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RECURRENCE SEQUENCES AND NORLUND-BERNOULLI POLYNOMIALS

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Abstract. The purpose of this paper is to establish some identities containing Norlund-Bernoulli polynomials, which as one application, yield some results of Toscano [8], Kelisky [5] and Zhang & Guo [10] as special cases, as well as other identities involving Bernoulli-Euler and Fibonacci-Lucas or Pell and Pell-Lucas numbers.

1. Definition and Notation

Definition 1. The general two order linear recurrence sequences are defined by

$$S_n(p,q) = pS_{n-1}(p,q) - qS_{n-2}(p,q)$$

with S_0, S_1; p, q arbitrary, provided that $\triangle = p^2 - 4q > 0$.

In particular, if $S_0 = 0$, $S_1 = 1$ or $S_0 = 2$, $S_1 = p$, we have generalized Fibonacci and Lucas sequences, respectively, in symbols $U_n(p,q)$, $V_n(p,q)$.

Let α, β ($\alpha > \beta$) are the roots of equation $x^2 - px + q = 0$, then we have (see [3])

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad V_n(p,q) = \alpha^n + \beta^n \tag{1}$$

$$S_n(p,q) = \left(S_1 - \frac{1}{2}S_0\right) U_n(p,q) + \frac{1}{2}S_0 V_n(p,q) \tag{2}$$

We assume that

$$\begin{aligned} S_0 &= \omega \\ S_1 &= \frac{1}{2}p\omega + \left(x - \frac{1}{2}\omega\right)\triangle^{\frac{1}{2}} \end{aligned}$$

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and, according to (1) and (2), we deduce

$$S_n(x;p,q) = \left(x - \frac{1}{2}\omega\right)\Delta^{\frac{1}{2}}U_n(p,q) - \frac{1}{2}\omega V_n(p,q) \tag{3}$$

$$S_n(x;p,q) = x\alpha^n + (\omega - x)\beta^n \tag{4}$$

From this point on, we use U_n, V_n and $S_n(x)$ to denote $U_n(p,q)$, $V_n(p,q)$ and $S_n(x;p,q)$, respectively.

Definition 2. The Norlund-Bernoulli polynomials $B_n^{(k)}(x|\omega_1, \omega_2, \ldots, \omega_k)$ are defined as (see [1], [7])

$$\sum_{r=0}^{\infty} \frac{t^r}{r!} B_n^{(k)}(x|\omega_1, \omega_2, \ldots, \omega_k) =$$

$$\frac{\omega_1, \omega_2, \ldots, \omega_k t^k}{(\exp(\omega_1 t) - 1)(\exp(\omega_2 t) - 1)\ldots(\exp(\omega_k t) - 1)} \exp(tx) \tag{5}$$

In particular, $B_n^{(k)}(x|1,1,\ldots,1) = B_n^{(k)}(x)$ (the Bernoulli polynomials of higher order); $B_n^{(1)}(x) = B_n(x)$ (the ordinary Bernoulli polunomials); $B_n(0) = B_n$ (the Bernoulli numbers) (see [2]).

From this definition, it is easy to deduce the following properties (see [1]):

$$B_{2n+1} = 0 \qquad (n > 0) \tag{6}$$

$$B_n^{(k)}(\omega_1 + \omega_2 + \cdots + \omega_k - x|\omega_1, \omega_2, \cdots, \omega_k) =$$

$$= (-1)^n B_n^{(k)}(x|\omega_1, \omega_2, \cdots, \omega_k) \tag{7}$$

(7) is called the complementary argument theorem of Norlund-Bernoulli polynomials.

2. Some Lemmas

First we introduce the following results from [9].

$$S_n^m(x) + S_n^m(\omega - x) = \frac{1}{2^{m-1}} \sum_{r=0}^{[m/2]} \binom{m}{2r} \Delta^r U_n^{2r} \omega^{m-2r} V_n^{m-2r} (2x - \omega)^{2r} \tag{8}$$

$$(9)\quad S_n^{2m}(x)+S_n^{2m}(\omega-x)=2\binom{2m}{n}q^m x^m(\omega-x)^m+$$

$$+\sum_{r=0}^{m-1}\binom{2m}{r}q^{nr}V_{2(m-r)}\left[x^r(\omega-x)^{2m-r}+x^{2m-r}(\omega-x)^r\right]$$

$$(10)\quad S_n^{2m+1}(x)-S_n^{2m+1}(\omega-x)=\sum_{r=0}^{m}\binom{2m+1}{r}q^{nr}V_{n(2m-2r+1)}\cdot$$

$$\cdot\quad\left[x^r(\omega-x)^{2m-r+1}+x^{2m-r+1}(\omega-x)^r\right]$$

$$(11)\quad S_n^m(x)-S_n^m(\omega-x)=\frac{1}{2^{m-1}}\sum_{n=0}^{[m/2]}\binom{m}{2r+1}\Delta^{r+\frac{1}{2}}U_n^{2r+1}\cdot$$

$$\cdot\quad\omega^{m-2r-1}V_n^{m-2r-1}(2x-\omega)^{2r-1}$$

$$(12)\quad S_n^m(x)-S_n^m(\omega-x)=\Delta^{\frac{1}{2}}\sum_{r=0}^{[\frac{m-1}{2}]}\binom{m}{r}q^{nr}U_{n(m-2r)}\cdot$$

$$\left[x^{mr}(\omega-x)^r-x^r(\omega-x)^{m-r}\right]$$

and the generating functions and the generating functions

$$(13)\qquad \sum_{r=0}^{\infty}\frac{t^r}{r!}U_{nr}=\frac{1}{\Delta^{\frac{1}{2}}}\left[\exp(t\alpha^n)-\exp(t\beta^n)\right]$$

$$(14)\qquad \sum_{r=0}^{\infty}\frac{t^r}{r!}V_{nr}=\exp(t\alpha^n)+\exp(t\beta^n)$$

3. The Main Results

Theorem 1.

$$\sum_{r=0}^{[\frac{m-1}{2}]}\binom{m}{2r}\Delta^r U_n^{2r}B_{2r}^{(k)}(x|\omega_1,\omega_2,\ldots,\omega_k)$$

$$(15)\qquad (m-2r)!\sum_{r_1+r_2+\cdots+r_k=m-2r}\frac{\omega_1^{r_1}U_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}U_{nr_k}}{r_k!}$$

$$=\frac{(m)_k}{2^{m-k}}\omega_1\omega_2\ldots\omega_k U_n^k\sum_{r=0}^{[\frac{m-k}{2}]}\binom{m-k}{2r}\Delta^r U_n^{2r}(\omega_1+\omega_2+\ldots+\omega_k)^{m-k-2r}$$

$$V_n^{m-k-2r}\left(2x-(\omega_1+\omega_2+\ldots+\omega_k)\right)^{2r}=(m)_kU_n^k\frac{1+(-1)^m}{2}\omega_1\omega_2\ldots\omega_kq^{\frac{n(m-k)}{2}}$$

$$\binom{m-k}{\frac{m-k}{2}}x^{\frac{m-k}{2}}(\omega_1+\omega_2+\ldots+\omega_k-x)^{\frac{m-k}{2}}$$

(16)
$$=\frac{(m)_k}{2^{m-k}}\omega_1,\omega_2,\ldots,\omega_kU_n^k\sum_{r=0}^{\left[\frac{m-k-1}{2}\right]}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}$$

$$\left[x^r(\omega_1+\cdots+\omega_k-x)^{m-k-r}+x^{m-k-r}(\omega_1+\cdots+\omega_k-x)^r\right]$$

Theorem 2.

$$\sum_{r=0}^{\left[\frac{m-2}{2}\right]}\binom{m}{2r+1}\Delta^rU_n^{2r+1}B_r^{(k)}(x|\omega_1,\omega_2,\ldots,\omega_k)$$

(15)
$$(m-2r-1)!\sum_{r_1+r_2+\cdots+r_k=m-2r-1}\frac{\omega_1^{r_1}U_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}U_{nr_k}}{r_k!}$$

$$=\frac{(m)_k}{2^{m-k}}\omega_1,\omega_2,\ldots,\omega_kU_n^k\sum_{r=0}^{\left[\frac{m-k-1}{2}\right]}\binom{m-k}{2r+1}\Delta^rU_n^{2r+1}(\omega_1+\omega_2+\cdots+\omega_k)^{m-k-2r-1}$$

$$V_n^{m-k-2r-1}\left(2x-(\omega_1+\omega_2+\cdots+\omega_k)\right)^{2r+1}$$

(16)
$$=\frac{(m)_k}{2}\omega_1,\omega_2,\ldots,\omega_kU_n^k\sum_{r=0}^{\left[\frac{m-k-1}{2}\right]}\binom{m-k}{r}q^{nr}V_{n(m-k-2r)}$$

$$\left[x^r(\omega_1+\cdots+\omega_k-x)^{m-k-r}-x^{m-k-r}(\omega_1+\cdots+\omega_k-x)^r\right]$$

4. The Proofs of Theorems

From (5), replacing t by $\Delta^{\frac{1}{2}}U_nt$, we have

$$\sum_{r=0}^{\infty}B_r^{(k)}(x|\omega_1,\omega_2,\ldots,\omega_k)\frac{\left(\Delta^{\frac{1}{2}}U_nt\right)}{r!}$$

$$=\frac{\omega_1\omega_2\ldots\omega_k\left(\Delta^{1/2}U_nt\right)^k\cdot\exp\left(x\Delta^{\frac{1}{2}}U_nt\right)}{\left(\exp\left(\omega_1\Delta^{\frac{1}{2}}U_nt\right)-1\right)\left(\exp\left(\omega_2\Delta^{\frac{1}{2}}U_nt\right)-1\right)\cdots\left(\exp\left(\omega_k\Delta^{\frac{1}{2}}U_nt\right)-1\right)}.$$

$$= \frac{\omega_1\omega_2\ldots\omega_k\left(\Delta^{1/2}U_n t\right)^k\cdot\exp\left(t\left(x\alpha^n+(\omega_1+\cdots+\omega_k)\beta^n\right)\right)}{\left(\exp\left(\omega_1 t\alpha^n\right)-\exp\left(\omega_1 t\beta^n\right)\right)\cdots\left(\exp\left(\omega_k t\alpha^n\right)-\exp\left(\omega_k t\beta^n\right)\right)}$$

therefore

$$\sum_{r_1=0}^{\infty}\frac{\omega_1^{r_1}t^{r_1}}{r_1!}U_{nr_1}\cdots\sum_{r_k=0}^{\infty}\frac{\omega_k^{r_k}t^{r_k}}{r_k!}U_{nr_k}\sum_{r=0}^{\infty}B_r^{(k)}(x|\omega_1,\ldots,\omega_k)\frac{\left(\Delta^{\frac{1}{2}}U_n t\right)^r}{r!}=$$
$$=\omega_1\ldots\omega_k\left(U_n t\right)^k\exp(tS_n(x))\,.$$

Hence

$$\left[\sum_{r=0}^{\infty}\left(\sum_{r_1+r_2+\cdots+r_k=r}\frac{\omega_1^{r_1}t^{r_1}}{r_1!}\cdots\sum_{r_k=0}^{\infty}\frac{\omega_k^{r_k}t^{r_k}}{r_k!}\right)t^r\right]\cdot$$
$$\cdot\left(\sum_{r=0}^{\infty}B_r^{(k)}(x|\omega_1,\ldots,\omega_k)\frac{\left(\Delta^{\frac{1}{2}}U_n t\right)^r}{r!}\right)=\omega_1\ldots\omega_k\left(U_n t\right)^k\exp(tS_n(x))\,.$$

We expand the product figuring in the first term into a power series of t, and compare with the expansion of the second therm, we obtain

$$\sum_{r=0}^{m-1}\binom{m}{r}\Delta^{\frac{r}{2}}U_n^rB_r^{(k)}(x|\omega_1,\omega_2,\ldots,\omega_k)(m-r)!$$
$$\sum_{r_1+\cdots+r_k=m-r}\frac{\omega_1^{r_1}U_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}U_{nr_k}}{r_k!}=(m)_kU_n^k\omega_1\ldots\omega_kS_n^{m-k}(x)\,. \tag{17}$$

If we replace x by $\omega_1+\cdots+\omega_k-x$ in (17), and using (7), we find

$$\sum_{r=0}^{m-1}\binom{m}{r}\Delta^{\frac{r}{2}}U_n^r(-1)^rB_r^{(k)}(x|\omega_1,\omega_2,\ldots,\omega_k)(m-r)!$$
$$\sum_{r_1+\cdots+r_k=m-r}\frac{\omega_1^{r_1}U_{nr_1}}{r_1!}\cdots\frac{\omega_k^{r_k}U_{nr_k}}{r_k!}= \tag{18}$$
$$=(m)_kU_n^k\omega_1\ldots\omega_kS_n^{m-k}(\omega_1+\cdots+\omega_k-x)$$

(17)–(18), using (8), (9) or (17)–(18), using (11), (12), we get the proofs of Theorem 1 and 1, respectively.

5. Some Consequences

If we take $x = (\omega_1 + \cdots + \omega_k)/2$ in (15), then

$$\sum_{r=0}^{[\frac{m-1}{2}]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)} \left(\frac{\omega_1 \ldots \omega_k}{2}\middle|\omega_1, \omega_2, \ldots, \omega_k\right)(m-2r)!$$

(19)
$$\sum_{r_1+\cdots+r_k=m-2r} \frac{\omega_1^{r_1} U_{nr_1}}{r_1!} \cdots \frac{\omega_k^{r_k} U_{nr_k}}{r_k!} =$$

$$= \frac{(m)_k}{2^{m-k}} \omega_1\omega_2 \ldots \omega_k U_n^k (\omega_1 + \omega_2 + \cdots + \omega_k)^{m-k} V_n^{m-k}$$

Again taking $\omega_1 = \cdots = \omega_k = 1$ in (19), we get the following results from [10]

$$\sum_{r=0}^{[\frac{m-1}{2}]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)} \left(\frac{k}{2}\right)(m-2r)! \sum_{r_1+\cdots+r_k=m-2r} \frac{U_{nr_1}}{r_1!} \cdots \frac{U_{nr_k}}{r_k!} =$$

(20)
$$= \frac{(m)_k}{2^{m-k}} U_n^k k^{m-k} V_n^{m-k}$$

If we take in (20) and recalling that $B_{2n}\left(\frac{1}{2}\right) = \left(2^{1-2n} - 1\right) B_{2n}$ (see [8]), we have

(21)
$$\sum_{r=0}^{[\frac{m-1}{2}]} \binom{m}{2r} \Delta^r U_n^{2r} \left(2^{1-2r} - 1\right) B_{2r} U_{n(m-2r)} = \frac{1}{2^{m-1}} m U_n V_n^{m-1},$$

where (21) is the result of paper [8].

If we take $x = 0$ in (16), then

$$\sum_{r=0}^{[\frac{m-1}{2}]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)} (0|\omega_1, \omega_2 \ldots, \omega_k)(m-2r)!$$

(21)
$$\sum_{r_1+\cdots+r_k=m-2r} \frac{\omega_1^{r_1} U_{nr_1}}{r_1!} \cdots \frac{\omega_k^{r_k} U_{nr_k}}{r_k!} = \frac{(m)_k}{2^{m-k}} \omega_1\omega_2 \ldots \omega_k U_n^k V_{n(m-k)}$$

Taking $\omega_1 = \cdots = \omega_k = 1$ in (21), then we obtain (21) from [11]:

$$\sum_{r=0}^{[\frac{m-1}{2}]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r}^{(k)} (m-2r)!$$

$$\sum_{r_1+\cdots+r_k=m-2r} \frac{U_{nr_1}}{r_1!}\cdots\frac{U_{nr_k}}{r_k!} = \frac{(m)_k}{2}U_n^k V_{n(m-k)} \tag{22}$$

From (22), using that $B_k^{(n+1)} = \left(1-\frac{k}{n}\right)B_k^{(n)} - kB_{k-1}^{(n)}$ (see [6]) and taking $k=1,2$ we get

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} \Delta^r U_n^{2r} B_{2r} U_{n(m-2r)} = \frac{m}{2}U_n V_{n(m-1)} \tag{23}$$

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} \Delta^r U_n^{2r} \left((1-2r)B_{2r} - 2rB_{2r-1}\right)\cdot$$

$$\cdot\sum_{t=0}^{m-2r} \binom{m-2r}{t} U_m U_{n(m-2r-1)} = \frac{m(m-1)}{2}U_n^2 V_{n(m-2)} \tag{24}$$

If we take $p=1$, $q=-1$, then

$U_n(1,-1) = F_n$ Fibonacci numbers

$V_n(1,-1) = L_n$ Lucas numbers

and from (21) (23), it follows

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} 5^r F_n^{2r}\left(2^{1-2r}-1\right) B_{2r} F_{n(m-2r)} = \frac{m}{2^{m-1}} F_n L_n^{m-1} \tag{25}$$

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} 5^r F_n^{2r} B_{2r} F_{n(m-2r)} = \frac{m}{2} F_n L_{n(m-1)} \tag{26}$$

where (26) is Kelisky's result given in [5].

If we take $p=1$, $q=-1$, then

$U_n(2,-1) = P_n$ Pell numbers

$V_n(2,-1) = Q_r$ Pell-Lucas numbers (see [4])

and from (21), (23), it follows

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} 8^r P_n^{2r}\left(2^{1-2r}-1\right) B_{2r} P_{n(m-2r)} = \frac{m}{2^{m-1}} P_n Q_n^{m-1} \tag{27}$$

$$\sum_{r=0}^{\left[\frac{m-1}{2}\right]} \binom{m}{2r} 8^r P_n^{2r} B_{2r} P_{n(m-2r)} = \frac{m}{2} P_n Q_{n(m-1)}. \tag{28}$$

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