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# Explicit Formulas for the Nörlund Polynomials $B_n^{(x)}$ and $b_n^{(x)}$

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**Abstract**—In this paper, the authors establish some explicit formulas and representations for the Nörlund polynomial $B_n^{(x)}$ and $b_n^{(x)}$. Several identities involving Bernoulli numbers, Nörlund numbers, Stirling numbers and the associated Stirling numbers are also presented. 

**Keywords**—Nörlund numbers, Nörlund polynomials, Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, Stirling numbers, Associated Stirling numbers.

## 1. INTRODUCTION AND DEFINITIONS

For a real or complex parameter $\alpha$, the *generalized Bernoulli polynomials* $B_n^{(\alpha)}(x)$ and the *generalized Euler polynomials* $E_n^{(\alpha)}(x)$, *each* of degree $n$ in $x$ *as well as* in $\alpha$, are defined by means of the following generating functions (see, for details, [1, p. 253 *et seq.*; 2, Section 2.8; 3, Section 1.6]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \qquad \left(|t|<2\pi;\ 1^{\alpha}:=1\right) \tag{1.1}$$

and

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \qquad \left(|t|<\pi;\ 1^{\alpha}:=1\right), \tag{1.2}$$

respectively. Clearly, we have

$$B_n^{(1)}(x)=B_n(x) \quad \text{and} \quad E_n^{(1)}(x)=E_n(x) \qquad \left(n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right), \tag{1.3}$$

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in terms of the classical *Bernoulli polynomials* $B_n(x)$ and the classical *Euler polynomials* $E_n(x)$, $\mathbb{N}$ being the set of positive integers. When $x = 0$, we also have

$$B_n^{(\alpha)} := B_n^{(\alpha)}(0) \quad \text{and} \quad E_n^{(\alpha)} := E_n^{(\alpha)}(0) \qquad (n \in \mathbb{N}_0), \tag{1.4}$$

where $B_n^{(\alpha)}$ and $E_n^{(\alpha)}$ denote the *Bernoulli numbers of order* $\alpha$ and the *Euler numbers of order* $\alpha$, respectively. Thus, the classical *Bernoulli numbers* $B_n$ and the classical *Euler numbers* $E_n$ are given by

$$B_n := B_n(0) = B_n^{(1)} \quad \text{and} \quad E_n := 2^n \, E_n\left(\frac{1}{2}\right) \qquad (n \in \mathbb{N}_0). \tag{1.5}$$

Numerous interesting (and useful) properties and relationships involving each of these families of polynomials and numbers can be found in many books and tables (see, for example, [1–3]). For various explicit representations and other results for these and their closely-related families, the reader may be referred to some recent works by (for example) Srivastava *et al.* ([4–6]) and Luo and Srivastava [7] (see also many of the references cited in each of these recent works). The main subjects of our investigation in this paper are the so-called Nörlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$, which are defined by (see [8–10])

$$\left(\frac{t}{e^t - 1}\right)^x = \sum_{n=0}^{\infty} B_n^{(x)} \frac{t^n}{n!} \tag{1.6}$$

and

$$\left(\frac{t}{\log(1+t)}\right)^x = \sum_{n=0}^{\infty} b_n^{(x)} t^n, \tag{1.7}$$

respectively. These polynomials and numbers have many important applications. In fact, $B_n^{(k)}$ ($k \in \mathbb{N}$) are the above-mentioned Bernoulli numbers of order $k$ ($k \in \mathbb{N}$) (see also [11,12]), $b_n^{(k)}$ ($k \in \mathbb{N}$) are the Bernoulli numbers of the second kind of order $k$ ($k \in \mathbb{N}$) (see [13]). The numbers

$$B_n^{(1)} = B_n \qquad \text{and} \qquad b_n^{(1)} = b_n$$

are the ordinary Bernoulli numbers given by (1.5) and the Bernoulli numbers of the second kind, respectively, and $B_n^{(n)}$ are called the Nörlund numbers (see [10,12,14]).

We now turn to the Stirling numbers $s(n,k)$ of the first kind, which are usually defined by (see, for example, [3, p. 56 *et seq.*; 10,14,15])

$$x(x-1)(x-2)\dots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k \tag{1.8}$$

or by means of the following generating function:

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n,k) \, \frac{x^n}{n!}. \tag{1.9}$$

It follows from (1.8) or (1.9) that

$$s(n,k) = s(n-1,k-1) - (n-1)s(n-1,k) \tag{1.10}$$

and that

$$s(n,0) = \delta_{n,0} \qquad (n \in \mathbb{N}_0), \quad s(n,n) = 1,$$

$$s(n,1) = (-1)^{n-1}\,(n-1)! \qquad (n \in \mathbb{N}) \quad \text{and} \quad s(n,k) = 0 \qquad (k > n \;\text{ or }\; k < 0),$$

where (*and in what follows*) $\delta_{m,n}$ denotes the Kronecker symbol.

The *associated* Stirling numbers $d(n,k)$ of the first kind and the *associated* Stirling numbers $b(n,k)$ of the second kind are defined, respectively, by (see [10,15])

$$(-\log(1-x)-x)^k = k! \sum_{n=2k}^{\infty} d(n,k)\frac{x^n}{n!} \tag{1.11}$$

and

$$(e^x - 1 - x)^k = k! \sum_{n=2k}^{\infty} b(n,k)\frac{x^n}{n!}. \tag{1.12}$$

It follows from (1.11) that

$$d(n,k) = (n-1)d(n-2,k-1) + (n-1)d(n-1,k) \tag{1.13}$$

and that

$$d(n,0) = \delta_{n,0} \quad (n \in \mathbb{N}_0), \qquad d(n,1) = (n-1)! \quad (n \in \mathbb{N} \setminus \{1\}),$$

and

$$d(n,k) = 0 \quad (2k > n \;\text{ or }\; k < 0).$$

Similarly, we find from (1.12) that

$$b(n,k) = (n-1)b(n-2,k-1) + kb(n-1,k) \tag{1.14}$$

and that

$$\begin{aligned} b(n,0) &= \delta_{n,0} \qquad && (n \in \mathbb{N}_0), \\ b(n,1) &= 1 && (n \in \mathbb{N} \setminus \{1\}), \end{aligned}$$

and

$$b(n,k) = 0 \qquad (2k > n \text{ or } k < 0).$$

The main purpose of this paper is to prove several explicit formulas and representations for the Nörlund polynomials $B_n^{(x)}$ and $b_n^{(x)}$. We also obtain some identities involving Bernoulli numbers, Nörlund numbers, Stirling numbers, and the associated Stirling numbers.

## 2. A SET OF MAIN RESULTS

One of our main results is contained in Theorem 1 below.

THEOREM 1. *Let $n \geqq k$ $(n,k \in \mathbb{N})$ and*

$$\sigma(n,k) := (-1)^k \sum_{j=k}^{n} \frac{n!}{(n+j)!}\, s(j,k)\, b(n+j,j). \tag{2.1}$$

*Then*

$$B_n^{(x)} = \sum_{k=1}^{n} \sigma(n,k)x^k. \tag{2.2}$$

REMARK 1. By setting $x = 1, -1$ in Theorem 1 and noting that

$$B_n^{(-1)} = \frac{1}{n+1},$$

we immediately deduce the following consequence of Theorem 1.

COROLLARY 1. *Let $n \in \mathbb{N}$. Then,*

$$\sum_{k=1}^{n} \sigma(n,k) = B_n \quad \textit{and} \quad \sum_{k=1}^{n} (-1)^k \, \sigma(n,k) = \frac{1}{n+1} \,. \tag{2.3}$$

THEOREM 2. *Let $n \geqq k$ $(n,k \in \mathbb{N})$. Suppose also that $\sigma(n,k)$ is defined by (2.1). Then*

$$\sigma(n,k) = (-1)^{n-k} \, \frac{n!}{k!} \sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1} \ldots B_{v_k}}{(v_1 \ldots v_k) \, v_1! \ldots v_k!}. \tag{2.4}$$

REMARK 2. By means of (2.1) and (2.4), we can easily deduce the following interesting summation identity involving the Bernoulli numbers, the Stirling numbers of the first kind, and the associated Stirling numbers of the second kind.

COROLLARY 2. *Let $n \geqq k$ $(n,k \in \mathbb{N})$. Then*

$$\sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1} \ldots B_{v_k}}{(v_1 \ldots v_k) \, v_1! \ldots v_k!} = (-1)^n \, k! \sum_{j=k}^{n} \frac{s(j,k) \, b(n+j,j)}{(n+j)!}. \tag{2.5}$$

REMARK 3. Upon setting $n = k, k+1, k+2$ in (2.5), if we note that (see [3,10,12,14])

$$B_1 = -\frac{1}{2}, \qquad B_2 = \frac{1}{6}, \qquad B_3 = 0,$$

and that

$$s(k,k) = 1, \qquad s(k+1,k) = -\binom{k+1}{2}, \qquad \text{and} \qquad s(k+2,k) = 2\binom{k+2}{3} + 3\binom{k+2}{4},$$

we get

$$b(2k,k) = \frac{(2k)!}{2^k \, k!}, \qquad b(2k+1,k) = \frac{4 \cdot (2k+1)!}{3 \cdot 2^{k+2} \, (k-1)!},$$

and

$$b(2k+2,k) = \frac{(2k+1) \cdot (2k+2)!}{9 \cdot 2^{k+2} \, (k-1)!}.$$

THEOREM 3. *Let $n \geqq k$ $(n,k \in \mathbb{N})$ and*

$$\tau(n,k) := (-1)^{n-k} \sum_{j=k}^{n} \frac{s(j,k) \, d(n+j,j)}{(n+j)!}. \tag{2.6}$$

*Then*

$$b_n^{(x)} = \sum_{k=1}^{n} \tau(n,k) x^k. \tag{2.7}$$

REMARK 4. Setting $x = 1, -1$ in Theorem 3 and noting that

$$b_n^{(-1)} = \frac{(-1)^n}{n+1},$$

we deduce the following immediate consequence of Theorem 3.

COROLLARY 3. *Let $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n} \tau(n,k) = b_n \quad \textit{and} \quad \sum_{k=1}^{n} (-1)^k \tau(n,k) = \frac{(-1)^n}{n+1} \,. \tag{2.8}$$

THEOREM 4. *Let $n \geqq k$ $(n, k \in \mathbb{N})$. Then*

$$\tau(n,k) = \frac{(-1)^k}{k!} \sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1}^{(v_1)} \ldots B_{v_k}^{(v_k)}}{(v_1 \ldots v_k)\, v_1! \ldots v_k!}, \tag{2.9}$$

*where $\tau(n,k)$ is defined as in Theorem 3.*

REMARK 5. By applying (2.6) and (2.9), we can readily derive the following consequence of Theorem 4.

COROLLARY 4. *Let $n \geqq k$ $(n, k \in \mathbb{N})$. Then,*

$$\sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1}^{(v_1)} \ldots B_{v_k}^{(v_k)}}{(v_1 \ldots v_k)\, v_1! \ldots v_k!} = (-1)^n\, k! \sum_{j=k}^{n} \frac{s(j,k)\, d(n+j,j)}{(n+j)!}. \tag{2.10}$$

REMARK 6. By setting $n = k,\ k+1,\ k+2$ in (2.10) and noting that (see [3,10,12,14])

$$B_1 = -\frac{1}{2}, \qquad B_2^{(2)} = \frac{5}{6}, \qquad B_3^{(3)} = -\frac{9}{4},$$

and that

$$s(k,k) = 1, \qquad s(k+1,k) = -\binom{k+1}{2}, \qquad \text{and} \qquad s(k+2,k) = 2\binom{k+2}{3} + 3\binom{k+2}{4},$$

we get

$$d(2k,k) = \frac{(2k)!}{2^k\, k!}, \qquad d(2k+1,k) = \frac{(2k+1)!}{3 \cdot 2^{k-1}\, (k-1)!},$$

and

$$d(2k+2,k) = \frac{(4k+5) \cdot (2k+2)!}{9 \cdot 2^{k+1}\, (k-1)!}.$$

REMARK 7. Setting $n = 1,2,3,4$ in Theorem 1, we get

$$B_1^{(x)} = -\frac{1}{2}x, \qquad B_2^{(x)} = -\frac{1}{12}x + \frac{1}{4}x^2, \qquad B_3^{(x)} = \frac{1}{8}x^2 - \frac{1}{8}x^3,$$

and

$$B_4^{(x)} = \frac{1}{120}x + \frac{1}{48}x^2 - \frac{1}{8}x^3 + \frac{1}{16}x^4.$$

REMARK 8. Setting $n = 1,2,3,4$ in Theorem 3, we get

$$b_1^{(x)} = \frac{1}{2}x, \qquad b_2^{(x)} = -\frac{5}{24}x + \frac{1}{8}x^2, \qquad b_3^{(x)} = \frac{1}{8}x - \frac{5}{48}x^2 + \frac{1}{48}x^3,$$

and

$$b_4^{(x)} = -\frac{251}{2880}x + \frac{97}{1152}x^2 - \frac{5}{192}x^3 + \frac{1}{384}x^4.$$

REMARK 9. Setting $k = 1$ in Corollary 2 and Corollary 4, we obtain

$$B_n = n \cdot n! \sum_{j=1}^{n} (-1)^{n-j-1} \frac{(j-1)!\, b(n+j,j)}{(n+j)!} \qquad (n \in \mathbb{N}) \tag{2.11}$$

and

$$B_n^{(n)} = n \cdot n! \sum_{j=1}^{n} (-1)^{n-j-1} \frac{(j-1)!\, d(n+j,j)}{(n+j)!} \qquad (n \in \mathbb{N}), \tag{2.12}$$

respectively.

## 3. PROOFS OF THEOREMS 1–4

In this section, we present an outline of the proof of each of our main results stated in Section 2.

PROOF OF THEOREM 1. By (1.6) and (1.12), we have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(x)} \frac{t^n}{n!} &= \left(\frac{t}{e^t-1}\right)^x = \left(\frac{1}{1+(1/t)(e^t-1-t)}\right)^x \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{x+j-1}{j}(e^t-1-t)^j\, t^{-j} \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{x+j-1}{j} j! \sum_{n=2j}^{\infty} b(n,j)\, \frac{t^{n-j}}{n!} \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{x+j-1}{j} j! \sum_{n=j}^{\infty} b(n+j,j)\, \frac{t^n}{(n+j)!} \\ &= \sum_{n=0}^{\infty}\sum_{j=0}^{n}(-1)^j\, j!\binom{x+j-1}{j} b(n+j,j)\, \frac{t^n}{(n+j)!},\end{aligned} \tag{3.1}$$

which readily yields

$$\begin{aligned}B_n^{(x)} &= \sum_{j=0}^{n}(-1)^j\, \frac{n!\, j!}{(n+j)!}\binom{x+j-1}{j} b(n+j,j) \\ &= \sum_{j=0}^{n}(-1)^j\, \frac{n!}{(n+j)!}\, b(n+j,j)\cdot(x+j-1)(x+j-2)\dots(x+1)x \\ &= \sum_{j=0}^{n}(-1)^j\, \frac{n!}{(n+j)!}\, b(n+j,j)\sum_{k=1}^{j}(-1)^{j-k}\, s(j,k)x^k \\ &= \sum_{k=1}^{n}(-1)^k \sum_{j=k}^{n}\frac{n!}{(n+j)!}\, s(j,k)\, b(n+j,j)x^k = \sum_{k=1}^{n}\sigma(n,k)x^k.\end{aligned}$$

This completes the proof of Theorem 1.

PROOF OF THEOREM 2. By applying Theorem 1, we have

$$k!\,\sigma(n,k) = \frac{d^k}{dx^k}\left\{B_n^{(x)}\right\}\bigg|_{x=0}. \tag{3.2}$$

On the other hand, it follows from (1.6) that

$$\sum_{n=k}^{\infty}\frac{d^k}{dx^k}\left\{B_n^{(x)}\right\}\bigg|_{x=0}\frac{t^n}{n!} = \left(\log\left(\frac{t}{e^t-1}\right)\right)^k. \tag{3.3}$$

Thus, by (3.2) and (3.3), we have

$$k!\sum_{n=k}^{\infty}\sigma(n,k)\,\frac{t^n}{n!} = \left(\log\left(\frac{t}{e^t-1}\right)\right)^k, \tag{3.4}$$

which, in view of the following known result (see [9]),

$$\sum_{n=1}^{\infty}\frac{B_n}{n}\,\frac{(-t)^n}{n!} = \log\left(\frac{e^t-1}{t}\right),$$

yields

$$
\begin{aligned}
k! \sum_{n=k}^{\infty} \sigma(n,k)\, \frac{t^n}{n!} &= \left( - \sum_{n=1}^{\infty} \frac{B_n}{n}\, \frac{(-t)^n}{n!} \right)^k \\
&= \sum_{n=k}^{\infty} \left( (-1)^{n-k}\, n! \sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1} \ldots B_{v_k}}{(v_1 \ldots v_k)\, v_1! \ldots v_k!} \right) \frac{t^n}{n!}.
\end{aligned}
\tag{3.5}
$$

The assertion (2.4) of Theorem 2 would now follow easily from (3.5).

PROOF OF THEOREM 3. By (1.7) and (1.11), we have

$$
\begin{aligned}
\sum_{n=0}^{\infty} b_n^{(x)}\, t^n &= \left( \frac{t}{\log(1+t)} \right)^x = \left( \frac{1}{1 + (1/t)\,(\log(1+t) - t)} \right)^x \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{x+j-1}{j} (\log(1+t) - t)^j\, t^{-j} \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{x+j-1}{j} j! \sum_{n=2j}^{\infty} (-1)^{n-j}\, d(n,j)\, \frac{t^{n-j}}{n!} \\
&= \sum_{j=0}^{\infty} (-1)^j \binom{x+j-1}{j} j! \sum_{n=j}^{\infty} (-1)^n\, d(n+j,j)\, \frac{t^n}{(n+j)!} \\
&= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^{n-j}\, j! \binom{x+j-1}{j} d(n+j,j)\, \frac{t^n}{(n+j)!},
\end{aligned}
\tag{3.6}
$$

which leads us easily to

$$
\begin{aligned}
b_n^{(x)} &= \sum_{j=0}^{n} (-1)^{n-j}\, \frac{j!}{(n+j)!} \binom{x+j-1}{j} d(n+j,j) \\
&= \sum_{j=0}^{n} (-1)^{n-j}\, \frac{1}{(n+j)!}\, d(n+j,j) \cdot (x+j-1)(x+j-2) \ldots (x+1)x \\
&= \sum_{j=0}^{n} (-1)^{n-j}\, \frac{1}{(n+j)!}\, d(n+j,j) \sum_{k=1}^{j} (-1)^{j-k}\, s(j,k)\, x^k \\
&= \sum_{k=1}^{n} (-1)^{n-k} \sum_{j=k}^{n} \frac{s(j,k)\, d(n+j,j)}{(n+j)!}\, x^k = \sum_{k=1}^{n} \tau(n,k)\, x^k.
\end{aligned}
$$

This completes the proof of Theorem 3.

PROOF OF THEOREM 4. By applying Theorem 3, we have

$$
k!\, \tau(n,k) = \frac{d^k}{dx^k} \left\{ b_n^{(x)} \right\} \bigg|_{x=0}. \tag{3.7}
$$

On the other hand, it follows from (1.7) that

$$
\sum_{n=k}^{\infty} \frac{d^k}{dx^k} \left\{ b_n^{(x)} \right\} \bigg|_{x=0} t^n = \left( \log \left( \frac{t}{\log(1+t)} \right) \right)^k. \tag{3.8}
$$

By means of (3.7) and (3.8), we get

$$
k! \sum_{n=k}^{\infty} \tau(n,k) t^n = \left( \log \left( \frac{t}{\log(1+t)} \right) \right)^k. \tag{3.9}
$$

Since (see [11,12])

$$\sum_{n=0}^{\infty} B_n^{(n)} \frac{t^n}{n!} = \frac{t}{(1+t)\log(1+t)},$$

it is easily seen that

$$\sum_{n=1}^{\infty} \left(-\frac{B_n^{(n)}}{n}\right) \frac{t^n}{n!} = \log\left(\frac{t}{\log(1+t)}\right). \tag{3.10}$$

Therefore, by comparing (3.10) with (3.9), we have

$$\begin{aligned} k! \sum_{n=k}^{\infty} \tau(n,k) t^n &= \left(-\sum_{n=1}^{\infty} \frac{B_n^{(n)}}{n} \frac{t^n}{n!}\right)^k \\ &= \sum_{n=k}^{\infty} \left((-1)^k \sum_{v_1,\ldots,v_k \in \mathbb{N}}^{v_1+\cdots+v_k=n} \frac{B_{v_1}^{(v_1)} \ldots B_{v_k}^{(v_k)}}{(v_1 \ldots v_k)\, v_1! \ldots v_k!}\right) t^n, \end{aligned} \tag{3.11}$$

which readily yields the assertion (2.9) of Theorem 4.

Numerous further results involving the polynomials and numbers considered in this paper can also be derived by using the methods and techniques described here.

# REFERENCES

1. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions, Volume I*, McGraw-Hill Book Company, New York, (1953).
2. Y.L. Luke, *The Special Functions and Their Approximations, Volume 1*, Academic Press, New York, (1969).
3. H.M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, (2001).
4. H.M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129**, 77–84, (2000).
5. H.M. Srivastava and Á. Pintér, Remarks on some relationships between the Bernoulli and Euler polynomials, *Appl. Math. Lett.* **17** (4), 375–380, (2004).
6. H.M. Srivastava and P.G. Todorov, An explicit formula for the generalized Bernoulli polynomials, *J. Math. Anal. Appl.* **130**, 509–513, (1988).
7. Q.-M. Luo and H.M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, *Computers Math. Applic.* **51** (3/4), 631–642, (2006).
8. A. Adelberg, Arithemetic properties of the Nörlund polynomial $B_n^{(x)}$, *Discrete Math.* **284**, 5–13, (1999).
9. L. Carlitz, Some properties of the Nörlund polynomial $B_n^{(x)}$, *Math. Nachr.* **33**, 297–311, (1967).
10. L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, (Translated from the French by J.W. Nienhuys), Reidel, Dordrecht, (1974).
11. F.T. Howard, Congruences and recurrences for Bernoulli numbers of higher order, *Fibonacci Quart.* **32**, 316–328, (1994).
12. N.E. Nörlund, *Vorlesungenüber Differenzenrechnung*, Springer-Verlag, Berlin, (1924); Reprinted by Chelsea Publishing Company, Bronx, New York, (1954).
13. F.H. Howard, The power of 2 dividing the coefficients of certain power series, *Fibonacci Quart.* **39**, 358–364, (2001).
14. C. Jordan, *Calculus of Finite Differences*, Third Edition, Chelsea Publishing Company, Bronx, New York, (1965).
15. F.T. Howard, Congruences for the Stirling numbers and associated Stirling numbers, *Acta Arith.* **55**, 29–41, (1990).