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Discrete Mathematics 204 (1999) 5–13

DISCRETE MATHEMATICS

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Arithmetic properties of the Nörlund polynomial $B_n^{(x)}$

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Received 12 March 1997; revised 21 February 1998; accepted 21 February 1998

Dedicated to H.W. Gould on the occasion of his 70th birthday

Abstract

For any prime p we establish a congruence of Kummer type for the Nörlund polynomial $B_n^{(x)}$, and determine the highest power of p which divides the coefficient denominators of the Nörlund polynomial. This improves the result of Carlitz (Math, Nachr. 33 (1967) 297–311), where only an upper bound was proven. We deduce a simple formula for the least common denominator of the coefficients. Applications are made for the Stirling polynomials.

Keywords: Nörlund polynomials; Bernoulli numbers and polynomials; Stirling polynomials; Kummer congruences

1. Introduction

We are pleased to contribute to this volume dedicated to Gould, whose collection of binomial identities [6] led us into the field of Bernoulli and related polynomials [2].

The Nörlund polynomials $B_n^{(x)}$ are defined by [10, Ch. 6]

$$\sum_n B_n^{(x)} \frac{t^n}{n!} = \left(\frac{t}{e^t - 1}\right)^x. \tag{1}$$

They have many important applications, e.g. $B_n^{(k)}$ is the Bernoulli number of order k and degree n, and in particular $B_n^{(1)} = B_n$ is the ordinary Bernoulli number, and $B_n^{(n)}$ is the Nörlund number [7]. Many other important polynomials for combinatorial algebra can be expressed in terms of the Nörlund polynomials, including the Stirling polynomials of both kinds [2,5] and the Stefani polynomials [4].

$B_n^{(x)}$ is a rational polynomial of degree n, with highest coefficient $(-1/2)^n$. Some examples are $B_0^{(x)} = 1$, $B_1^{(x)} = -x/2$, $B_2^{(x)} = x(3x-1)/12$, $B_3^{(x)} = -x^2(x-1)/8$, $B_4^{(x)} = x(15x^3 - 30x^2 + 5x + 2)/240$, and $B_5^{(x)} = -x^2(x-1)(3x^2 - 7x - 2)/96$.

We proved [2, Lemmas 9.2, 9.4] that $x = 1$ is a simple root of $B_n^{(x)}$ if n is odd>1, and that $x = 0$ is a simple root if n is even>0, and a double root if n is odd>1.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00363-X

Carlitz [4] determined many arithmetic properties of the Nörlund polynomials, including an upper bound for the highest power of a prime p dividing the denominator of a coefficient of $B_n^{(x)}$. We prove in this paper that his upper bound is actually a *maximum*, and can even locate the highest degree term in which this prime power occurs. Our result is similar to the description we gave in [1] of the p-adic pole pattern of the Bernoulli polynomial $B_n^{(k)}(x)$, where the order k can be an arbitrary p-adic integer. (In that paper, we made the unnecessary assumption that k was a rational integer from 0 to n.) If $[x]$ is the integer part function, Theorem 1 can be restated as follows.

Theorem 1. *If* $m=[n/(p-1)]$ *then* $(mp)!B_n^{(x)}/n!$ *has p-adic integer coefficients, not all divisible by p.*

A simple necessary and sufficient condition for $B_n^{(x)}$ to be p-integral (denominators prime to p) is an immediate consequence, namely $B_n^{(x)}$ is p-integral iff $n \geqslant [n/(p-1)]p$. It follows that if $B_n^{(x)}$ is p-integral then $n \leqslant p(p-2)$. Considering all primes, we deduce a simple formula for the integer least common denominator of the coefficients of $B_n^{(x)}$, and it follows easily that the numerators are then relatively prime.

We have also found a p-adic mod p congruence of Kummer type for $B_n^{(x)}$ multiplied by the highest power of p in a denominator. Theorem 3 can be restated as follows.

Theorem 3. *If* $m=[n/(p-1)]$ *and* $\sigma = n - m(p-1) = n \bmod p-1$, *then*

$$(mp)!\frac{B_n^{(x)}}{n!} \equiv (-1)^m \sum_k \binom{m}{k} \frac{1}{2^k}(x-n-1)_{k(p-1)+m} \frac{B_{\sigma-k}^{(x)}}{(\sigma-k)!} \pmod{p},$$

where the effective range of summation is $0 \leqslant k \leqslant \min\{\sigma, m \bmod p\}$.

This general congruence has many interesting special cases, including the congruence

$$2^n B_n^{(x)} \equiv (x-n-1)_n \pmod{2}$$

which is a variant of one which Carlitz found [4, (6.3)], by a more complicated method. Since there are simple formulas expressing the Stirling polynomials in terms of the Nörlund polynomials, our theorems have immediate applications to Stirling polynomials, i.e. Theorem 1 holds as stated if $B_n^{(x)}/n!$ is replaced by a Stirling polynomial $f_n(x)$ or $g_n(x)$, and Theorem 3 holds with obvious minor modifications.

In this paper, we give proofs independent of those given by Carlitz, to make this paper essentially self-contained, and also because we believe that our methods are more elementary and in some cases more powerful. They are certainly more constructive. Carlitz makes heavy use of properties of Bernoulli numbers and of the theory of Hurwitz series. We, on the other hand, rely on an explicit formula for the 'terms' of a Nörlund polynomial and use only standard p-divisibility results for factorials.

It was pointed out to us by a referee that Lundell in [8] proved results equivalent to our Theorem 1 and some of its corollaries. Although his method has much in common with ours, his proof is much longer and lacks the characterization of the relevant terms that is provided by our simple, combinatorial, Lemma 2. Our main result, Theorem 3, follows easily form this explicit characterization.

2. Preliminaries

If p is prime and $n \in N$, let $r(n) = r_p(n) =$ the highest power of p dividing $n!$. It should be noted that this does not conform to the notation of [4], where the author uses $v(n)$ for what we call $r(n)$.

If $n = \sum a_i p^i$ is the base p expansion of n (with $0 \leqslant a_i \leqslant p-1$), let $S(n) = S_p(n) = \sum a_i =$ the digit sum. Note that $S(n) \equiv n (\mathrm{mod}\ p-1)$.

The following properties are elementary and well known (cf. [9]):

(i) If $mp \leqslant n < (m+1)p$ then $r(n) = r(mp) = m + r(m)$.

(ii) $(mp+k)! \equiv (-1)^m k! m! p^m (\mathrm{mod}\ p^{r(mp)+1})$ if $0 \leqslant k < p$.

(iii) $r(n) = (n - S(n))/(p-1)$.

(iv) $r(n+m) \geqslant r(n) + r(m)$, with equality iff $p \nmid \binom{n+m}{n}$.

(v) $p \nmid \binom{n}{m}$ iff $S(n) = S(m) + S(n-m)$.

Recall the following explicit formula for $B_n^{(x)}$ [3, Section 2]:

$$B_n^{(x)} = (-1)^n n! \sum_{w \leqslant n} t_u(x - n - 1) \tag{2}$$

summed over all non-negative integer sequences $(u) = (u_1, \ldots, u_n)$, where $w = w(u) = \sum i u_i$ is the weight, and $d = d(u) = \sum u_i$ is the degree of the term

$$t_u(s) = \binom{s}{d} \binom{d}{u_1 \cdots u_n} / (2^{u_1} \cdots (n+1)^{u_n}) = (s)_d / (u! \Lambda^u), \tag{3}$$

with the shorthand $u! = u_1! \cdots u_n!$, $\Lambda^u = 2^{u_1} \cdots (n+1)^{u_n}$ and $(s)_d = s(s-1) \cdots (s-d+1)$.

If (u) is as above, let $v_1(u) =$ highest power of p dividing $u! \Lambda^u$, i.e. $v_1(u) = \sum r(u_i) + \sum u_i \alpha_i$ where $\alpha_i =$ the highest power of p dividing $i+1$.

Lemma 1. *Suppose that* (a) $u_i > 0$ *for some* $i > p-1$ *or* (b) $u_i \geqslant p-1$ *for some* $1 < i < p-1$. *Then there exists* (u') *such that* $w(u') \leqslant w(u)$ *and* $v_1(u') > v_1(u)$.

Proof. First consider (a), and let $i = kp^\alpha - 1$ where $\alpha \geqslant 0$, $(k, p) = 1$, and $i \geqslant p$. Let $u'_j = u_j$ if $j \neq i$ or $p-1$, $u'_i = 0$, and $u'_{p-1} = u_{p-1} + (\alpha + 1) u_i$. It is easy to verify that (u') has the desired properties.

For case (b), let $u'_j = u_j$ if $j \neq i$ or $p-1$, $u'_i = 0$, but now $u'_{p-1} = u_{p-1} + [u_i/(p-1)] + 1$. Again the verification is straightforward. (What is needed is that $r(u_i) \leqslant [u_i/(p-1)]$ and $(p-1)([u_i/(p-1)] + 1) \leqslant i u_i$ if $i \geqslant 2$ and $u_i \geqslant p-1$.) $\square$

Thus if $v_1(u)$ is maximal for all $w(u) \leqslant n$ then $u_i = 0$ for all $i > p-1$ and $u_i < p-1$ for all $1 < i < p-1$. It follows that if $p > 2$ then

$$v_1(u) = r(u_1) + r(u_{p-1}) + u_{p-1} = r(u_1) + r(pu_{p-1}). \tag{4}$$

For the remainder of this paper, let $n = m(p-1) + \sigma$ with $0 \leqslant \sigma < p-1$, so $m = [n/(p-1)]$ and $\sigma = n \bmod p-1$.

Lemma 2. *The maximum value of $v_1(u)$ for all $w(u) \leqslant n$ is $v_1(u) = r(mp)$. Furthermore, if $w(u) \leqslant n$ then $v_1(u) = r(mp)$ iff $u_{p-1} = m-k$, $u_1 \geqslant kp$, and $p \nmid \binom{m}{k}$, with $k \geqslant 0$.*

Proof. First observe that $w(u) \leqslant n$ implies that $u_{p-1} \leqslant m$, i.e. $u_{p-1} = m-k$ with $k \geqslant 0$. But $u_1 + (m-k)(p-1) \leqslant n = m(p-1) + \sigma$, so $u_1 \leqslant k(p-1) + \sigma < (k+1)p$. Thus $r(u_1) \leqslant r(kp)$, and by (4), if $v_1(u)$ is maximal then $v_1(u) = r(u_1) + r(m-k) + m - k \leqslant r(kp) + r(m-k) + m - k \leqslant m + r(m)$, if $p > 2$, with $v_1(u) = m + r(m) = r(mp)$ iff $u_1 \geqslant kp$ and $p \nmid \binom{m}{k}$. Clearly if $p = 2$, then by Lemma 1, $v_1(u)$ is maximal iff $u_1 = m = n$. $\square$

Remark. Since $w(u) \leqslant n$, we have $u_1 \geqslant kp$ iff $u_1 = k(p-1) + \delta$ where $k \leqslant \delta \leqslant \sigma$. Hence if $u_1 \geqslant kp$ then $k \leqslant p-1$, so $p \nmid \binom{m}{k}$ iff $p \nmid (m)_k$ iff $k \leqslant m \bmod p$. In particular all (u) such that $w(u) \leqslant n$ and $v_1(u) = r(mp)$ have $u_{p-1} = m$ iff $\sigma = 0$ or $p|m$. Of course any (u) with $w(u) \leqslant n$ and $u_{p-1} = m$ satisfies the maximality condition.

Corollary. *Among all terms with $w(u) \leqslant n$ and maximal $v_1(u)$ there is a unique term with minimum degree $d = m$ and a unique term with maximum degree $d = m + K(p-2) + \sigma$, where $K = \min\{n \bmod p-1, m \bmod p\}$.*

Proof. $d(u) \geqslant u_1 + u_{p-1} \geqslant kp + (m-k) = m + k(p-1)$, so the minimum degree is m, when $u_{p-1} = m$ and $u_i = 0$ if $i \neq p-1$.

For the maximum degree, if $u_i > 0$ for some $1 < i < p-1$, we can increase $d(u)$ by taking $u_i' = 0$ and $u_1' = u_1 + iu_i$, which is impossible. Thus we can assume $u_i = 0$ if $i \neq 1$ or $p-1$, so $u_{p-1} = m-k$ and $u_1 = k(p-1) + \sigma$, giving degree $m + k(p-2) + \sigma$, with $k \leqslant K$. Taking $k = K$ clearly gives the unique maximal degree term. $\square$

3. Main results

Observe first that since $m = [n/(p-1)]$, we have

$$n < (m+1)p, \text{ so } r(n) \leqslant r(mp). \tag{5}$$

Theorem 1. *$p^{r(mp)-r(n)} B_n^{(x)}$ has p-adic integer coefficients, and $r(mp) - r(n)$ is the smallest exponent of p with this property, where $m = [n/(p-1)]$.*

Proof. $p^{r(mp)-r(n)}B_n^{(x)} = (-1)^n n!\, p^{-r(n)} p^{r(mp)} \sum_{w \leqslant n} t_u(x-n-1)$. Since $v_1(u) \leqslant r(mp)$ for all (u), the coefficients are p-adic integers. Furthermore, by the corollary of Section 2, the term of greatest degree such that $v_1(u)$ is maximal gives a unit coefficient for that degree, and all higher coefficients are divisible by p. Thus $p^{r(mp)-r(n)}B_n^{(x)}$ is a primitive p-adic integer polynomial, and the corollary locates highest degree unit coefficient. □

Remark. There is an equivalent formulation of Theorem 1, that does not explicitly show the p-power but is slightly simpler, namely $(mp)!B_n^{(x)}/n!$ has p-adic integer coefficients, not all divisible by p. We can also replace $(mp)!$ by $(n+m)!$ since $n+m = mp+\sigma$.

Carlitz [4] defined $P_n(x) = n!B_n^{(-x)}$ and $e(n) =$ the smallest integer such that $p^{e(n)}P_n(x)$ is a p-adic integer polynomial. He proved that if $p=2$ then $e(n) = n - r(n)$ [4, Theorem 3], which coincides with our formula since $e(n) = r(2n) - 2r(n) = n - r(n)$, in case $p=2$. However, for $p>2$ he only proved that $e(n) \leqslant r(m+n) - 2r(n)$. Since obviously $r(m+n) = r(mp)$, the preceding theorem gives the equality

$$e(n) = r(m+n) - 2r(n), \tag{6}$$

which is clear improvement.

Corollary 1. *$B_n^{(x)}$ has p-adic integer coefficients iff $n \geqslant [n/(p-1)]p$.*

Proof. By the theorem, $B_n^{(x)}$ has p-adic integer coefficients iff $r(mp) \leqslant r(n)$. But as previously noted, $n < (m+1)p$, so $r(mp) \leqslant r(n)$ iff $n \geqslant mp$. Observe $n \geqslant mp$ iff $m \leqslant \sigma$. □

Corollary 2. *If $B_n^{(x)}$ has p-adic integer coefficients, then $n \leqslant (p-2)p$.*

Proof. Since $\sigma < p-1$, if $m \leqslant \sigma$ then $n \leqslant (p-2)(p-1) + p - 2 = (p-2)p$. □

Putting the p-adic information together for all p, we get a simple formula for the least common denominator of the coefficients of $B_n^{(x)}$, i.e. we can find the smallest positive integer d such that $dB_n^{(x)}$ has rational integer coefficients. If p_i ranges over all primes, then we have the following corollary.

Corollary 3. *The least common denominator of the coefficients of $B_n^{(x)}$ is*

$$d = \prod p_i^{r_i(m_i p_i)}/n!\,, \text{ where } m_i = [n/(p_i - 1)] \text{ and } r_i \text{ is } r_{p_i}. \tag{7}$$

This d is a positive integer, and $dB_n^{(x)}$ is a primitive integer polynomial.

Proof. First note that only primes $p_i \leqslant n+1$ must be considered, so the product is finite. By (5), d is a positive integer, and by the theorem, $dB_n^{(x)}$ has no denominators primes, so is an integer polynomial. Since for each prime p, some coefficient of $dB_n^{(x)}$

is not divisible by p, it is primitive, i.e. when put over the common denominator d, the numerators of the coefficients of $B_n^{(x)}$ are relatively prime. □

The following theorem is essentially the same as [4, Theorem 4], expect that we give more information about the prime $p=3$. However, our proof is considerably shorter.

Theorem 2. *If p is an odd prime then $n!B_n^{(x)}$ is p-integral, except if $n=p-1$ or $p=3$ and $n=8$.*

Proof. By the preceding theorem, we must show that if

$$n \neq p-1 \quad \text{and} \quad (n,p) \neq (8,3)$$

and

$$m=[n/(p-1)] \quad \text{then } 2r(n) \geqslant r(mp). \tag{8}$$

Observe that if $n=p-1$ then $m=1$, and if $(n,p)=(8,3)$ then $m=4$, and in both cases $2r(n)-r(mp)=-1$. Also note that inequality (8) is trivial if $m=0$, if $m=1$ and $n>p-1$, if $n=2(p-1)$, or if $p=3$ and $n>8$.

Replacing n by $m(p-1)$, we can assume $p-1|n$, and consider the equivalent inequality $2(n-S(n)) \geqslant mp-S(mp)$, for which it will suffice to prove that $(p-2)n \geqslant 2(p-1)S(n)$.

Consider $k=S(n)/(p-1)$. If $k=0$, then $n=0$. If $k=1$, since $n=2(p-1)$ has been done, we can assume $n \geqslant 3(p-1)$, in which case we have $3(p-1)(p-2) \geqslant 2(p-1)$ since $p \geqslant 3$. Clearly, if $S(n) \geqslant k(p-1)$ then $n \geqslant (p-1)(p^{k-1}+\cdots+1)=p^k-1$. Thus if $k=2$ then $n \geqslant p^2-1$, and $(p^2-1)(p-2) \geqslant 4(p-1)$ for $p \geqslant 5$ is obvious. Finally, if $k \geqslant 3$, then we need the inequality $(p^{k-1}+\cdots+1)(p-2) \geqslant 2k(p-1)$, which is clearly true if $p \geqslant 3$ and $k \geqslant 3$. □

Corollary. *$(n+1)!B_n^{(x)}$ has denominators that are all powers of 2.*

Proof. We must show that $(n+1)!B_n^{(x)}$ is p-integral for every odd prime. But the extra factor $n+1$ takes care of all the exceptions in the theorem. □

The following theorem gives a mod p congruence for $B_n^{(x)}$, with denominators cleared, in terms of low-degree Nörlund polynomials. We follow the standard convention that two polynomials are congruent (p-adically) if their corresponding coefficients are. The theorem is analogous to one that we have previously proved about p-adic integer order Bernoulli numbers [3, Theorem 1(i)].

Theorem 3. *If $m=[n/(p-1)]$ and $\sigma=n \bmod p-1$, then*

$$p^{r(mp)}\frac{B_n^{(x)}}{n!} \equiv \frac{p^{r(m)}}{m!}\sum_k \binom{m}{k}\frac{1}{2^k}(x-n-1)_{k(p-1)+m}\frac{B_{\sigma-k}^{(x)}}{(\sigma-k)!} \pmod{p}.$$

Proof. First observe that k is unrestricted in the summation, but terms are zero mod p if $k>\sigma$ or if $p|\binom{m}{k}$, so the effective range is $0\leqslant k\leqslant K$, where $K=\min\{\sigma, m \bmod p\}$. By (2) and Lemma 2, $p^{r(mp)}(-1)^n B_n^{(x)}/n! \equiv p^{r(mp)}\sum t_u(x-n-1)$, summed over all (u) such that $w(u)\leqslant n$ and $v_1(u)=r(mp)$, and these terms are characterized by the lemma.

For fixed k with $0\leqslant k\leqslant K$, let $S=\{(u)\,|\,w(u)\leqslant n, v_1(u)=r(mp)$ and $u_{p-1}=m-k\}$. If $(u)\in S$, let $u_1'=u_1-kp$, $u_{p-1}'=0$, and $u_i'=u_i$ if $i\neq 1,p-1$. Then $w(u')=w(u)-m(p-1)-k$ and $d(u')=d(u)-k(p-1)-m$. By Lemma 2, $(u)\to(u')$ gives a one–one correspondence of S with $\{(u)\,|\,w(u)\leqslant\sigma-k\}$. Furthermore, since $r(mp)=r(m-k)+r(kp)+m-k$ for $(u)\in S$,

$$p^{r(mp)}\sum_{u\in S}t_u(s)\equiv\frac{p^{r(m-k)}}{(m-k)!}\frac{p^{r(kp)}}{(kp)!}\frac{1}{2^{kp}}(s)_{k(p-1)+m}\sum_{w\leqslant\sigma-k}t_u(s-k(p-1)-m)\ (\mathrm{mod}\ p).$$

But $s=x-n-1$, so $s-k(p-1)-m=x-(\sigma-k)-1-(m+k)p$. Since the denominators of the terms of the right-hand side are all $<p$, for the mod p, congruence we can ignore the $(m+k)p$ summand in $s-k(p-1)-m$, and since $(kp)!/p^k\equiv(-1)^k k!\ (\mathrm{mod}\ p^{r(k)+1})$ by Wilson's Theorem, and $2^p\equiv 2\ (\mathrm{mod}\ p)$ by Fermat's Little Theorem, the result follows for the summand with index k. Now sum over all k. □

Remark. By (ii) of Section 2, we can rewrite the theorem as

$$(mp)!\frac{B_n^{(x)}}{n!}\equiv(-1)^m\sum_k\binom{m}{k}\frac{1}{2^k}(x-n-1)_{k(p-1)+m}\frac{B_{\sigma-k}^{(x)}}{(\sigma-k)!}\ (\mathrm{mod}\,p).$$

Observe that the degree of the polynomial on the right-hand side of the congruence is precisely $m+K(p-2)+\sigma$, so this theorem refines the corollary in Section 2. We can also replace the factor $(mp)!$ by $(m+n)!/\sigma!$ in the congruence.

Example. We will consider all cases where $m=1$, i.e. $n=p-1+\sigma$ where $0\leqslant\sigma\leqslant p-1$. For $\sigma=0$, we get $pB_{p-1}^{(x)}\equiv -x\ (\mathrm{mod}\,p)$, while for $\sigma>0$ we get

$$B_{p+\sigma-1}^{(x)}\equiv-((x-\sigma)B_\sigma^{(x)}/\sigma+(1/2)(x-\sigma)_pB_{\sigma-1}^{(x)})\ (\mathrm{mod}\ p)\quad\text{if}\quad p>2.$$

Since $(x-\sigma)_p\equiv x^p-x\ (\mathrm{mod}\ p)$, for $p>2$, we get $B_p^{(x)}\equiv-(1/2)(x^p-x^2)\ (\mathrm{mod}\,p)$, which since $x^2|B_p^{(x)}$, gives a new proof of the p-adic congruence $B_p^{(l)}\equiv l^2/2\ (\mathrm{mod}\ pl^2)$ if l is a p-adic integer and $p|l$, which we had previously found [3, Corollary 6] by a different method.

Corollary 1. *If*

$$p-1|n\ \text{then}\ p^{r(mp)}\frac{B_n^{(x)}}{n!}\equiv p^{r(m)}\binom{x-n-1}{m}\ (\mathrm{mod}\ p).$$

Proof. $\sigma=0$ implies $K=0$, so the sum collapses. Observe that in this case only the one term such that $u_{p-1}=m$ has to be considered, and then $p^{r(mp)}t_u(s)=(p^{r(m)}/m!)(x-n-1)_m$. □

Remark. If $m=n/(p-1)$, the preceding congruence can be restated as

$$(n+m)!B_n^{(x)}/n! \equiv (-1)^m (x-n-1)_m \pmod p.$$

Carlitz [4] establishes congruences for $P_n(x)$ for $p-1 \leqslant n \leqslant 2p-1$, mostly trivial consequences of Theorem 1. His congruence (7.11) for $n=2p-2$ is incorrect. The correct congruence, which follows from the preceding corollary with $m=2$, is

$$P_{2p-2}(x) \equiv (1/2)x(x-1) \pmod p \quad \text{if} \quad p>2.$$

Carlitz, who is using a cumbersome n-term recursion for $P_n(x)$, apparently overlooked a significant term in the recursion.

The following special case $(p=2)$ of Corollary 1 essentially restates [4, (6.3)].

Corollary 2.

$$2^n B_n^{(x)} \equiv (x-n-1)_n \pmod 2.$$

Proof. Since $r(2n)=n+r(n)$ for $p=2$, this follows from the previous corollary. □

Corollary 3. *If*

$$p|m \quad \textit{then} \quad p^{r(mp)} \frac{B_n^{(x)}}{n!} \equiv p^{r(m)} \binom{x-n-1}{m} \frac{B_\sigma^{(x)}}{\sigma!} \pmod p.$$

Proof. Again $K=0$, so the sum collapses. □

Remark. By the remarks after Lemma 2, the preceding corollaries give all the cases where the sum collapses. If $p|m$, we can rewrite the preceding congruence as

$$(m+n)!B_n^{(x)}/n! \equiv (-1)^m (x-n-1)_m B_\sigma^{(x)} \pmod p.$$

If $m=qp$, then using some elementary number theory, we get

$$p^{r(mp)} \frac{B_n^{(x)}}{n!} \equiv (-1)^q \frac{p^{r(q)}}{q!} (x^p - x)^q \frac{B_\sigma^{(x)}}{\sigma!} \pmod p. \tag{9}$$

Again we give the special case for $p=2$, which is actually contained in Corollary 2.

Corollary 4. *If n is even then* $2^n B_n^{(x)} \equiv (x^2-x)^{n/2} \pmod 2$.

Applications to Stirling polynomials: Recall the definition of the Stirling numbers.

$$(x)_n = \sum S_1(n,k)x^k \quad \text{and} \quad x^n = \sum S_2(n,k)(x)_k.$$

In [5], the stirling polynomials $f_n(x)$ and $g_n(x)$ are defined, where

$$f_n(m) = S_2(m+n,m) \quad \text{and} \quad g_n(m) = (-1)^n S_1(m, m-n).$$

It is not hard to see that $f_n(x)$ and $g_n(x)$ are rational polynomials of degree $2n$, which are linked by the 'Stirling duality' $g_n(x)=f_n(-x)$.

The relations between the Stirling and Nörlund polynomials are [2, Remark 9.3]

$$f_n(x)=\binom{x+n}{n}B_n^{(-x)} \quad \text{and} \quad g_n(x)=(-1)^n\binom{x-1}{n}B_n^{(x)}. \tag{10}$$

Since $(x+n)_n$ and $(x-1)_n$ are monic integer polynomials, we deduce immediately from Theorem 1, with the same notations, that the p-adic least common denominator of the coefficients of $f_n(x)$ and of $g_n(x)$ is $p^{r(mp)}$ or $(mp)!$, and with the notations of Corollary 3 to Theorem 1, the integer least common denominators is $\prod p_i^{r_i(m_i p_i)}$.

Also we can get congruences for $p^{r(mp)}f_n(x)$ and $p^{r(mp)}g_n(x)$ from Theorem 3, e.g.

$$p^{r(mp)}g_n(x)\equiv(-1)^n\frac{p^{r(m)}}{m!}\sum_k\binom{m}{k}\frac{1}{2^k}(x-1)_{k(p-1)+m+n}\frac{B_{\sigma-k}^{(x)}}{(\sigma-k)!} \pmod p. \tag{11}$$

Finally, as before, we have the special cases $p-1|n$ and $p|[n/(p-1)]$, where the sums collapse, with several equivalent formulations.

Remark. It was pointed out to us by a referee that if $p|m$ then the congruences in Theorem 3 and several of its corollaries appear to hold mod higher powers of p. We have been able to prove that these congruences hold mod $p^{1+v(m)}$, where $v(m)$ is the highest power of p dividing m. Since the proof of this refinement is similar to other proofs in this paper, we will not include it here.

Acknowledgements

We extend our thanks to the Centre for Number Theory Research of Macquarie University for its hospitality during the genesis of this paper.

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