Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Barnes-type Narumi polynomials

Dae San Kim[1] and Taekyun Kim[2*]

*Correspondence: tkkim@kw.ac.kr
[2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea
Full list of author information is available at the end of the article

Abstract
In this paper, we study the Barnes-type Narumi polynomials with umbral calculus viewpoint. From our study, we derive various identities of the Barnes-type Narumi polynomials.
MSC: 05A19; 05A40; 11B68

Keywords: Barnes-type Narumi polynomial; umbral calculus

1 Introduction

As is well known, the Narumi polynomials of order α are defined by the generating function to be

$$\left(\frac{t}{\log(1+t)}\right)^{\alpha}(1+t)^x=\sum_{n=0}^{\infty}N_n^{(\alpha)}(x)\frac{t^n}{n!}\quad\text{(see [1])}.\tag{1}$$

Let $r\in\mathbb{Z}_{>0}$. We consider the polynomials $N_n(x|a_1,\ldots,a_r)$ and $\hat{N}_n(x|a_1,\ldots,a_r)$, respectively, called the Barnes-type Narumi polynomials of the first kind and those of the second kind and respectively given by

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^x=\sum_{n=0}^{\infty}N_n(x|a_1,\ldots,a_r)\frac{t^n}{n!}\tag{2}$$

and

$$\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}}\right)(1+t)^x=\sum_{n=0}^{\infty}\hat{N}_n(x|a_1,\ldots,a_r)\frac{t^n}{n!},\tag{3}$$

where $a_1,a_2,\ldots,a_r\neq 0$.

When $x=0$,

$$N_n(a_1,\ldots,a_r)=N_n(0|a_1,\ldots,a_r)$$

and

$$\hat{N}_n(a_1,\ldots,a_r)=\hat{N}_n(0|a_1,\ldots,a_r)$$

are respectively called the Barnes-type Narumi numbers of the first kind and those of the second kind.

Note that

$$N_n(x|\underbrace{1,\ldots,1}_{r}) = N_n^{(r)}(x),$$

$$\hat{N}_n(x|\underbrace{1,\ldots,1}_{r}) = \hat{N}_n^{(r)}(x)$$

and

$$\hat{N}_n(x|\underbrace{1,\ldots,1}_{r}) = N_n^{(r)}(x-r).$$

In the previous paper [2], $N_n^{(\alpha)}(x)$ was denoted by $N_n^{(-\alpha)}$ and called the Narumi polynomial of order α.

The Bernoulli polynomials are defined by the generating function to be

$$\frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad \text{(see [3–6])}. \tag{4}$$

When $x = 0$, $B_n = B_n(0)$ are called the Bernoulli numbers. In [7], it is known that the Cauchy numbers are given by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n\frac{t^n}{n!}. \tag{5}$$

It is well known that the Stirling number of the first kind is given by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{\infty} S_1(n,l)x^l \quad (n \geq 0) \text{ (see [1, 2, 7–11])}. \tag{6}$$

From (6), we have

$$\bigl(\log(1+t)\bigr)^n = n!\sum_{l=n}^{\infty} S_1(l,n)\frac{t^l}{l!} \quad (n \geq 0). \tag{7}$$

Let $\mathbb{C}$ be the complex number field and let $\mathcal{F}$ be the set of all formal power series in the variable t:

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} a_k\frac{t^k}{k!} \,\middle|\, a_k \in \mathbb{C} \right\}.$$

Let $\mathbb{P} = \mathbb{C}[x]$ and let $\mathbb{P}^*$ be the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on $p(x)$ which satisfies $\langle L+M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle$, and $\langle cL|p(x)\rangle = c\langle L|p(x)\rangle$, where c is a complex constant. The linear functional $\langle f(t)|\cdot\rangle$ on $\mathbb{P}$ is defined by $\langle f(t)|x^n\rangle = a_n$ $(n \geq 0)$, where $f(t) \in \mathcal{F}$. Thus, we note that

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n,k \geq 0), \tag{8}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [12–18]).

For $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$, we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ denotes both the algebra of formal power series in t and the vector space of all linear functionals on $\mathbb{P}$, and so an element $f(t)$ of $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the umbral algebra. The order $o(f(t))$ of a power series $f(t) \neq 0$ is the smallest integer for which the coefficient of t^k does not vanish. If $o(f(t)) = 1$, then $f(t)$ is called a delta series; if $o(f(t)) = 0$, then $g(t)$ is called an invertible series. Let $f(t), g(t) \in \mathcal{F}$ with $o(f(t)) = 1$ and $o(g(t)) = 0$. Then there exists a unique sequence $s_n(x)$ ($\deg s_n(x) = n$) such that $\langle g(t)f(t)^k|s_n(x)\rangle = n!\delta_{n,k}$ for $n, k \geq 0$. The sequence $s_n(x)$ is called the Sheffer sequence for $(g(t), f(t))$ which is denoted by $s_n(x) \sim (g(t), f(t))$.

For $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle \tag{9}$$

and

$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \langle f(t)|x^k\rangle \frac{t^k}{k!}, \\ p(x) &= \sum_{k=0}^{\infty} \langle t^k|p(x)\rangle \frac{x^k}{k!}. \end{aligned} \tag{10}$$

From (10), we can derive the following equation (11):

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k}, \qquad e^{yt} p(x) = p(x+y) \quad \text{(see [1])}. \tag{11}$$

Let $s_n(x) \sim (g(t), f(t))$. Then the following will be used:

$$\frac{ds_n(x)}{dx} = \sum_{l=0}^{n-1} \binom{n}{l} \langle \bar{f}(t)|x^{n-l}\rangle s_l(x), \tag{12}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$ with $\bar{f}(f(t)) = f(\bar{f}(t)) = t$,

$$\frac{1}{g(\bar{f}(t))} e^{x\bar{f}(t)} = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!} \quad \text{for all } x \in \mathbb{C}, \tag{13}$$

$$f(t)s_n(x) = ns_{n-1}(x) \quad (n \geq 1), \qquad s_n(x) = \sum_{j=0}^{n} \frac{\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j|x^n\rangle}{j!} x^j, \tag{14}$$

$$s_n(x+y) = \sum_{j=0}^{n} \binom{n}{j} s_j(x) p_{n-j}(y), \quad \text{where } p_n(x) = g(t)s_n(x), \tag{15}$$

$$\langle f(t)|xp(x)\rangle = \langle \partial_t f(t)|p(x)\rangle, \quad \text{where } \partial_t f(t) = \frac{df(t)}{dt} \tag{16}$$

and

$$s_{n+1}(x) = \left(x - \frac{g'(t)}{g(t)}\right) \frac{1}{f'(t)} s_n(x) \quad (n \geq 0) \text{ (see [1, 19])}. \tag{17}$$

Let us assume that $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), l(t))$. Then we have

$$s_n(x) = \sum_{m=0}^{n} c_{n,m} r_m(x) \quad (n \geq 0), \tag{18}$$

where

$$c_{n,m} = \frac{1}{m!} \left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))} l(\bar{f}(t))^m \middle| x^n \right\rangle \quad \text{(see [1, 5])}. \tag{19}$$

From (2), (3) and (13), we note that

$$N_n(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{t}{e^{a_j t} - 1} \right), e^t - 1 \right) \tag{20}$$

and

$$\hat{N}_n(x|a_1, \ldots, a_r) \sim \left(\prod_{j=1}^{r} \left(\frac{t e^{a_j t}}{e^{a_j t} - 1} \right), e^t - 1 \right). \tag{21}$$

In this paper, we study the Barnes-type Narumi polynomials with umbral calculus viewpoint. From our study, we derive various identities of the Barnes-type Narumi polynomials.

2 Barnes-type Narumi polynomials

From (21), we note that

$$\prod_{j=1}^{r} \left(\frac{t}{e^{a_j t} - 1} \right) N_n(x|a_1, \ldots, a_r) \sim \left(1, e^t - 1\right) \tag{22}$$

and

$$(x)_n \sim \left(1, e^t - 1\right). \tag{23}$$

Thus, by (22) and (23), we get

$$\begin{aligned} N_n(x|a_1, \ldots, a_r) &= \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) (x)_n \\ &= \sum_{m=0}^{n} S_1(n, m) \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) x^m. \end{aligned} \tag{24}$$

Note that

$$\begin{aligned} \prod_{j=1}^{r} \left(\frac{e^{a_j t} - 1}{t} \right) &= \left(\sum_{l_1=0}^{\infty} \frac{a_1^{l_1+1}}{(l_1+1)!} t^{l_1} \right) \times \cdots \times \left(\sum_{l_r=0}^{\infty} \frac{a_r^{l_r+1}}{(l_r+1)!} t^{l_r} \right) \\ &= \sum_{l_1, \ldots, l_r=0}^{\infty} \sum_{l_1+\cdots+l_r=i} \frac{a_1^{l_1+1} \cdots a_r^{l_r+1}}{(l_1+1)! \cdots (l_r+1)!} t^i. \end{aligned} \tag{25}$$

From (24) and (25), we have

$$\begin{aligned} N_n(x|a_1,\ldots,a_r) &= \sum_{m=0}^{\infty} S_1(n,m) \sum_{i=0}^{m} \sum_{l_1+\cdots+l_r=i} \frac{a_1^{l_1+1}\cdots a_r^{l_r+1}}{(l_1+1)!\cdots(l_r+1)!} t^i x^m \\ &= \sum_{m=0}^{n} S_1(n,m) \sum_{i=0}^{m} \frac{i!}{(i+r)!} \sum_{l_1+\cdots+l_r=i} \binom{i+r}{l_1+1,\ldots,l_r+1}\binom{m}{i} \\ &\quad\times a_1^{l_1+1}\cdots a_r^{l_r+1} x^{m-i} \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1}\binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} x^i. \end{aligned} \tag{26}$$

By (21), we see that

$$\prod_{j=1}^{r} \left(\frac{te^{a_jt}}{e^{a_jt}-1}\right) \hat{N}_n(x|a_1,\ldots,a_r) \sim \left(1, e^t-1\right), \tag{27}$$

and we recall (23).

Thus, we have

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)(x)_n = e^{-\sum_{j=1}^{r} a_jt} \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{t}\right)(x)_n \\ &= e^{-\sum_{j=1}^{r} a_jt} N_n(x|a_1,\ldots,a_r) \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1}\binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} e^{-\sum_{j=1}^{r} a_jt} x^i \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \right. \\ &\quad \left. \times \binom{m-i+r}{l_1+1,\ldots,l_r+1}\binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} \right\} \left(x - \sum_{j=1}^{r} a_j\right)^i \end{aligned} \tag{28}$$

or

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \prod_{j=1}^{r}\left(\frac{e^{a_jt}-1}{te^{a_jt}}\right)(x)_n = \prod_{j=1}^{r}\left(\frac{e^{-a_jt}-1}{-t}\right)(x)_n \\ &= \sum_{m=0}^{n} S_1(n,m) \sum_{i=0}^{m} (-1)^i \frac{i!}{(i+r)!} \\ &\quad\times \sum_{l_1+\cdots+l_r=i} \binom{i+r}{i_1+1,\ldots,i_r+1}\binom{m}{i} a_1^{l_1+1}\cdots a_r^{l_r+1} x^{m-i} \end{aligned}$$

$$= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} (-1)^{m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i. \tag{29}$$

Therefore, by (26), (28) and (29), we obtain the following theorem.

Theorem 1 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i$$

and

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} S_1(n,m) \frac{(m-i)!}{(m-i+r)!} \times \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{i} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} \left(x - \sum_{j=1}^{r} a_j \right)^i \\ &= \sum_{i=0}^{n} \left\{ \sum_{m=i}^{n} \sum_{l_1+\cdots+l_r=m-i} (-1)^{m-i} S_1(n,m) \times \frac{(m-i)!}{(m-i+r)!} \binom{m-i+r}{l_1+1,\ldots,l_r+1} \binom{m}{j} a_1^{l_1+1} \cdots a_r^{l_r+1} \right\} x^i. \end{aligned}$$

From (14), we can derive the following equation (30):

$$N_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n} \frac{1}{j!} \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \left(\log(1+t)\right)^j \middle| x^n \right\rangle x^j, \tag{30}$$

where

$$\begin{aligned} &\left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \left(\log(1+t)\right)^j \middle| x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| \left(\log(1+t)\right)^j x^n \right\rangle \\ &= \left\langle \prod_{j=1}^{r} \left(\frac{(1+t)^{a_j}-1}{\log(1+t)} \right) \middle| j! \sum_{l=j}^{\infty} S_1(l,j) \frac{t^l}{l!} x^n \right\rangle \\ &= j! \sum_{l=j}^{n} \binom{n}{l} S_1(l,j) N_{n-l}(a_1,\ldots,a_r). \end{aligned} \tag{31}$$

Thus, by (30) and (31), we obtain the following theorem.

Theorem 2 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)N_{n-l}(a_1,\ldots,a_r)\right\}x^j.$$

By the same methods as in (28), (29) and (30), we get

$$\begin{aligned}
&\hat{N}_n(x|a_1,\ldots,a_r) \\
&\quad= \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)N_{n-l}(a_1,\ldots,a_r)\right\}\left(x-\sum_{i=1}^{r}a_i\right)^j \\
&\quad= \sum_{j=0}^{n}\left\{\sum_{l=j}^{n}\binom{n}{l}S_1(l,j)\hat{N}_{n-l}(a_1,\ldots,a_r)\right\}x^j.
\end{aligned} \tag{32}$$

By (8), we get

$$\begin{aligned}
N_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} N_i(y|a_1,\ldots,a_r)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^y \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\middle| \sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}x^m\right\rangle \\
&= \sum_{m=0}^{n}(y)_m\binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\middle| x^{n-m}\right\rangle \\
&= \sum_{m=0}^{n}(y)_m\binom{n}{m}N_{n-m}(a_1,\ldots,a_r)
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\hat{N}_n(y|a_1,\ldots,a_r) &= \left\langle \sum_{i=0}^{\infty} \hat{N}_i(y|a_1,\ldots,a_r)\frac{t^i}{i!}\middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}}\right)(1+t)^y \middle| x^n\right\rangle \\
&= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}}\right)\middle| \sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}x^n\right\rangle \\
&= \sum_{m=0}^{n}(y)_m\binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)(1+t)^{a_j}}\right)\middle| x^{n-m}\right\rangle \\
&= \sum_{m=0}^{n}\binom{n}{m}(y)_m\hat{N}_{n-m}(a_1,\ldots,a_r).
\end{aligned} \tag{34}$$

Therefore, by (33) and (34), we obtain the following theorem.

Theorem 3 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} \binom{n}{m} N_{n-m}(a_1,\ldots,a_r)(x)_m$$

and

$$\hat{N}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} \binom{n}{m} \hat{N}_{n-m}(a_1,\ldots,a_r)(x)_m.$$

From (15), we note that

$$N_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n} \binom{n}{j} N_j(x|a_1,\ldots,a_r)(y)_{n-j} \tag{35}$$

and

$$\hat{N}_n(x+y|a_1,\ldots,a_r) = \sum_{j=0}^{n} \binom{n}{j} \hat{N}_j(x|a_1,\ldots,a_r)(y)_{n-j}. \tag{36}$$

By (14), we get

$$\left(e^t - 1\right) N_n(x|a_1,\ldots,a_r) = nN_{n-1}(x|a_1,\ldots,a_r) \tag{37}$$

and

$$\begin{aligned}\left(e^t - 1\right) N_n(x|a_1,\ldots,a_r) &= e^t N_n(x|a_1,\ldots,a_r) - N_n(x|a_1,\ldots,a_r) \\ &= N_n(x+1|a_1,\ldots,a_r) - N(x|a_1,\ldots,a_r).\end{aligned} \tag{38}$$

From (37) and (38), we have

$$N_n(x+1|a_1,\ldots,a_r) - N_n(x|a_1,\ldots,a_r) = nN_{n-1}(x|a_1,\ldots,a_r). \tag{39}$$

By the same method as (39), we get

$$\hat{N}_n(x+1|a_1,\ldots,a_r) - \hat{N}_n(x|a_1,\ldots,a_r) = n\hat{N}_{n-1}(x|a_1,\ldots,a_r). \tag{40}$$

Recall that $N_n(x|a_1,\ldots,a_r) \sim (\prod_{j=1}^{r}(\frac{t}{e^{a_j t}-1}), e^t - 1)$.
From (17), we can derive the following equation (41):

$$N_{n+1}(x|a_1,\ldots,a_r) = xN_n(x-1|a_1,\ldots,a_r) - e^{-t}\frac{g'(t)}{g(t)} N_n(x|a_1,\ldots,a_r). \tag{41}$$

Now, we observe that

$$\begin{aligned}\frac{g'(t)}{g(t)} &= \left(\log g(t)\right)' = \left(r \log t - \sum_{j=1}^{r} \log\left(e^{a_j t} - 1\right)\right)' = \frac{r}{t} - \sum_{j=1}^{r} \frac{a_j e^{a_j t}}{e^{a_j t} - 1} \\ &= \frac{\sum_{j=1}^{r} \prod_{i \neq j}^{r}(e^{a_i t} - 1)\{e^{a_j t} - 1 - t a_j e^{a_j t}\}}{t \prod_{j=1}^{r}(e^{a_j t} - 1)},\end{aligned} \tag{42}$$

where

$$r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}=\frac{\sum_{j=1}^{r}\prod_{i\neq j}(e^{a_it}-1)\{e^{a_jt}-1-a_jte^{a_jt}\}}{\prod_{j=1}^{r}(e^{a_jt}-1)}$$
$$=\frac{-\frac{1}{2}(\sum_{j=1}^{r}a_1a_2\cdots a_{j-1}a_j^2a_{j+1}\cdots a_r)t^{r+1}+\cdots}{(a_1a_2\cdots a_r)t^r}$$
$$=-\frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots \tag{43}$$

has at least the order 1.

By (42) and (43), we get

$$\begin{aligned}
&\frac{g'(t)}{g(t)}N_n(x|a_1,\ldots,a_r)\\
&=\frac{r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}\left(\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}x^i\right)\\
&=\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\frac{r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}}{t}x^i\\
&=\sum_{i=0}^{n}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\left(r-\sum_{j=1}^{r}\frac{a_jte^{a_jt}}{e^{a_jt}-1}\right)\frac{x^{i+1}}{i+1}\\
&=\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\left(rx^{i+1}-\sum_{j=1}^{r}\sum_{m=0}^{\infty}B_m\frac{(-a_jt)^m}{m!}x^{i+1}\right)\\
&=-\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\\
&\quad\times\sum_{j=1}^{r}\sum_{m=1}^{i+1}(-1)^m\binom{i+1}{m}B_ma_j^mx^{i+1-m}\\
&=-\sum_{i=0}^{n}\frac{1}{i+1}\left\{\sum_{l=i}^{n}\binom{n}{l}S_1(l,i)N_{n-l}(a_1,\ldots,a_r)\right\}\\
&\quad\times\sum_{j=1}^{r}\sum_{m=0}^{i}(-1)^{i+1-m}\binom{i+1}{m}a_j^{i+1-m}B_{i+1-m}x^m.
\end{aligned}\tag{44}$$

Therefore, by (41) and (44), we obtain the following theorem.

Theorem 4 *For $n\geq 0$, we have*

$$\begin{aligned}
&N_{n+1}(x|a_1,\ldots,a_r)\\
&=xN_n(x-1|a_1,\ldots,a_r)+\sum_{m=0}^{n}\left\{\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\
&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}N_{n-l}(a_1,\ldots,a_r)\right\}(x-1)^m.
\end{aligned}$$

By the same method as the proof of Theorem 4, we get

$$\begin{aligned}\hat{N}_{n+1}(x|a_1,\ldots,a_r) &= \left(x-\sum_{j=1}^{r}a_j\right)\hat{N}_n(x-1|a_1,\ldots,a_r)\\&\quad+\sum_{m=0}^{n}\left\{\sum_{i=m}^{n}\sum_{l=i}^{n}\sum_{j=1}^{r}\frac{1}{i+1}\binom{n}{l}\binom{i+1}{m}S_1(l,i)\right.\\&\quad\left.\times B_{i+1-m}(-a_j)^{i+1-m}\hat{N}_{n-l}(a_1,\ldots,a_r)\right\}(x-1)^m.\end{aligned}\tag{45}$$

From (12) and (20), we can derive the following equation (46):

$$\langle\overline{f}(t)|x^{n-l}\rangle=\langle\log(1+t)|x^{n-l}\rangle=\left\langle\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}t^m\middle|x^{n-l}\right\rangle=(-1)^{n-l-1}(n-l-1)!.\tag{46}$$

Thus, by (46), we get

$$\begin{aligned}\frac{d}{dx}N_n(x|a_1,\ldots,a_r)&=\sum_{l=0}^{n-1}\binom{n}{l}(-1)^{n-l-1}(n-l-1)!N_l(x|a_1,\ldots,a_r)\\&=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}N_l(x|a_1,\ldots,a_r).\end{aligned}\tag{47}$$

By the same method as (47), we get

$$\frac{d}{dx}\hat{N}_n(x|a_1,\ldots,a_r)=n!\sum_{l=0}^{n-1}\frac{(-1)^{n-l-1}}{l!(n-l)}\hat{N}_l(x|a_1,\ldots,a_r).\tag{48}$$

From (8), we note that, for $n\geq 1$,

$$\begin{aligned}&N_n(y|a_1,\ldots,a_r)\\&\quad=\left\langle\sum_{i=0}^{\infty}N_i(y|a_1,\ldots,a_r)\frac{t^i}{i!}\middle|x^n\right\rangle\\&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^y\middle|x^n\right\rangle\\&\quad=\left\langle\partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^y\right)\middle|x^{n-1}\right\rangle\\&\quad=\left\langle\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\left(\partial_t(1+t)^y\right)\middle|x^{n-1}\right\rangle\\&\qquad+\left\langle\left(\partial_t\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y\middle|x^{n-1}\right\rangle\\&\quad=yN_{n-1}(y-1|a_1,\ldots,a_r)+\left\langle\left(\partial_t\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y\middle|x^{n-1}\right\rangle.\end{aligned}\tag{49}$$

Now, we observe that

$$
\begin{aligned}
&\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \\
&= \sum_{j=1}^{r}\prod_{i\neq j}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{a_j(1+t)^{a_j-1}\log(1+t)-((1+t)^{a_j}-1)\frac{1}{1+t}}{(\log(1+t))^2} \\
&= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\sum_{j=1}^{r}\left\{\frac{a_j(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{1}{\log(1+t)}\right\} \\
&= \frac{1}{1+t}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{\sum_{j=1}^{r}\{\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t},
\end{aligned}
\tag{50}
$$

where

$$
\sum_{j=1}^{r}\left\{\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\right\} = \frac{1}{2}\left(\sum_{j=1}^{r}a_j\right)t+\cdots
\tag{51}
$$

is a series with order greater than or equal to 1.

By (50) and (51), we get

$$
\begin{aligned}
&\left\langle\left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)(1+t)^y\,\middle|\, x^{n-1}\right\rangle \\
&= \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)\frac{(1+t)^y}{1+t}\,\middle|\,\frac{\sum_{j=1}^{r}\{\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t}x^{n-1}\right\rangle \\
&= \frac{1}{n}\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1}\,\middle|\,\sum_{j=1}^{r}\left\{\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\right\}x^n\right\rangle \\
&= \frac{1}{n}\left\{\sum_{j=1}^{r}a_j\left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y+a_j-1}\,\middle|\,\frac{t}{\log(1+t)}x^n\right\rangle\right. \\
&\quad \left. - r\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1}\,\middle|\,\frac{t}{\log(1+t)}x^n\right\rangle\right\} \\
&= \frac{1}{n}\sum_{j=1}^{r}a_j\sum_{l=0}^{n}\binom{n}{l}C_l\left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1}\prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y+a_j-1}\,\middle|\, x^{n-l}\right\rangle \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_l\left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{y-1}\,\middle|\, x^{n-l}\right\rangle \\
&= \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jC_lN_{n-l}(y+a_j-1|a_1,\ldots,\hat{a}_j,\ldots,a_r) \\
&\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_lN_{n-l}(y-1|a_1,\ldots,a_r),
\end{aligned}
\tag{52}
$$

where $\hat{a}_j$ means that a_j is omitted.

Therefore, by (49) and (52), we obtain the following theorem.

Theorem 5 *For $n \geq 1$, we have*

$$\begin{aligned} N_n(x|a_1,\ldots,a_r) &= xN_{n-1}(x-1|a_1,\ldots,a_r) \\ &\quad + \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jC_lN_{n-l}(x+a_j-1|a_1,\ldots,\hat{a}_j,\ldots,a_r) \\ &\quad - \frac{r}{n}\sum_{l=0}^{r}\binom{n}{l}C_lN_{n-l}(x-1|a_1,\ldots,a_r), \end{aligned}$$

where C_n are the Cauchy numbers with the generating function given by

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n\frac{t^n}{n!}.$$

By the same method as the proof of Theorem 5, we get

$$\begin{aligned} \hat{N}_n(x|a_1,\ldots,a_r) &= \left(x - \sum_{j=1}^{r}a_j\right)\hat{N}_{n-1}(x-1|a_1,\ldots,a_r) \\ &\quad - \frac{r}{n}\sum_{l=0}^{n}\binom{n}{l}C_l\hat{N}_{n-l}(x-1|a_1,\ldots,a_r) \\ &\quad - \frac{1}{n}\sum_{j=1}^{r}\sum_{l=0}^{n}\binom{n}{l}a_jC_l\hat{N}_{n-l}(x-1|a_1,\ldots,\hat{a}_j,\ldots,a_r). \end{aligned} \tag{53}$$

Now we compute the following formula (54) in two different ways:

$$\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle. \tag{54}$$

On the one hand,

$$\begin{aligned} &\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle \\ &\quad = \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| m!\sum_{l=m}^{\infty}S_1(l,m)\frac{t^l}{l!}x^n \right\rangle \\ &\quad = m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| x^{n-l} \right\rangle \\ &\quad = m!\sum_{l=m}^{n}\binom{n}{l}S_1(l,m)N_{n-l}(a_1,\ldots,a_r) \\ &\quad = m!\sum_{l=0}^{n-m}\binom{n}{l}S_1(n-l,m)N_l(a_1,\ldots,a_r). \end{aligned} \tag{55}$$

On the other hand,

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^n \right\rangle \\
&\quad= \left\langle \partial_t\left(\prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m\right) \middle| x^{n-1} \right\rangle \\
&\quad= \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\partial_t\bigl(\bigl(\log(1+t)\bigr)^m\bigr)\bigr) \middle| x^{n-1} \right\rangle \\
&\qquad+ \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-1} \right\rangle.
\end{aligned} \tag{56}$$

Note that

$$\begin{aligned}
&\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\partial_t\bigl(\bigl(\log(1+t)\bigr)^m\bigr)\bigr) \middle| x^{n-1} \right\rangle \\
&\quad= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\frac{1}{1+t} \middle| \log(1+t)^{m-1}x^{n-1} \right\rangle \\
&\quad= m\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| (m-1)!\sum_{l=m-1}^{\infty} S_1(l,m-1)\frac{t^l}{l!}x^{n-1} \right\rangle \\
&\quad= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| x^{n-1-l} \right\rangle \\
&\quad= m!\sum_{l=m-1}^{n-1}\binom{n-1}{l}S_1(l,m-1)N_{n-1-l}(-1|a_1,\ldots,a_r) \\
&\quad= m!\sum_{l=0}^{n-m}\binom{n-1}{l}S_1(n-1-l,m-1)N_l(-1|a_1,\ldots,a_r)
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
&\left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-1} \right\rangle \\
&\quad= \left\langle \left(\partial_t \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\right) \middle| m!\sum_{l=m}^{\infty} S_1(l,m)\frac{t^l}{l!}x^{n-1} \right\rangle \\
&\quad= m!\sum_{l=m}^{n-1}\binom{n-1}{l}S_1(l,m) \\
&\qquad\times \left\langle \prod_{i=1}^{r}\left(\frac{(1+t)^{a_i}-1}{\log(1+t)}\right)(1+t)^{-1} \middle| \frac{\sum_{j=1}^{r}\{\frac{a_jt(1+t)^{a_j}}{(1+t)^{a_j}-1}-\frac{t}{\log(1+t)}\}}{t}x^{n-1-l} \right\rangle \\
&\quad= m!\sum_{l=m}^{n-1}\binom{n-1}{l}\frac{S_1(l,m)}{n-l}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{\log(1+t)} \right) (1+t)^{-1} \middle| \sum_{j=1}^{r} \left\{ \frac{a_j t (1+t)^{a_j}}{(1+t)^{a_j}-1} - \frac{t}{\log(1+t)} \right\} x^{n-l} \right\rangle \\
&= \frac{m!}{n} \sum_{l=m}^{n-1} \binom{n}{l} S_1(l,m) \\
&\quad \times \left\{ \sum_{j=1}^{r} a_j \left\langle \frac{\log(1+t)}{(1+t)^{a_j}-1} \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{\log(1+t)} \right) (1+t)^{a_j-1} \middle| \frac{t}{\log(1+t)} x^{n-l} \right\rangle \right. \\
&\quad \left. - r \left\langle \prod_{i=1}^{r} \left(\frac{(1+t)^{a_i}-1}{\log(1+t)} \right) (1+t)^{-1} \middle| \frac{t}{\log(1+t)} x^{n-l} \right\rangle \right\} \\
&= \frac{m!}{n} \sum_{l=m}^{n-1} \binom{n}{l} S_1(l,m) \left\{ \sum_{j=1}^{r} a_j \sum_{k=0}^{n-l} C_k \binom{n-l}{k} N_{n-l-k}(a_j-1 | a_1, \ldots, \hat{a}_j, \ldots, a_r) \right. \\
&\quad \left. - r \sum_{k=0}^{n-l} C_k \binom{n-l}{k} N_{n-l-k}(-1 | a_1, \ldots, a_r) \right\}.
\end{aligned} \tag{58}
$$

Therefore, by (55), (56), (57) and (58), we obtain the following theorem.

Theorem 6 *For $n-1 \geq m \geq 1$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) N_l(a_1, \ldots, a_r) \\
&\quad = \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) N_l(-1 | a_1, \ldots, a_r) \\
&\qquad + \frac{1}{n} \sum_{l=m}^{n-1} \sum_{k=0}^{n-l} \sum_{j=1}^{r} \binom{n}{l} \binom{n-l}{k} a_j C_{n-l-k} S_1(l,m) N_k(a_j-1 | a_1, \ldots, \hat{a}_j, \ldots, a_r) \\
&\qquad - \frac{r}{n} \sum_{l=m}^{n-1} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} C_{n-l-k} S_1(l,m) N_k(-1 | a_1, \ldots, a_r).
\end{aligned}
$$

By the same method as the proof of Theorem 6, we get

$$
\begin{aligned}
&\sum_{l=0}^{n-m} \binom{n}{l} S_1(n-l,m) \hat{N}_l(a_1, \ldots, a_r) \\
&\quad = \sum_{l=0}^{n-m} \binom{n-1}{l} S_1(n-l-1, m-1) \hat{N}_l(-1 | a_1, \ldots, a_r) \\
&\qquad + \frac{1}{n} \sum_{j=1}^{r} \sum_{l=m}^{n-1} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} a_j C_{n-l-k} S_1(l,m) \hat{N}_k(-1 | a_1, \ldots, \hat{a}_j, \ldots, a_r) \\
&\qquad - \frac{r}{n} \sum_{l=m}^{n-1} \sum_{k=0}^{n-l} \binom{n}{l} \binom{n-l}{k} C_{n-l-k} S_1(l,m) \hat{N}_k(-1 | a_1, \ldots, a_r) \\
&\qquad - \sum_{j=1}^{r} a_j \sum_{l=0}^{n-m-1} \binom{n-1}{l} S_1(n-l-1, m) \hat{N}_k(-1 | a_1, \ldots, a_r),
\end{aligned} \tag{59}
$$

where $n-1 \geq m \geq 1$.

Let us consider the following two Sheffer sequences:

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right), e^t-1\right) \tag{60}$$

and (23).

We let

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m}(x)_m. \tag{61}$$

From (18) and (19), we note that

$$\begin{aligned} C_{n,m} &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)t^m \middle| x^n\right\rangle \\ &= \binom{n}{m}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right) \middle| x^{n-m}\right\rangle \\ &= \binom{n}{m} N_{n-m}(a_1,\ldots,a_r). \end{aligned} \tag{62}$$

Therefore, by (61) and (62), we obtain the following theorem.

Theorem 7 *For $n \geq 0$, we have*

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\binom{n}{m} N_{n-m}(a_1,\ldots,a_r)(x)_m.$$

By the same method as the proof of Theorem 7, we get

$$\hat{N}_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n}\binom{n}{m} \hat{N}_{n-m}(a_1,\ldots,a_r)(x)_m. \tag{63}$$

For

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right), e^t-1\right)$$

and

$$H_n^{(s)}(x|\lambda) \sim \left(\left(\frac{e^t-\lambda}{1-\lambda}\right)^s, t\right), \quad \lambda \in \mathbb{C} \text{ with } \lambda \neq 1,$$

let us assume that

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m} H_m^{(s)}(x|\lambda), \tag{64}$$

where $H_m^{(s)}(x|\lambda)$ are the Frobenius-Euler polynomials of order s defined by the generating function as

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^s e^{xt}=\sum_{n=0}^{\infty}H_n^{(s)}(x|\lambda)\frac{t^n}{n!}.$$

From (18) and (19), we note that

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m(1-\lambda+t)^s \middle| x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \sum_{j=0}^{\min\{s,n\}}\binom{s}{j}(1-\lambda)^{s-j}t^j x^n\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\lambda)^{s-j}(n)_j \\
&\quad \times \left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1_t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-j}\right\rangle \\
&= \frac{1}{m!(1-\lambda)^s}\sum_{j=0}^{n-m}\binom{s}{j}(1-\lambda)^{s-j}(n)_j m! \\
&\quad \times \sum_{l=0}^{n-j-m}\binom{n-j}{l}S_1(n-j-l,m)N_l(a_1,\ldots,a_r) \\
&= \sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l} \\
&\quad \times (n)_j(1-\lambda)^{-j}S_1(n-j-l,m)N_l(a_1,\ldots,a_r).
\end{aligned} \tag{65}$$

Therefore, by (64) and (65), we obtain the following theorem.

Theorem 8 *For $n \geq 0$, we have*

$$\begin{aligned}
&N_n(x|a_1,\ldots,a_r) \\
&\quad = \sum_{m=0}^{n}\left\{\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j(1-\lambda)^{-j}\right. \\
&\quad\quad \left. \times S_1(n-j-l,m)N_l(a_1,\ldots,a_r)\right\}H_m^{(s)}(x|\lambda).
\end{aligned}$$

By the same method as the proof of Theorem 8, we get

$$\begin{aligned}
&\hat{N}_n(x|a_1,\ldots,a_r) \\
&\quad = \sum_{m=0}^{n}\left\{\sum_{j=0}^{n-m}\sum_{l=0}^{n-m-j}\binom{s}{j}\binom{n-j}{l}(n)_j\right. \\
&\quad\quad \left. \times (1-\lambda)^{-j}S_1(n-j-l,m)\hat{N}_l(a_1,\ldots,a_r)\right\}H_m^{(s)}(x|\lambda).
\end{aligned} \tag{66}$$

Now, we consider the following two Sheffer sequences:

$$N_n(x|a_1,\ldots,a_r) \sim \left(\prod_{j=1}^{r}\left(\frac{t}{e^{a_j t}-1}\right), e^t - 1\right) \tag{67}$$

and

$$B_n^{(s)}(x) \sim \left(\left(\frac{e^t-1}{t}\right)^s, e^t - 1\right),$$

where $B_n^{(s)}(x)$ are the Bernoulli polynomials of order s given by the generating function as

$$\left(\frac{t}{e^t-1}\right)^s e^{xt} = \sum_{n=0}^{\infty} B_n^{(s)}(x)\frac{t^n}{n!}.$$

Let us assume that

$$N_n(x|a_1,\ldots,a_r) = \sum_{m=0}^{n} C_{n,m} B_m^{(s)}(x). \tag{68}$$

By (18) and (19), we get

$$\begin{aligned}
C_{n,m} &= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \left(\frac{t}{\log(1+t)}\right)^s \middle| x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \left(\frac{t}{\log(1+t)}\right)^s x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| \sum_{k=0}^{\infty} \mathbb{C}_k^{(s)}\frac{t^k}{k!} x^n \right\rangle \\
&= \frac{1}{m!}\sum_{k=0}^{n-m}\binom{n}{k}\mathbb{C}_k^{(s)}\left\langle \prod_{j=1}^{r}\left(\frac{(1+t)^{a_j}-1}{\log(1+t)}\right)\bigl(\log(1+t)\bigr)^m \middle| x^{n-k} \right\rangle \\
&= \frac{1}{m!}\sum_{k=0}^{n-m}\binom{n}{k}\mathbb{C}_k^{(s)} m! \sum_{l=0}^{n-m-k}\binom{n-k}{l} S_1(n-l-k,m) N_l(a_1,\ldots,a_r) \\
&= \sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)} S_1(n-k-l,m) N_l(a_1,\ldots,a_r),
\end{aligned} \tag{69}$$

where $\mathbb{C}_k^{(s)}$ are the Cauchy numbers of the first kind of order s defined by the generating function as

$$\left(\frac{t}{\log(1+t)}\right)^s = \sum_{n=0}^{\infty} \mathbb{C}_n^{(s)}\frac{t^n}{n!}.$$

Therefore, by (68) and (69), we obtain the following theorem.

Theorem 9 *For $n \geq 0$, we have*

$$N_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left\{\sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)} \times S_1(n-k-l,m)N_l(a_1,\dots,a_r)\right\}B_m^{(s)}(x).$$

By the same method as the proof of Theorem 9, we get

$$\hat{N}_n(x|a_1,\dots,a_r) = \sum_{m=0}^{n}\left\{\sum_{k=0}^{n-m}\sum_{l=0}^{n-m-k}\binom{n}{k}\binom{n-k}{l}\mathbb{C}_k^{(s)} \times S_1(n-k-l,m)\hat{N}_l(a_1,\dots,a_r)\right\}B_m^{(s)}(x).$$

Recently, several authors have studied umbral calculus (see [1–5, 7–18, 20]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Author details
[1]Department of Mathematics, Sogang University, Seoul, 121-742, Republic of Korea. [2]Department of Mathematics, Kwangwoon University, Seoul, 139-701, Republic of Korea.

Acknowledgements
This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MOE) (No. 2012R1A1A2003786). The authors would like to thank the referees for their valuable comments.

Received: 30 April 2014 Accepted: 9 June 2014 Published: 22 Jul 2014

References
1. Roman, S: The Umbral Calculus. Dover, New York (2005)
2. Kim, T, Kim, DS: Some identities involving associated sequences of special polynomials. J. Comput. Anal. Appl. **16**, 626-642 (2014)
3. Kim, DS, Kim, T, Lee, S-H, Rim, S-H: Umbral calculus and Euler polynomials. Ars Comb. **112**, 293-306 (2013)
4. Kim, DS, Kim, T: Some identities of Bernoulli and Euler polynomials arising from umbral calculus. Adv. Stud. Contemp. Math. (Kyungshang) **23**(1), 159-171 (2013)
5. Kim, DS, Kim, T, Dolgy, DV, Rim, S-H: Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus. Adv. Differ. Equ. **2013**, 73 (2013)
6. Kim, T, Mansour, T, Rim, S-H, Lee, S-H: Apostol-Euler polynomials arising from umbral calculus. Adv. Differ. Equ. **2013**, 301 (2013)
7. Kim, DS, Kim, T: Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **23**(21), 621-636 (2013)
8. Kim, T, Kim, DS, Mansour, T, Rim, S-H, Schork, M: Umbral calculus and Sheffer sequences of polynomials. J. Math. Phys. **54**(8), 083504 (2013)
9. Kim, T: Identities involving Laguerre polynomials derived from umbral calculus. Russ. J. Math. Phys. **21**(1), 36-45 (2014)
10. Lu, D-Q, Xiang, Q-M, Luo, C-H: Some results for Apostol-type polynomials associated with umbral algebra. Adv. Differ. Equ. **2013**, 201 (2013)
11. Maldonado, M, Prada, J, Senosiain, MJ: Appell bases on sequence spaces. J. Nonlinear Math. Phys. **18**, suppl. 1, 189-194 (2011)
12. Araci, S, Kong, X, Acikgoz, M, Şen, E: A new approach to multivariate q-Euler polynomials using the umbral calculus. J. Integer Seq. **17**(1), Article 14.1.2 (2014)
13. Dere, R, Simsek, Y: Applications of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. (Kyungshang) **22**(3), 433-438 (2012)
14. Ernst, T: Examples of a q-umbral calculus. Adv. Stud. Contemp. Math. (Kyungshang) **16**(1), 1-22 (2008)
15. Fang, Q, Wang, T: Umbral calculus and invariant sequences. Ars Comb. **101**, 257-264 (2011)
16. Gessel, IM: Applications of the classical umbral calculus. Dedicated to the memory of Gian-Carlo Rota. Algebra Univers. **49**(4), 397-434 (2003)

17. Kim, DS, Kim, T: Some identities of Frobenius-Euler polynomials arising from umbral calculus. Adv. Differ. Equ. **2012**, 196 (2012)
18. Kim, DS, Kim, T, Ryoo, CS: Sheffer sequences for the powers of Sheffer pairs under umbral composition. Adv. Stud. Contemp. Math. (Kyungshang) **23**(2), 275-285 (2013)
19. Ryoo, CS, Song, H, Agarwal, RP: On the roots of the q-analogue of Euler-Barnes' polynomials. Adv. Stud. Contemp. Math. **9**(2), 153-163 (2004)
20. Taylor, BD: Umbral presentations for polynomial sequences. Umbral calculus and its applications (Cortona, 1998). Comput. Math. Appl. **41**(9), 1085-1098 (2001)

10.1186/1687-1847-2014-182
Cite this article as: Kim and Kim: Barnes-type Narumi polynomials. *Advances in Difference Equations* 2014, 2014:182