^5+14k^4+72k^3-14k^2+20k-1)}{(k^2-k-1)^4} \tag{2.20}$$

$$\sum_{n=0}^{\infty} \frac{n^4 L_n}{k^n} = \frac{k(k^8 + 43k^7 + 89k^6 + 422k^5 + 422k^3 - 89k^2 + 43k - 1)}{(k^2 - k - 1)^5} \tag{2.21}$$

...

Proof of (2.20):

$$\sum \frac{n^3 L_n}{k^n} = \sum \left[\left(\frac{\alpha}{k}\right)^n + \left(\frac{\beta}{k}\right)^n \right] = \left[\frac{k\alpha(k^2 + 4\alpha k + \alpha^2)}{(k-\alpha)^4} + \frac{k\beta(k^2 + 4\beta k + \beta^2)}{(k-\beta)^4} \right]$$

$$= \frac{k}{(k^2 - k - 1)^4} \left[\alpha(k^2 + 4\alpha k + \alpha^2)(k-\beta)^4 + \beta(k^2 + 4\beta k + \beta^2)(k-\alpha)^4 \right]$$

The bracketed expression in the numerator becomes

$$(\alpha+\beta)k^6 + [8 + 4(\alpha^2+\beta^2)]k^5 + [-6(\alpha+\beta) + 16(\alpha+\beta) + \alpha^3 + \beta^3]k^4 +$$
$$[4(\alpha^2+\beta^2) + 48 + 4(\alpha^2+\beta^2)]k^3 + [-(\alpha^3+\beta^3) - 16(\alpha+\beta) + 6(\alpha+\beta)]$$
$$k^2 + [4(\beta^2+\alpha^2) + 8]k - (\alpha+\beta)].$$

Thus the sum simplifies to

$$\sum \frac{n^3 L_n}{k^n} = \frac{k(k^6 + 20k^5 + 14k^4 + 72k^3 - 14k^2 + 20k - 1)}{(k^2 - k - 1)^4}. \qquad \square$$

(2.17) to (2.21) form sequences $\{L_k^{(m)}(n^m, k, L_n)\}$, $m \geq 0$, $k \geq 2$ from which the following sequences of sums are obtained.

$$L_k^{(0)} = \{1s_2^{(0)}, 1s_3^{(0)}, 1s_4^{(0)}, \dots\}$$
$$= \left\{ \frac{6}{1}, \frac{15}{5}, \frac{28}{11}, \frac{45}{19}, \frac{66}{29}, \frac{91}{41}, \frac{120}{55}, \frac{153}{71}, \frac{190}{89}, \dots \right\} \tag{2.22}$$

$$L_k^{(1)} = \{1s_2^{(1)}, 1s_3^{(1)}, 1s_4^{(1)}, \dots\}$$
$$= \left\{ \frac{22}{1}, \frac{60}{5^2}, \frac{124}{11^2}, \frac{220}{19^2}, \frac{354}{29^2}, \frac{532}{41^2}, \frac{760}{55^2}, \frac{1044}{71^2}, \frac{1390}{89^2}, \dots \right\} \tag{2.23}$$

$$L_k^{(2)} = \{1s_2^{(2)}, 1s_3^{(2)}, 1s_4^{(2)}, \dots\}$$
$$= \left\{\frac{210}{1}, \frac{1050}{5^3}, \frac{3468}{11^3}, \frac{8970}{19^3}, \frac{19758}{29^3}, \frac{38850}{41^3}, \frac{70200}{55^3}, \frac{118818}{71^3}, \frac{190890}{89^3}, \dots\right\} \quad (2.24)$$

$$L_k^{(3)} = \{1s_2^{(3)}, 1s_3^{(3)}, 1s_4^{(3)}, \dots\}$$
$$= \left\{\frac{2974}{1}, \frac{25800}{5^4}, \frac{130492}{11^4}, \frac{478120}{19^4}, \frac{1412922}{29^4}, \frac{3580864}{41^4}, \frac{8087800}{55^4}, \frac{16702272}{71^4}, \frac{32107990}{89^4}, \dots\right\} \quad (2.25)$$

$$L_k^{(4)} = \{1s_2^{(4)}, 1s_3^{(4)}, 1s_4^{(4)}, \dots\}$$
$$= \left\{\frac{56130}{1}, \frac{836250}{5^5}, \frac{6369900}{11^5}, \frac{32550570}{19^5}, \frac{127433550}{29^5}, \frac{412168050}{41^5}, \frac{1154595000}{55^5}, \frac{2891089170}{71^5}, \frac{6616135290}{89^5}, \dots\right\}. \quad (2.26)$$

...

Finally, for the Pell numbers, P_n (See [2]) since $|1 \pm \sqrt{2}| < 3$, the following converge for $k \geq 3$.

$$\sum_{n=0}^{\infty} \frac{P_n}{k^n} = \frac{k}{(k^2 - 2k - 1)} \quad (2.27)$$

$$\sum_{n=0}^{\infty} \frac{nP_n}{k^n} = \frac{k(k^2 + 1)}{(k^2 - 2k - 1)^2} \quad (2.28)$$

$$\sum_{n=0}^{\infty} \frac{n^2 P_n}{k^n} = \frac{k(k^4 + 2k^3 + 6k^2 - 2k + 1)}{(k^2 - 2k - 1)^3} \quad (2.29)$$

$$\sum_{n=0}^{\infty} \frac{n^3 P_n}{k^n} = \frac{k(k^6 + 8k^5 + 27k^4 + 27k^2 - 8k + 1)}{(k^2 - 2k - 1)^4} \quad (2.30)$$

$$\sum_{n=0}^{\infty} \frac{n^4 P_n}{k^n} = \frac{k(k^8 + 22k^7 + 120k^6 + 102k^5 + 270k^4 - 102k^3 + 120k^2 - 22k + 1)}{(k^2 - 2k - 1)^5} \quad (2.31)$$

...

Proof of (2.30):

Note here that $\alpha = 1+\sqrt{2}$, and $\beta = 1-\sqrt{2}$, so that $\alpha-\beta = 2\sqrt{2}$, $\alpha^2-\beta^2 = 4\sqrt{2}$ and $\alpha^3-\beta^3 = 10\sqrt{2}$.

$$\sum \frac{n^3 P_n}{k^n} = \frac{1}{2\sqrt{2}} \sum \left[\left(\frac{\alpha}{k}\right)^n - \left(\frac{\beta}{k}\right)^n\right] = \frac{1}{2\sqrt{2}} \left[\frac{k\alpha(k^2+4\alpha k+\alpha^2)}{(k-\alpha)^4} - \frac{k\beta(k^2+4\beta k+\beta^2)}{(k-\beta)^4}\right]$$

$$= \frac{k}{2\sqrt{2}(k^2-k-1)^4}\left[\alpha(k^2+4\alpha k+\alpha^2)(k-\beta)^4 - \beta(k^2+4\beta k+\beta^2)(k-\alpha)^4\right]$$

The bracketed expression in the numerator becomes

$$(\alpha-\beta)k^6 + 4(\alpha^2-\beta^2)k^5 + [6(\alpha-\beta)+16(\alpha-\beta)+\alpha^3-\beta^3]k^4 +$$
$$[\alpha^3-\beta^3+16(\alpha-\beta)+6(\alpha-\beta)]k^2 + 4(\beta^2-\alpha^2)k + \alpha - \beta.$$

Thus the sum simplifies to

$$\sum \frac{n^3 P_n}{k^n} = \frac{k(k^6+8k^5+27k^4+27k^2-8k+1)}{(k^2-k-1)^4}. \quad \square$$

(2.27) to (2.31) form sequences $\{P_k^{(m)}(n^m, k, P_n)\}$, $m \ge 0$, $k \ge 3$ from which the following sequences of sums are obtained.

$$P_k^{(0)} = \{ps_3^{(0)}, ps_4^{(0)}, ps_5^{(0)}, \dots\}$$
$$= \left\{\frac{3}{2}, \frac{4}{7}, \frac{5}{14}, \frac{6}{23}, \frac{7}{34}, \frac{8}{47}, \frac{9}{62}, \frac{10}{79}, \dots\right\} \tag{2.32}$$

$$P_k^{(1)} = \{ps_3^{(1)}, ps_4^{(1)}, ps_5^{(1)}, \dots\}$$
$$= \left\{\frac{30}{2^2}, \frac{68}{7^2}, \frac{130}{14^2}, \frac{222}{23^2}, \frac{350}{34^2}, \frac{520}{47^2}, \frac{738}{62^2}, \frac{1010}{79^2}, \dots\right\} \tag{2.33}$$

$$P_k^{(2)} = \{ps_3^{(2)}, ps_4^{(2)}, ps_5^{(2)}, \dots\}$$
$$= \left\{\frac{552}{2^3}, \frac{1892}{7^3}, \frac{5080}{14^3}, \frac{11598}{23^3}, \frac{23576}{34^3}, \frac{43912}{47^3}, \frac{76392}{62^3}, \frac{125810}{79^3}, \dots\right\} \tag{2.34}$$

$$P_k^{(3)} = \{ps_3^{(3)}, ps_4^{(3)}, ps_5^{(3)}, \dots\}$$

$$= \left\{\frac{15240}{2^4}, \frac{78404}{7^4}, \frac{290680}{14^4}, \frac{868686}{23^4}, \frac{2227400}{34^4}, \frac{5092360}{47^4}, \frac{10647864}{62^4},\right.$$

$$\left.\frac{20726210}{79^4}, \dots\right\} \tag{2.35}$$

$$P_k^{(4)} = \{ps_3^{(4)}, ps_4^{(4)}, ps_5^{(4)}, \dots\}$$

$$= \left\{\frac{561216}{2^5}, \frac{4345508}{7^5}, \frac{22310080}{14^5}, \frac{87372942}{23^5}, \frac{282337664}{34^5}, \frac{790203016}{47^5}, \frac{1977971328}{62^5},\right.$$

$$\left.\frac{4528097810}{79^5}, \dots\right\}. \tag{2.36}$$

...

3. K-ORESME ORESME NUMBERS

Generalizations can be obtained in more than one way. For example, if $A_n^{(k)} = \frac{n}{k^n}$, then it follows that

$$A_{n+2}^{(k)} = A_{n+1}^{(k)} - \frac{k-1}{k^2} A_n^{(k)} - \frac{k-2}{k^{n+2}} \text{ for } k \geq 2. \tag{3.1}$$

Investigating the solutions of this non-homogeneous equation will not be explored in this paper. Here the k-Oresme numbers are defined analogous to (1.2) as the solutions to the homogeneous equation

$$A_{n+2}^{(k)} = A_{n+1}^{(k)} - \frac{1}{k^2} A_n^{(k)} \text{ where } A_0^{(k)} = 0, \text{ and } A_1^{(k)} = \frac{1}{k} \text{ for } k \geq 3. \tag{3.2}$$

Solutions are found to be

$$A_n^{(k)} = \frac{1}{k^n \sqrt{k^2-4}} \left[\left(\frac{k + \sqrt{k^2-4}}{2} \right)^n - \left(\frac{k - \sqrt{k^2-4}}{2} \right)^n \right] = \frac{\alpha^n - \beta^n}{k^n \sqrt{k^2-4}}. \tag{3.3}$$

The numerators form sequences all beginning with $a_0 = 0$ and $a_1 = 1$. Additional members are shown for $k = 3$ through $k = 13$ in Table 1. Note that the sequence for $k = 3$ yields

Fibonacci numbers, F_{2n} (See 5.1). It will be shown in section 5 that several of these sequences form familiar patterns after a modification of the initial conditions.

k \ n	2	3	4	5	6	7	8	9	...
3	3	8	21	55	144	377	987	2584	...
4	4	15	56	209	780	2911	10864	40545	...
5	5	24	115	551	2640	12649	60605	290376	...
6	6	35	204	1189	6930	40391	235416	1372105	...
7	7	48	329	2255	15456	105937	726103	4976784	...
8	8	63	496	3905	30744	242047	1905632	15003009	...
9	9	80	711	6319	56160	499121	4435929	39424240	...
10	10	99	980	9701	96030	950599	9409960	93149001	...
11	11	120	1309	14279	155760	1699081	18534131	202176360	...
12	12	143	1704	20305	241956	2883167	34356048	409389409	...
13	13	168	2171	28055	362544	4685017	60542677	782369784	...

Table 1 Numerator sequences for k-Oresme numbers

4. SOME PROPERTIES OF K-ORESME NUMBERS

It is a routine, though sometimes laborious, exercise to establish identities analogous to Horadam's given in section 1 as (1.3) - (1.10).

Theorem:

$$\text{(a)}\quad A_{n+2}^{(k)} - \left(\frac{k^2-1}{k^2}\right) A_n^{(k)} + \frac{1}{k^2} A_{n-1}^{(k)} = 0 \tag{4.1}$$

$$\text{(b)}\quad A_{n+2}^{(k)} - \left(\frac{k^2-1}{k^2}\right) A_{n+1}^{(k)} + \frac{1}{k^4} A_{n-1}^{(k)} = 0 \tag{4.2}$$

$$\text{(c)}\quad \sum_{j=0}^{n-1} A_j^{(k)} = k^2 \left(\frac{1}{k} - A_{n+1}^{(k)}\right) \tag{4.3}$$

$$\text{(d)}\quad \sum_{j=0}^{\infty} A_j^{(k)} = k \tag{4.4}$$

$$\text{(e)}\quad \sum_{j=0}^{n-1} (-1)^j A_j^{(k)} = \frac{k^2}{2k^2+1} \left[-\frac{1}{k} + (-1)^n (A_{n+1}^{(k)} - 2A_n^{(k)})\right] \tag{4.5}$$

(f) $$\sum_{j=0}^{n-1} A_{2j}^{(k)} = \frac{k^2}{2k^2+1}\left[k + A_{2n-1}^{(k)} - (k^2+1)A_{2n}^{(k)}\right] \tag{4.6}$$

(g) $$\sum_{j=0}^{n-1} A_{2j+1}^{(k)} = \frac{k^2}{2k^2+1}\left[\frac{k^2+1}{k} + \frac{k^2+1}{k^2}(A_{2n-1}^{(k)} - k^2 A_{2n}^{(k)}\right] \tag{4.7}$$

(h) $$A_{n+1}^{(k)}A_{n-1}^{(k)} - (A_n^{(k)})^2 = -\left(\frac{1}{k^2}\right)^n. \tag{4.8}$$

Proof of (e): Using $\sum_{j=0}^{n-1}(-1)^j x^j = \frac{1+(-1)^{n-1}x^n}{1+x}$ and $A_n^{(k)} = \frac{1}{\sqrt{k^2-4}}\left[\left(\frac{\alpha}{k}\right)^n - \left(\frac{\beta}{k}\right)^n\right]$ and noting that $\alpha+\beta = k$ and $\alpha\cdot\beta = 1$, we find that

$$\begin{aligned}
\sum_{j=0}^{n-1}(-1)^j A_j^{(k)} &= \frac{1}{\sqrt{k^2-4}}\left[\frac{1+(-1)^{n-1}\alpha^n}{k^n\left(1+\frac{\alpha}{k}\right)} - \frac{1+(-1)^{n-1}\beta^n}{k^n\left(1+\frac{\beta}{k}\right)}\right] \\
&= \frac{k}{k^n\sqrt{k^2-4}(2k^2+1)}\left[-\sqrt{k^2-4} + (-1)^n\{k(\beta^n-\alpha^n) + (\beta^{n-1}-\alpha^{n-1})\}\right] \\
&= \frac{k}{2k^2+1}\left[-1 + (-1)^n(-kA_n^{(k)} - \frac{1}{k}A_{n-1}^{(k)})\right] \\
&= \frac{k^2}{2k^2+1}\left[-\frac{1}{k} + (-1)^n\left(-A_n^{(k)} - \frac{1}{k^2}A_{n-1}^{(k)}\right)\right] \\
&= \frac{k^2}{2k^2+1}\left[-\frac{1}{k} + (-1)^n\left(A_{n+1}^{(k)} - 2A_n^{(k)}\right)\right]. \quad \square
\end{aligned}$$

5. SOME VARIATIONS

Returning to the sequences of Table 1, we note that $k = 3$ yielded a Fibonacci number pattern. Thus it is seen from (3.3) that

$$A_n^{(3)} = \frac{1}{3^n\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right] = \frac{F_{2n}}{3^n}. \tag{5.1}$$

If the initial conditions and/or the coefficients of $A_n^{(k)}$ in (3.1) are varied slightly, other familiar sequences are obtained.

These can be represented as generalized Horadam Fibonacci numbers

$$w_n\left(a_0, a_1; 1, \frac{\pm 1}{k^2}\right) \tag{5.2}$$

but the emphasis here is to display the difference equation solutions as variations of k-Oresme numbers. For example, it is routine but in some cases tedious to show that the following hold.

Theorem:

$$\text{(a)}\ \frac{F_{2n-1}}{3^n} \text{ satisfies } A^{(3)}_{n+2} = A^{(3)}_{n+1} - \frac{1}{9}A^{(3)}_n;\ A^{(3)}_0 = \frac{1}{1},\ A^{(3)}_1 = \frac{1}{3} \tag{5.3}$$

$$\text{(b)}\ \frac{L_{3n-4}}{4^n} \text{ satisfies } A^{(4)}_{n+2} = A^{(4)}_{n+1} - \frac{1}{16}A^{(4)}_n;\ A^{(4)}_0 = \frac{7}{1},\ A^{(4)}_1 = \frac{-1}{4} \tag{5.4}$$

$$\text{(c)}\ \frac{P_{2n-1}}{6^n} \text{ satisfies } A^{(6)}_{n+2} = A^{(6)}_{n+1} - \frac{1}{36}A^{(6)}_n;\ A^{(6)}_0 = \frac{1}{1},\ A^{(6)}_1 = \frac{1}{6} \tag{5.5}$$

$$\text{(d)}\ \frac{F_{4n-1}}{7^{n+1}} \text{ satisfies } A^{(7)}_{n+2} = A^{(7)}_{n+1} - \frac{1}{49}A^{(7)}_n;\ A^{(7)}_0 = \frac{1}{7},\ A^{(7)}_1 = \frac{2}{49} \tag{5.6}$$

$$\text{(e)}\ \frac{F_{5n-2}}{11^{n+1}} \text{ satisfies } A^{(11)}_{n+2} = A^{(11)}_{n+1} + \frac{1}{121}A^{(11)}_n;\ A^{(11)}_0 = \frac{-1}{11},\ A^{(11)}_1 = \frac{2}{121}. \tag{5.7}$$

Proof of (c):

Using the recurrence relation for Pell numbers (See [2]) where needed, we have

$$A^{(6)}_{n+1} - \frac{1}{6^2}A^{(6)}_n = \frac{P_{2n+1}}{6^{n+1}} - \frac{P_{2n-1}}{6^{n+2}} = \frac{6P_{2n+1} - P_{2n-1}}{6^{n+2}} = \frac{5P_{2n+1} + 2P_{2n}}{6^{n+2}}$$

$$= \frac{P_{2n+1} + 2P_{2n+2}}{6^{n+2}} = \frac{P_{2n+3}}{6^{n+2}} = A^{(6)}_{n+2}. \quad \square$$

6. CONCLUSION

The interested reader may discover further properties analogous to those presented by Horadam in [1], and obtain sums involving higher powers of n, as well as sums involving other recurring sequences. Varying the initial conditions for the k-Oresme numbers difference equation may also yield some interesting sequences.

Finally note that additional historical and bibliographical information on Nicole Oresme and his work can be found on the ORESME Reading Group Web Page

⟨http://www.nku.edu/%7Ecurtin/oresme.html⟩. Or just search under Oresme for other interesting sites.

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