International conference PRIM 2006

HANKEL TRANSFORM OF NARAYANA POLYNOMIALS AND GENERALIZED CATALAN NUMBERS

Marko D. Petković,

Predrag M. Rajković,

Department of Mathematics

University of Niš

Serbia.

1 Introduction

First we will define Hankel transform of a integer sequence and show some examples.

Definition 1 The Hankel transform of a given sequence $A = \{a_0, a_1, a_2, ...\}$ is the sequence of Hankel determinants $\{h_0, h_1, h_2, \dots\}$ where $h_n = |a_{i+j-2}|_{i,j=1}^{n}$, i.e

$$A = \{a_n\}_{n\in\mathbb{N}_0} \;\to\; h = \{h_n\}_{n\in\mathbb{N}_0} : \; h_n = \begin{vmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{vmatrix} \tag{1.1}$$

Example 1 The Hankel transform of a Catalan sequence given by $c(n) = \frac{1}{n+1}\binom{2n}{n}$ is the sequence of all 1's. Thus each of the determinants has value 1:

$$|1|, \quad \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{vmatrix}, \quad \ldots \tag{1.2}$$

Example 2 Sequence of central binomial coefficients defined by $a_n = \binom{2n}{n}$ has Hankel transform $h_n = 2^n$, i.e.

$$|1| = 1, \quad \begin{vmatrix} 1 & 2 \\ 6 & 20 \end{vmatrix} = 2, \quad \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 20 \\ 20 & 70 & 252 \end{vmatrix} = 4, \quad \ldots \tag{1.3}$$

Example 3 In paper [3], A. Cvetković, P. Rajković and M. Ivković have proven that Hankel transform of sequence A005087 in On-Line Encyclopedia of Integer Sequences [11] defined by:

$$a_n = c(n) + c(n+1) = \frac{1}{n+1}\binom{2n}{n} + \frac{1}{n+2}\binom{2n+2}{n+1} \tag{1.4}$$

equals to the sequence A001906, i.e. bisection of Fibonacci sequence $F(2n+1)$.

We generalized previous result, i.e. computed the Hankel transform of the generalized sequence $a_n(L) = c(n;L) + c(n+1;L)$, where $c(n;L)$ is a sequence of generalized Catalan numbers.

2 Hankel transform of Narayana Polynomials

In this section we will present the metod for computing Hankel transform of the sequence of Narayana polynomials based on Krattenthaler formula in [6]. Similar method will be used also for generalized sequence from [3].

- First we will find the real measure whose moments are values of Narayana polynomials
- Then we will construct the sequence of orthogonal polynomials with respect to found measure
- Finally, from the three-terms recurrence relation we will derive Hankel transform in the closed form.

2.1 Narayana numbers and polynomials

We will consider the sequence of the Narayana and shifted Narayana numbers

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k+1}, \qquad \tilde{N}(n,k) = N(n+1,k).$$

To this sequence we can join the Narayana triangles

$$\mathbf{N} = \left[N(n,k)\right]_{n,k\in\mathbb{N}}, \qquad \tilde{\mathbf{N}} = \left[\tilde{N}(n,k)\right]_{n,k\in\mathbb{N}}.$$

and the Narayana polynomials

$$a(n;r) = \sum_{k=0}^{n} \tilde{N}(n,k) r^k, \qquad a_1(n;r) = \sum_{k=0}^{n} N(n,k) r^k.$$

It is valid

$$a(n;r) = a_1(n+1;r) \qquad (n \in \mathbb{N}).$$

Definition 2 For a given function $y = f(x)$, $f(0) = 0$, the series reversion is the sequence $\{s_k\}$ such that

$$x = f^{-1}(y) = s_0 + s_1 y + \cdots + s_n y^n + \cdots ,$$

where $x = f^{-1}(y)$ is the inverse function of $y = f(x)$.

In the paper [1], P. Barry showed the next facts.

Lemma 1 The series reversion of the next functions are Narayana and shifted Narayana numbers

$$y = f(x) = \frac{x}{1 + (r+1)x + rx^2} \quad \Rightarrow \quad f^{-1}(y) = \sum_{n=0}^{+\infty} a(n;r)y^n,$$

$$y = g(x) = \frac{x(1 - rx)}{1 - (r-1)x} \quad \Rightarrow \quad g^{-1}(y) = \sum_{n=0}^{+\infty} a_1(n;r)y^n.$$

From the previous Lemma, we can easily derive the generating functions of the sequences $a(n;r)$ and $a_1(n;r)$.

Corolary 1 The generating functions of the sequences $a(n;r)$ and $a_1(n;r)$ are given by:

$$\begin{aligned} A(x,r) &= \frac{-1+(r+1)x+\sqrt{(1-(r+1)x)^2-4rx^2}}{2rx^2} \\ A_1(x,r) &= \frac{A(x,r)-1}{x} \end{aligned} \tag{2.5}$$

2.2 Deriving the weight function of $\{a(n;r)\}_{n\in\mathbb{N}_0}$

Our goal is to find weight function w such that $a(n,r)$, $n=0,1,\dots$ are moments corresponding to this function, i.e. that holds $a(n,r)=\int_{\mathbb{R}} x^n w(x)dx$.

Theorem 1 **The weight function whose n-th moment is $a(n,r)$ is:**

$$w(x)=\begin{cases}\frac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, & x\in\left((\sqrt{r}-1)^2,(\sqrt{r}+1)^2\right);\\ 0, & \text{otherwise.}\end{cases} \tag{2.6}$$

Proof. **We will use Stieltjes inversion formula (see [2]). First define the function:**

$$F(z,r)=\frac{1}{z}A\left(\frac{1}{z},r\right)=-\frac{(r+1)-z+\sqrt{(z-r-1)^2-4r}}{2rz} \tag{2.7}$$

Then, from the theory of distributions, we have that distribution function $\psi(x)$ and measure (weight) $w(x)$ satisfies following relations:

$$\psi(t)-\psi(0)=-\frac{1}{\pi}\lim_{y\to 0+}\int_0^t \Im F(x+iy;L)dx,\quad w(t)=\frac{d\psi(t)}{dt}. \tag{2.8}$$

It can be shown that an integral of $F(z;r)$ is equal to:

$$\mathcal{F}(z;L) = \int F(z;L)dz = \frac{1}{4r}\Big[z^2+(1+r-z)\rho(z;r)-2z(r+1)\Big]+l_1(z;r), \quad (2.9)$$

Where we denoted:

$$\begin{aligned} \rho(z;r) &= \sqrt{(z-r-1)^2-4r} \\ l_1(z;r) &= \ln\left(-(r+1)+z+\rho(z;r)\right) \end{aligned} \quad (2.10)$$

We can notice that in the complex plane, function $\rho(z;r)$ has two branch points $z=(\sqrt{r}-1)^2$ and $z=(\sqrt{r}+1)^2$, and $l_1(z)$ has one more, $z=r+1$.

Now by choosing appropriate regular branches of $\rho(z;r)$ and $l_1(z;r)$ we can find the limits:

$$\lim_{y\to 0+} \Im\rho(x+iy;r) = \begin{cases} \sqrt{4r-(x-r-1)^2} & , x \in ((\sqrt{r}-1)^2, (\sqrt{r}+1)^2) \\ 0 & , \text{otherwise.} \end{cases},$$

and

$$\lim_{y\to 0+} \Im l_1(x+iy;r) = \begin{cases} \pi + \arctan \frac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)} & , x \in ((\sqrt{r}-1)^2, r+1) \\ \arctan \frac{\sqrt{4r-(x-r-1)^2}}{x-(r+1)} & , x \in (r+1, (\sqrt{r}+1)^2) \\ 0 & , \text{otherwise.} \end{cases}$$

Now to complete the proof we need to substitute these values into the formula for $w(x)$. $\square$

2.3 Orthogonal polynomials w.r.t. the weight $w(x)$

In the paper [6],C. Krattenthaler proved that Hankel transform h_n of a sequence $a_n = \int_{\mathbb{R}} x^n w(x)dx$ is given with the following relation $h_n = a_0^n \prod_{i=1}^{n-1} \beta_i^{n-i}$.

Coefficients β_i are from the three-terms recurrence relation between monic orthogonal polynomials with respect to the weight $\omega(x)$.

$$Q_{n+1}(x) = (x - \alpha_n)Q_n(x) - \beta_n Q_{n-1}(x), \tag{2.11}$$

Lemma 2 Coefficients α_n and β_n, $n = 0, 1, \ldots$ in the three-term recurrence relation (2.11) with respect to weight function:

$$w(x) = \begin{cases} \frac{\sqrt{4r-(x-r-1)^2}}{2\pi r}, & x \in \left((\sqrt{r}-1)^2, (\sqrt{r}+1)^2\right); \\ 0, & \text{otherwise.} \end{cases}$$

are given with:

$$\beta_0 = 1 \quad \beta_n = r, n \geq 1 \quad \alpha_n = r+1, n \geq 0$$

Proof. **Second kind Chebyshev polynomials are orthogonal w.r.t the weight** $w^{(1)}(x) = \sqrt{1-x^2}$.

$$Q_n^{(1)}(x) = S_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{2^n \cdot \sqrt{1-x^2}}$$

Corresponding coefficients are:

$$\beta_0^{(1)} = \frac{\pi}{2}, \quad \beta_n^{(1)} = \frac{1}{4},\ n \geq 1 \quad \alpha_n^{(1)} = 0,\ n \geq 0$$

Let we introduce new weight function:

$$w^{(2)}(x) = \sqrt{4r-(x-r-1)^2} = w\left(\frac{1}{2\sqrt{r}}x - \frac{r+1}{2\sqrt{r}}\right) = w(ax+b)$$

Using the transformation formulas from [4] we obtain new coefficients:

$$\beta_0^{(2)} = \sqrt{r}\pi, \quad \beta_n^{(2)} = \frac{\beta_n^{(1)}}{a^2} = r,\ n \geq 1 \quad \alpha_n^{(2)} = \frac{\alpha_n^{(1)} - b}{a^2} = r+1,\ n \geq 0$$

Finally by dividing weight function $w^{(2)}(x)$ with constant $\frac{1}{\pi\sqrt{r}}$ we have

$$\beta_n^{(3)} = \beta_n^{(2)} = r,\ n \geq 1, \quad \beta_0^{(3)} = 1, \quad \alpha_n^{(3)} = \alpha_n^{(2)} = r + 1$$

which completes the proof. $\square$

2.4 The Hankel transform of $\{a(n;r)\}_{n\in\mathbb{N}_0}$

Now we are ready to prove the main theorem of this section:

Theorem 2 **The Hankel transform of the sequence $a(n;r)$ is**

$$h(n;r) = r^{\binom{n}{2}}.$$

Proof. **Using Krattenthaler formula we have:**

$$h(n;r) = a(0;r)^n \prod_{i=1}^{n-1} \beta_i^{n-i} = 1^n \prod_{i=1}^{n-1} r^{n-i} = r^{\binom{n}{2}}.$$

□

3 Hankel Transform of sum of consecutive generalized Catalan numbers

In this section we will consider the generalized Catalan numbers and we will find the Hankel transform of a sequence $a_n(L)$, the generalization of the sequence A005087.

- First we will define the generalized binomial coefficients and generalized Catalan numbers and consider its basic properties.
- Then we will derive the generating function of $a_n(L)$.
- Finally we will find the Hankel transform similarly as in the case of Narayana polynomials.

3.1 Definitions and basic properties

Definition 3 For a given sequence $\{b_n\}_{n\in\mathbb{N}_0}$ define the generalized binomial coefficient with:

$$T(n,k,\{b_m\}) = \sum_j \binom{k}{j}\binom{n-k}{j} b_j.$$

Also define the sequence of generalized Catalan numbers with:

$$c(n;\{b_m\}) = T(2n,n;\{b_m\}) - T(2n,n-1;\{b_m\})$$

It can be directly verified that holds $T(n,k,\{b_m\}) = T(n,n-k,\{b_m\})$.

Example 4 For the sequence $b_m = 1$, we have that $T(n,k;\{1\}) = \binom{n}{k}$. This comes from the Vandermode convolution identity:

$$\binom{n}{k} = \sum_j \binom{k}{j}\binom{n-k}{j}.$$

In that case also holds $c(n) = c(n;\{1\})$.

Now consider the sequence $b_m = L^m$ where L is positive real number. To simplify notation, we will denote:

$$T(n,k;\{L^m\}) = T(n,k;L), \qquad c(n;\{L^m\}) = c(n;L)$$

Definition 4 Denote with $a_n(L)$ generalization of the sequence A005087 defined by:

$$a_n(L) = c(n+1;L) + c(n;L)$$

Our goal is to find the Hankel transform $h_n(L)$ of this sequence.

3.2 The generating function

Proposition 1 The generalized binomial coefficient $T(2n+a, n+a; L)$ can be rewritten using Jacobi polynomial $P_n^{(a,b)}(x)$ by:

$$T(2n+a, n; L) = (L-1)^n P_n^{(a,0)}\left(\frac{L+1}{L-1}\right)$$

The generating function of Jacobi polynomials is given by:

$$G^{(a,b)}(x,t) = \sum_{n=0}^{\infty} P_n^{(a,b)}(x)t^n = \frac{2^{a+b}}{\phi\cdot(1-t+\phi)^a\cdot(1+t+\phi)^b}, \tag{3.12}$$

where $\phi = \phi(x,t) = \sqrt{1-2xt+t^2}$.

Now we can derive the generating functions of $T(2n+a, n; L)$ and also $a_n(L)$:

$$\sum_{n=0}^{\infty} T(2n+a, n; L)\ t^n = G^{(a,0)}\left(\tfrac{L+1}{L-1}, (L-1)t\right), \tag{3.13}$$

$$\begin{aligned}\mathcal{G}(t;L) &= \sum_{n=0}^{+\infty} a_n(L)t^n \\ &= \frac{t+1}{t} G^{(0,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right) - (t+1)G^{(2,0)}\left(\tfrac{L+1}{L-1},(L-1)t\right) - \frac{1}{t} \\ &= \frac{t+1}{\rho(t;L)} \left\{\frac{1}{t} - \frac{4}{(1-(L-1)t+\rho(t;L))^2}\right\} - \frac{1}{t}\end{aligned} \tag{3.14}$$

where

$$\rho(t;L) = \phi\left(\tfrac{L+1}{L-1},(L-1)t\right) = \sqrt{1-2(L+1)t+(L-1)^2t^2} \tag{3.15}$$

3.3 The Hankel transform of $a_n(L)$

Theorem 3 Numbers $a_n(L)$ are the moments of the following weight function:

$$\omega(x;L) = \frac{\sqrt{L}}{\pi}\Big(1+\frac{1}{x}\Big)\sqrt{1-\Big(\frac{x-L-1}{2\sqrt{L}}\Big)^2} \tag{3.16}$$

Now we need to describe the orthogonal polynomials $\{Q_n(x)\}$ corresponding to this weight function.

Example 5 For $L=4$, we can find the first members

$$Q_0(x) = 1, \qquad \|Q_0\|^2 = 5,$$

$$Q_1(x) = x - \frac{24}{5}, \qquad \|Q_1\|^2 = \frac{104}{5},$$

$$Q_2(x) = x^2 - \frac{127}{13}x + \frac{256}{13}, \qquad \|Q_2\|^2 = \frac{1088}{13},$$

$$Q_3(x) = x^3 - \frac{541}{17}x^2 + \frac{1096}{17}x - \frac{1344}{17}, \qquad \|Q_3\|^2 = \frac{5696}{17},$$

We can again start from the second kind Chebyshev polynomials orthogonal w.r.t. the weight $p^{(1/2,1/2)}(x) = \sqrt{1-x^2}$. They satisfy the three-term recurrence relation [2]:

$$S_{n+1}(x) = (x - \alpha_n^*)\ S_n(x) - \beta_n^* S_{n-1}(x) \quad (n = 0, 1, \ldots), \tag{3.17}$$

with initial values

$$S_{-1}(x) = 0, \qquad S_0(x) = 1,$$

where

$$\alpha_n^* = 0 \quad (n \geq 0) \qquad \text{and} \qquad \beta_0^* = \frac{\pi}{2}, \quad \beta_n^* = \frac{1}{4} \qquad (n \geq 1).$$

Let we introduce new weight function $\hat{w}(x) = (x - c)\, p^{(1/2,1/2)}(x)$. Corresponding coefficients $\hat{\alpha}_n$ and $\hat{\beta}_n$ can be evaluated as follows [4]:

$$\begin{aligned} \lambda_n &= S_n(c), \\ \hat{\alpha}_n &= c - \frac{\lambda_{n+1}}{\lambda_n} - \beta^*_{n+1}\frac{\lambda_n}{\lambda_{n+1}}, \\ \hat{\beta}_n &= \beta^*_n \frac{\lambda_{n-1}\lambda_{n+1}}{\lambda_n^2} \qquad (n \in \mathbb{N}_0). \end{aligned} \tag{3.18}$$

If we choose $c = -\frac{L+2}{2\sqrt{L}}$, it can be shown that holds:

$$\lambda_n = \frac{(-1)^n}{2\cdot 4^n L^{\frac{n}{2}}\sqrt{L^2+4}}\,\psi_{n+1} \quad (n = -1, 0, 1, \ldots).$$

where:

$$\psi_n = \left(L + 2 + \sqrt{L^2+4}\right)^n - \left(L + 2 - \sqrt{L^2+4}\right)^n.$$

The next transformation will be $\tilde{w}(x) = \hat{w}(ax+b)$, where $a = \frac{1}{2\sqrt{L}}$ and $b = -\frac{L+1}{2\sqrt{L}}$. After exchanging we obtain:

$$\tilde{w}(x) = \hat{w}\left(\frac{x-L-1}{2\sqrt{L}}\right) = \frac{1}{2\sqrt{L}}(x+1)\sqrt{1-\left(\frac{x-L-1}{2\sqrt{L}}\right)^2}. \tag{3.19}$$

Coefficients of three-term relation are now:

$$\tilde{\alpha}_n = \frac{\hat{\alpha}_n - b}{a}, \qquad \tilde{\beta}_n = \frac{\hat{\beta}_n}{a^2} \qquad (n \geq 0). \tag{3.20}$$

Multiplying the weight function $\tilde{w}(x)$ with the constant $\frac{2L}{\pi}$ we are only changing $\tilde{\beta}_0$. Finally, we have that coefficients corresponding to the:

$$\breve{w}(x) = \frac{2L}{\pi}\tilde{w}(x) = \frac{\sqrt{L}}{\pi}(x+1)\sqrt{1-\left(\frac{x-L-1}{2\sqrt{L}}\right)^2} \tag{3.21}$$

are given with:

$$\begin{aligned} \breve{\beta}_0 &= L(L+2), \quad \breve{\beta}_n = \tilde{\beta}_n = L\frac{\psi_n\psi_{n+2}}{\psi_{n+1}^2} \quad (n \in \mathbb{N}), \\ \breve{\alpha}_n &= \tilde{\alpha}_n = -1 + \frac{1}{2}\cdot\frac{\psi_{n+2}}{\psi_{n+1}} + 2L\cdot\frac{\psi_{n+1}}{\psi_{n+2}} \quad (n \in \mathbb{N}_0). \end{aligned} \tag{3.22}$$

Final transformation will be $\omega(x;L) = \frac{\breve{w(x)}}{x}$. If we know all about the MOPS orthogonal with respect to $\breve{w}(x)$ what can we say about the sequence $\{Q_n(x)\}$ orthogonal w.r.t. a weight

$$w_d(x) = \frac{\breve{w}(x)}{x-d} \qquad (d \notin \text{support}(\breve{w}))\ ?$$

In the book [5], W. Gautshi has proved that, by the auxiliary sequence:

$$r_{-1} = -\int_{\mathbb{R}} w_d(x)\ dx, \qquad r_n = d - \breve{\alpha}_n - \frac{\breve{\beta}_n}{r_{n-1}} \quad (n = 0, 1, \ldots),$$

it can be determined:

$$\begin{aligned} \alpha_{d,0} &= \breve{\alpha}_0 + r_0, & \alpha_{d,k} &= \breve{\alpha}_k + r_k - r_{k-1}, \\ \beta_{d,0} &= -r_{-1}, & \beta_{d,k} &= \breve{\beta}_{k-1}\frac{r_{k-1}}{r_{k-2}} \qquad (k \in \mathbb{N}). \end{aligned}$$

We need the case $d = 0$. Next Lemma can be proved by induction:

Lemma 3 The parameters r_n have the explicit form

$$\begin{aligned} r_n &= -\frac{\psi_{n+1}}{\psi_{n+2}} \cdot \frac{L\psi_{n+2} + \xi\varphi_{n+2}}{L\psi_{n+1} + \xi\varphi_{n+1}} \qquad (n \in \mathbb{N}_0). \\ \varphi_n &= \left(L + 2 + \sqrt{L^2+4}\right)^n + \left(L + 2 - \sqrt{L^2+4}\right)^n, \quad \xi = \sqrt{L^2+4} \end{aligned} \tag{3.23}$$

Now we have the coefficients $\beta_n = \beta_{0,n}$. By exchanging and using Krattenthaler formula we finally obtain:

$$\begin{aligned} h_n(L) &= \beta_0\beta_1\beta_2\cdots\beta_{n-2}\beta_{n-1}\cdot h_{n-1}(L) = \beta_0\frac{r_{n-2}}{r_{-1}}\prod_{k=0}^{n-2}\breve{\beta}_k\cdot h_{n-1}(L) \\ &= \frac{L^{n-1}}{2}\cdot\frac{L\psi_n+\xi\varphi_n}{L\psi_{n-1}+\xi\varphi_{n-1}}\cdot h_{n-1}(L) = \frac{L^{n(n-1)/2}}{2^{n+1}\xi}\cdot(L\psi_n+\xi\varphi_n) \\ &= \frac{L^{(n^2-n)/2}}{2^{n+1}\sqrt{L^2+4}}\cdot \\ &\quad \left\{(\sqrt{L^2+4}+L)(\sqrt{L^2+4}+L+2)^n+(\sqrt{L^2+4}-L)(L+2-\sqrt{L^2+4})^n\right\}. \end{aligned} \tag{3.24}$$

References

[1] **P. Barry,** *On Ineger Sequences Based Constructions of Generalized Pascal Triangles*, **Preprint, Waterford Institute of Technology, 2005.**

[2] **T. S. Chihara,** *An Introduction to Orthogonal Polynomials*, **Gordon and Breach, New York, 1978.**

[3] **A. Cvetković, P. Rajković and M. Ivković,** *Catalan Numbers, the Hankel Transform and Fibonacci Numbers*, **Journal of Integer Sequences, 5, May 2002, Article 02.1.3.**

[4] **W. Gautschi,** *Orthogonal polynomials: applications and computations*, **in** *Acta Numerica, 1996*, **Cambridge University Press, 1996, pp. 45–119.**

[5] **W. Gautschi,** *Orthogonal Polynomials: Computation and Approximation*, **Clarendon Press - Oxford, 2003.**

[6] **C. Krattenthaler,** *Advanced Determinant Calculus,* **at http://www.mat.univie.ac.at/People/kratt/artikel/detsurv.html.**

[7] **J. W. Layman,** *The Hankel Transform and Some of its Properties,* **Journal of Integer Sequences, Article 01.1.5, Volume 4, 2001.**

[8] **P. Peart and W. J. Woan,** *Generating functions via Hankel and Stieltjes matrices,* **Journal of Integer Sequences, Article 00.2.1, Issue 2, Volume 3, 2000.**

[9] **P.M. Rajković, M.D. Petković, P. Barry,** *The Hankel Transform of the Sum of Consecutive Generalized Catalan Numbers,* **Journal of Integral Transforms, to appear. Available electronically at Mathematics ArXiv, 2006.**

[10] **W. J. Woan,** *Hankel Matrices and Lattice Paths,* **Journal of Integer Sequences, Article 01.1.2, Volume 4, 2001.**

[11] **N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at http://www.research.att.com/∼njas/sequences/**

[12] **R. A. Sulanke,** *Counting Lattice Paths by Narayana Polynomials* **The Electronic Journal of Combinatorics, Vol. 7(1), 2000.**

Thanks for your attention!