RECURRENCE SEQUENCES AND NÖRLUND-EULER POLYNOMIALS

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1. It is well known that a general linear sequence $S_n(p,q)$ $(n=0,1,2,\ldots)$ of order 2 is defined by the law of recurrence,

$$S_n(p,q)=pS_{n-1}(p,q)-qS_{n-2}(p,q),$$

with S_0, S_1, p, and q arbitrary, provided that $\Delta=p^2-4q>0$, see [1].

In particular, if $S_0=0$ and $S_1=1$ or if $S_0=2$ and $S_1=p$, we have generalized Fibonacci and Lucas sequences, respectively, in symbols $U_n(p,q)$ and $V_n(p,q)$.

By the roots $x_1>x_2$ of the generating equation $x^2-px+q=0$, it is proved that

$$U_n(p,q)=\frac{x_1^n-x_2^n}{x_1-x_2} \quad \text{and} \quad V_n(p,q)=x_1^n+x_2^n; \tag{1}$$

moreover, the general term of the recurrence sequence $S_n(p,q)$ is expressed as a sum of the general terms of generalized Fibonacci and Lucas sequences by the formula

$$S_n(p,q)=\left(S_1-\frac{1}{2}pS_0\right)U_n(p,q)+\frac{1}{2}S_0V_n(p,q). \tag{2}$$

We assume

$$S_0=\omega,$$
$$S_1=\frac{1}{2}p\omega+\left(x-\frac{1}{2}\omega\right)\Delta^{\frac{1}{2}},$$

and, according to (1) and (2), we deduce

$$S_n(x;p,q)=\left(x-\frac{1}{2}\omega\right)\Delta^{\frac{1}{2}}\cdot U_n(p,q)+\frac{1}{2}\omega V_n(p,q) \tag{3}$$

and

$$S_n(x;p,q)=xx_1^n+(\omega-x)x_2^n. \tag{4}$$

From this point on, we shall use the brief notation U_n, V_n, and $S_n(x)$ to denote $U_n(p,q)$, $V_n(p,q)$, and $S_n(x;p,q)$, respectively.

2. From (3), we have

$$S_n^m(x)+S_n^m(\omega-x)=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m}{2}\right]}\left[\begin{matrix}m\\2r\end{matrix}\right]\Delta^rU_n^{2r}V_n^{m-2r}(2x-\omega)^{2r}, \tag{5}$$

and from (4), we have

$$\begin{aligned}S_n^m(x)+S_n^m(\omega-x)&=\sum_{r=0}^{m}\begin{bmatrix}m\\ r\end{bmatrix}[x_1^{nr}x_2^{n(m-r)}+x_1^{n(m-r)}x_2^{nr}]x^r(\omega-x)^{m-r}\\ &=\sum_{r=0}^{m}\begin{bmatrix}m\\ r\end{bmatrix}q^{nr}[x_1^{n(m-2r)}+x_2^{n(m-2r)}]x^r(\omega-x)^{m-r}.\end{aligned}$$

Then we have

$$\begin{aligned}&S_n^{2m}(x)+S_n^{2m}(\omega-x)\\ &=\sum_{r=0}^{m}\begin{bmatrix}2m\\ r\end{bmatrix}q^{nr}[x_1^{2n(m-r)}+x_2^{2n(m-r)}]x^r(\omega-x)^{2m-r}+\sum_{s=0}^{m-1}\begin{bmatrix}2m\\ s\end{bmatrix}q^{ns}[x_1^{2n(m-s)}+x_2^{2n(m-s)}]x^{2m-s}(\omega-x)^s\\ &=2\begin{bmatrix}2m\\ m\end{bmatrix}q^{mn}x^m(\omega-x)^m+\sum_{r=0}^{m-1}\begin{bmatrix}2m\\ r\end{bmatrix}q^{nr}[x_1^{2n(m-r)}+x_2^{2n(m-r)}][x^r(\omega-x)^{2m-r}+x^{2m-r}(\omega-x)^r]\\ &=2\begin{bmatrix}2m\\ m\end{bmatrix}q^{mn}x^m(\omega-x)^m+\sum_{r=0}^{m-1}\begin{bmatrix}2m\\ r\end{bmatrix}q^{nr}V_{2n(m-r)}[x^r(\omega-x)^{2m-r}+x^{2m-r}(\omega-x)^r].\end{aligned}\tag{6}$$

Similarly, we have the analogous formula

$$S_n^{2m+1}(x)+S_n^{2m+1}(\omega-x)=\sum_{r=0}^{m}\begin{bmatrix}2m+1\\ r\end{bmatrix}q^{nr}V_{n(2m-2r+1)}[x^r(\omega-x)^{2m-r+1}+x^{2m-r+1}(\omega-x)^r].\tag{7}$$

We now have the difference formulas

$$S_n^m(x)-S_n^m(\omega-x)=\frac{\Delta^{\frac{1}{2}}}{2^{m-1}}\sum_{r=0}^{[\frac{m-1}{2}]}\begin{bmatrix}m\\ 2r+1\end{bmatrix}\Delta^r U_n^{2r+1}V^{m-2r-1}\omega^{m-2r-1}(2x-\omega)^{2r+1},\tag{8}$$

and

$$S_n^m(x)-S_n^m(\omega-r)=\Delta^{\frac{1}{2}}\sum_{r=0}^{[\frac{m-1}{2}]}\begin{bmatrix}m\\ r\end{bmatrix}q^{nr}U_{n(m-2r)}[x^{m-r}(\omega-x)^r-x^r(\omega-x)^{m-r}].\tag{9}$$

We shall end this section by giving the generating functions

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}U_{nr}=\frac{1}{\Delta^{\frac{1}{2}}}\left(\exp(tx_1^n)-\exp(tx_2^n)\right)\tag{10}$$

and

$$\sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}=\exp(tx_1^n)+\exp(tx_2^n).\tag{11}$$

3. First, we recall the Nörlund-Euler polynomials $E_n^{(k)}(x|\omega,\dots,\omega_k)$ defined by the generating expansion (see [2], [6]):

$$\sum_{r=0}^{\infty}E_r^{(k)}(x|\omega,\dots,\omega_k)\frac{t^r}{r!}=\frac{2^ke^{xt}}{(e^{\omega_1t}+1)\cdots(e^{\omega_kt}+1)}.\tag{12}$$

In particular, the Nörlund-Euler numbers of order k are given by

$$E_n^{(k)}[\omega_1, \ldots, \omega_k] = 2^n E_n^{(k)}\left(\frac{\omega_1 + \cdots + \omega_k}{2} \,\middle|\, \omega_1, \ldots, \omega_k\right).$$

If $\omega_1 = \cdots = \omega_k = 1$, then $E_n^{(k)}[1, 1, \ldots, 1] = E_n^{(k)}$ (the Euler numbers of order k, see [3]), and we note that

$$E_n^{(k)}(\omega_1 + \cdots + \omega_k - x \,|\, \omega_1, \ldots, \omega_k) = (-1)^n E_n^{(k)}(x \,|\, \omega_1, \ldots, \omega_k). \tag{13}$$

From (12), replacing t by $\Delta^{\frac{1}{2}} U_n t$, we have

$$\sum_{r=0}^{\infty} \frac{(\Delta^{\frac{1}{2}} U_n t)^r}{r!} E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k) = \frac{2^k e^{x \Delta^{\frac{1}{2}} U_n t}}{\left(e^{\omega_1 \Delta^{\frac{1}{2}} U_n t} + 1\right) \cdots \left(e^{\omega_k \Delta^{\frac{1}{2}} U_n t} + 1\right)}$$
$$= \frac{2^k}{\left(e^{\omega_1 t x_1^n} + e^{\omega_1 t x_2^n}\right) \cdots \left(e^{\omega_k t x_1^n} + e^{\omega_k t x_2^n}\right)} e^{t S_n(x)};$$

therefore,

$$\left(e^{\omega_1 t x_1^n} + e^{\omega_1 t x_2^n}\right) \cdots \left(e^{\omega_k t x_1^n} + e^{\omega_k t x_2^n}\right) \sum_{r=0}^{\infty} \frac{(\Delta^{\frac{1}{2}} U_n t)^r}{r!} E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k) = 2^k e^{t S_n(x)}.$$

Using (11), we obtain

$$\sum_{r_1=0}^{\infty} \frac{\omega_1^{r_1} t^{r_1}}{r_1!} V_{nr_1} \cdots \sum_{r_k=0}^{\infty} \frac{\omega_k^{r_k} t^{r_k}}{r_k!} V_{nr_k} \sum_{r=0}^{\infty} \frac{(\Delta^{\frac{1}{2}} U_n t)^r}{r!} E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k) = 2^k e^{t S_n(x)}$$

i.e.,

$$\left[\sum_{r=0}^{\infty} \left(\sum_{r_1 + \cdots + r_k = r} \frac{\omega_1^{r_1} V_{nr_1}}{r_1!} \cdots \frac{\omega_k^{r_k} V_{nr_k}}{r_k!} \right) t^r \right] \sum_{r=0}^{\infty} \frac{(\Delta^{\frac{1}{2}} U_n t)^r}{r!} E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k) = 2^k e^{t S_n(x)}.$$

Expanding the product, figuring in the first member, into a power series of t, and comparing with the expansion of the second member, we find

$$\sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \Delta^{\frac{r}{2}} U_n^r E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k)(m-r)! \sum_{r_1 + \cdots + r_k = m-r} \frac{\omega_1^{r_1} V_{nr_1}}{r_1!} \cdots \frac{\omega_k^{r_k} V_{nr_k}}{r_k!} = 2^k S_n^m(x). \tag{14}$$

And if we replace x by $\omega_1 + \cdots + \omega_k - x$ in (14), and using (13), we have

$$\sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix} \Delta^{\frac{r}{2}} U_n^r (-1)^r E_r^{(k)}(x \,|\, \omega_1, \ldots, \omega_k)(m-r)! \sum_{r_1 + \cdots r_k = m-r} \frac{\omega_1^{r_1} V_{nr_1}}{r_1!} \cdots \frac{\omega_k^{r_k} v_{nr_k}}{r_k!} \tag{15}$$
$$= 2^k S_n^m(\omega_1 + \cdots \omega_k - x).$$

Taking (14) + (15), and using $\omega_1 + \cdots + \omega_k$ to replace ω in (5), (6), and (7), we obtain

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}\begin{bmatrix} m \\ 2r \end{bmatrix}\Delta^r U_n^{2r} E_{2r}^{(k)}(x|\omega_1, \ldots, \omega_k)(m-2r)! \sum_{r_1+\cdots+r_k=m-2r} \frac{\omega_1^{r_1}V_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}V_{nr_k}}{r_k!}$$
$$=\frac{1}{2^{m-k}}\sum_{r=0}^{\left[\frac{m}{2}\right]}\begin{bmatrix} m \\ 2r \end{bmatrix}\Delta^r U_n^{2r} V_n^{m-2r}(\omega_1+\cdots+\omega_k)^{m-2r}(2x-(\omega_1+\cdots+\omega_k))^{2r} \tag{16}$$

$$=(1+(-1)^m)2^{k-1}\begin{bmatrix} m \\ m/2 \end{bmatrix} q^{mn/2}(x(\omega_1+\cdots+\omega_k-x))^{m/2}+2^{k-1}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix} m \\ r \end{bmatrix} q^{nr}$$
$$\cdot V_{n(m-2r)}[x^r(\omega_1+\cdots+\omega_k-x)^{m-r}+x^{m-r}(\omega_1+\cdots+\omega_k-x)^r]. \tag{17}$$

Taking (14) – (15), and using $\omega_1+\cdots+\omega_k$ to replace ω in (8) and (9), we get

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix} m \\ 2r+1 \end{bmatrix}\Delta^r U_n^{2r+1} E_{2r+1}^{(k)}(x|\omega_1, \ldots, \omega_k)(m-2r-1)! \sum_{r_1+\cdots+r_k=m-2r-1}\left(\frac{\omega_1^{r_1}V_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}V_{nr_k}}{r_k!}\right)$$
$$=\frac{1}{2^{m-k}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix} m \\ 2r+1 \end{bmatrix}\Delta^r U_n^{2r+1} V_n^{m-2r-1}(\omega_1+\cdots+\omega_k)^{m-2r-1}(2x-(\omega_1+\cdots+\omega_k))^{2r+1} \tag{18}$$

$$=2^{k-1}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix} m \\ r \end{bmatrix} q^{nr} U_{n(m-2r)}[x^{m-r}(\omega_1+\cdots+\omega_k-x)^r-x^r(\omega_1+\cdots+\omega_k-x)^{m-r}]. \tag{19}$$

4. If we take $x=\frac{\omega_1+\cdots+\omega_k}{2}$ in (16), then

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}\begin{bmatrix} m \\ 2r \end{bmatrix}\Delta^r U_n^{2r}\frac{1}{2^{2r}}E_{2r}^{(k)}[\omega_1, \ldots, \omega_k](m-2r)! \sum_{r_1+\cdots r_k=m-2r}\frac{\omega_1^{r_1}V_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}v_{nr_k}}{r_k!}$$
$$=\frac{1}{2^{m-k}}(\omega_1+\cdots+\omega_k)^m V_n^m. \tag{20}$$

Now, setting $\omega_1=\cdots=\omega_k=1$ in (20), we have

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}\begin{bmatrix} m \\ 2r \end{bmatrix}\Delta^r U_n^{2r}\frac{E_{2r}^{(k)}}{2^{2r}}(m-2r)! \sum_{r_1+\cdots+r_k=m-2r}\frac{V_{nr_1}}{r_1!}\cdots\frac{V_{nr_k}}{r_k!}=\frac{1}{2^{m-k}}k^m V_n^m. \tag{21}$$

Again, if we take $k=1$, then

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}\begin{bmatrix} m \\ 2r \end{bmatrix}\Delta^r U_n^{2r}\frac{E_{2r}}{2^{2r}}V_{n(m-2r)}=\frac{1}{2^{m-1}}V_n^m. \tag{22}$$

If we set $k=1$ in (18), we obtain

$$\begin{aligned}&\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}(x\,|\,\omega_1)V_{n(m-2r-1)}\omega_1^{m-2r-1}\\ &=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}V_n^{m-2r-1}\omega_1^{m-2r-1}(2x-\omega_1)^{2r+1}.\end{aligned}\tag{23}$$

Now, taking $\omega_1 = 1$ and $x = 0$ or $x = \frac{1}{3}$, and using the following relations (see [1]),

$$E_{n-1}(0) = 2(1-2^n)\frac{B_n}{n},$$

$$E_{2n-1}\left(\frac{1}{3}\right) = (2^{2n}-1)\left(\frac{1}{3^{2n}-1}\right)\frac{B_{2n}}{2n},$$

where B_n is a Bernoulli number, we have

$$\begin{aligned}&\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}\frac{1}{r+1}(2^{2r+2}-1)B_{2r+2}V_{n(m-2r-1)}\\ &=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}V_n^{m-2r-1},\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}(2^{2r+2}-1)\left(1-\frac{1}{3^{2r+1}}\right)\frac{B_{2r+2}}{2r+2}V_{n(m-2r-1)}\\ &=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]\Delta^r U_n^{2r+1}V_n^{m-2r-1}\frac{1}{3^{2r+1}}.\end{aligned}\tag{25}$$

Assuming $p = 1$ and $q = -1$, we have the so-called Fibonacci and Lucas sequences

$$U_n = F_n \quad\text{and}\quad V_n = L_n,$$

respectively. And from (22), (24), and (25), it follows that

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}\left[\begin{matrix} m \\ 2r\end{matrix}\right]5^r F_n^{2r}\frac{E_{2r}}{2^{2r}}L_{n(m-2r)} = \frac{1}{2^{m-1}}L_n^m,\tag{26}$$

$$\begin{aligned}&\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]5^r F_n^{2r+1}\frac{1}{r+1}(2^{2r+2}-1)B_{2r+2}L_{n(m-2r-1)}\\ &=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\left[\begin{matrix} m \\ 2r+1\end{matrix}\right]5^r F_n^{2r+1}L_n^{m-2r-1},\end{aligned}\tag{27}$$

and

$$\begin{aligned}&\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix}m\\2r+1\end{bmatrix}5^r F_n^{2r+1}(2^{2r+2}-1)\left(1-\frac{1}{3^{2r+1}}\right)\frac{B_{2r+2}}{2r+2}L_{n(m-2r-1)}\\&=\frac{1}{2^{m-1}}\sum_{r=0}^{\left[\frac{m-1}{2}\right]}\begin{bmatrix}m\\2r+1\end{bmatrix}5^r F_n^{2r+1}L_n^{(m-2r-1)}\frac{1}{3^{2r+1}},\end{aligned}\tag{28}$$

where (26) is a generalization of P. F. Byrd's result (see [5]):

$$\sum_{r=0}^{\left[\frac{m}{2}\right]}5^r\begin{bmatrix}m\\2r\end{bmatrix}B_{2r}F_n^{2r}F_{n(m-2r)}=\frac{m}{2}F_nL_{n(m-1)}.$$

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