Available online at www.sciencedirect.com
ScienceDirect
Discrete Applied Mathematics 156 (2008) 2781–2789
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Some identities on the Catalan, Motzkin and Schröder numbers

Eva Y.P. Deng[a,*], Wei-Jun Yan[b]

[a] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, PR China
[b] Department of Foundation Courses, Neusoft Institute of Information, Dalian 116023, PR China

Received 19 July 2006; received in revised form 1 September 2007; accepted 18 November 2007
Available online 3 January 2008

Abstract

In this paper, some identities between the Catalan, Motzkin and Schröder numbers are obtained by using the Riordan group. We also present two combinatorial proofs for an identity related to the Catalan numbers with the Motzkin numbers and an identity related to the Schröder numbers with the Motzkin numbers, respectively.

Keywords: Catalan number; Motzkin number; Schröder number; Riordan group

1. Introduction

The Catalan, Motzkin and Schröder numbers have been widely encountered and widely investigated. They appear in a large number of combinatorial objects, see [11] and The On-Line Encyclopedia of Integer Sequences [8] for more details. Here we describe them in terms of certain lattice paths.

A *Dyck path* of semilength n is a lattice path from the origin $(0, 0)$ to $(2n, 0)$ consisting of up steps $(1, 1)$ and down steps $(1, -1)$ that never goes below the x-axis. The set of Dyck paths of semilength n is enumerated by the n-th *Catalan number*

$$c_n = \frac{1}{n+1}\binom{2n}{n}$$

(sequence A000108 in [8]). The generating function for the Catalan numbers $(c_n)_{n\in N}$ is

$$c(x) = \frac{1-\sqrt{1-4x}}{2x}.$$

A *Motzkin path* of length n is a lattice path from the origin $(0, 0)$ to $(n, 0)$ consisting of horizontal steps $(1, 0)$, up steps $(1, 1)$ and down steps $(1, -1)$ that never goes below the x-axis. The set of Motzkin paths of length n is enumerated by the n-th *Motzkin number*, denoted by m_n (sequence A001006 in [8]). The generating function for the

* Corresponding author.
E-mail addresses: ypdeng@dlut.edu.cn (E.Y.P. Deng), yanweijun@neusoft.edu.cn (W.-J. Yan).

0166-218X/$ - see front matter
doi:10.1016/j.dam.2007.11.014

Motzkin numbers $(m_n)_{n\in N}$ is

$$m(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}.$$

For convenience, we define $m_{-1} = 1$ and let

$$\widehat{m}(x) = \sum_{n\geq 0} m_{n-1}x^n = 1 + xm(x) = \frac{1+x-\sqrt{1-2x-3x^2}}{2x}.$$

A *Schröder path* of semilength n is a lattice path from the origin $(0, 0)$ to $(2n, 0)$ consisting of double horizontal steps $(2, 0)$, up steps $(1, 1)$ and down steps $(1, -1)$ that never goes below the x-axis. The set of Schröder paths of semilength n is enumerated by the n-th *large Schröder number* (in what follows just Schröder number), denoted by s_n (sequence A006318 in [8]). The generating function for the Schröder numbers $(s_n)_{n\in N}$ is

$$s(x) = \frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

The three sequences are closely related. For example, the Catalan numbers are related to the Motzkin numbers by

$$c_{n+1} = \sum_{k=0}^{n} \binom{n}{k} m_k, \quad n \geq 0; \qquad c_0 = 1. \tag{1}$$

The formula has been derived by Bernhart [1] using a difference operator. By using 2-Motzkin path, a combinatorial proof is implied by Deutsch [2]. The Schröder numbers are related to the Catalan numbers by

$$s_n = \sum_{k=0}^{n} \binom{2n-k}{k} c_{n-k}, \quad n \geq 0. \tag{2}$$

The formula is directed. Suppose that a Schröder path of semilength n contains k horizontal steps. We can reduce a Dyck path of semilength $n-k$ by removing all the horizontal steps. Conversely, given a Dyck path of semilength $n-k$, we can reconstruct $\binom{2n-k}{k}$ Schröder paths of semilength n by inserting k horizontal steps. The reader can find more relations in Bernhart [1], Donaghey and Shapiro [3] and Sulanke [12].

In the present paper, we will give more relations between the Catalan, Motzkin and Schröder numbers by the Riordan group theory in Section 2. In Section 3, we will give another combinatorial proof for the identity (1). In Section 4, we will give a combinatorial interpretation for an identity relating the Schröder and Motzkin numbers.

2. Riordan group and some identities

In 1978, Rogers [6] introduced the concept of renewal array. Shapiro et al. [7], and Sprugnoli [9,10] further generalized the concept under the name of Riordan array, and developed the Riordan group theory. Now we give a brief introduction to the related notations.

A *Riordan array* $R = (r_{i,j})_{i,j\geq 0}$ is an infinite lower triangular matrix whose k-th column has generating function $g(x)f^k(x)$, where $k = 0, 1, 2, \ldots$ and $g(x)$, $f(x)$ are generating functions, denoted by $R = (g(x), f(x))$. Suppose we multiply the matrix $R = (g(x), f(x))$ by a column vector $(a_0, a_1, \ldots)^{\mathrm{T}}$ and get a column vector $(b_0, b_1, \ldots)^{\mathrm{T}}$. Let $a(x)$ and $b(x)$ be the generating functions for the sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$, respectively. The method of Riordan arrays asserts that $b(x) = g(x)a(f(x))$, or equivalently, $b_i = \sum_{j\geq 0} r_{i,j}a_j$.

Riordan group $\mathcal{R} = \{R | R = (g(x), f(x))$ is a Riordan array, and $g(0) = 1, f(0) = 0, f(1) \neq 0\}$. The multiplication in $\mathcal{R}$ is

$$(g_1(x), f_1(x)) * (g_2(x), f_2(x)) = (g_1(x)g_2(f_1(x)), f_2(f_1(x))).$$

The identity is $(1, x)$. The inverse is

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right),$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$, i.e. $f(\bar{f}(x)) = \bar{f}(f(x)) = x$.

Riordan arrays provide a method to solve many combinatorial sums by means of generating functions. For example, we review the identities (1) and (2) from the Riordan array point of view. Let the Riordan array

$$A = \left(1, \frac{x}{1-x}\right).$$

Since the generating function of sequences $(m_{n-1})_{n\in N}$ is $\widehat{m}(x)$, it follows that

$$1 \cdot \widehat{m}\left(\frac{x}{1-x}\right) = c(x).$$

Thus we obtain the identity (1). Let the Riordan array

$$B = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right).$$

Using the same method, we can obtain the identity (2). Now we further set up the relation between the Schröder and Motzkin numbers by using the method of Riordan array.

Theorem 1. *For $n \geq 0$, we have the following formula:*

$$s_n = \sum_{k=0}^{n} s_{n,k} m_{k-1}, \tag{3}$$

where the generating function of $(s_{n,k})_{n\in N}$ equals $\frac{1}{1-x} \cdot \left(\frac{x}{1-3x+x^2}\right)^k$, $k = 0, 1, 2, \ldots$.

Proof. By the multiplication of the Riordan array, we obtain a new Riordan array by

$$C = B * A = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right) * \left(1, \frac{x}{1-x}\right) = \left(\frac{1}{1-x}, \frac{x}{1-3x+x^2}\right).$$

Then we have the desired result since

$$s(x) = \frac{1}{1-x} \cdot \widehat{m}\left(\frac{x}{1-3x+x^2}\right). \quad \blacksquare$$

For the general term $s_{n,k}$ of Riordan array C, we give a combinatorial representation in Section 4. In the case of $k = 1$, it is the sequences A027941 in [8], that is, $s_{n,1} = f_{2n+1} - 1$, where f_n is the Fibonacci number.

The problem of inverting combinatorial sums is interesting, for example, Riordan [5], Gould and Hsu [4] studied some inverse relations. The Riordan group allows us to invert the elements in the group, and give the invertible transformations. Now we consider the inversion of the identities (1)–(3).

Let the Riordan array

$$D = A^{-1} = \left(1, \frac{x}{1-x}\right)^{-1} = \left(1, \frac{-x}{1+x}\right).$$

Then we have the Motzkin numbers related with the Catalan numbers by

$$m_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} c_{k+1}. \tag{4}$$

In fact, the formula (1) and (4) are the special case of invertible transformation

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \Longleftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k,$$

where a_n, b_n are two given sequences. The transformation is widely used in the study of integer sequences and is called the binomial transform. When used to accelerate the speed of convergence of a sequence, it is called the Euler transform.

Theorem 2. *For the given sequences a_n, b_n, we have the transformation*

$$b_n = \sum_{k=0}^{n} \binom{2n-k}{k} a_{n-k} \Longleftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} c_{n,k} b_k$$

where $c_{n,k} = \frac{2k+1}{n+k+1}\binom{2n}{n+k}$.

Proof. Consider the inverse of Riordan array B, we have

$$E = B^{-1} = \left(\frac{1}{1-x}, \frac{x}{(1-x)^2}\right)^{-1} = (c(-x), xc^2(-x)),$$

where $c(x)$ is the generating function of the Catalan numbers. Then we get

$$b_n = \sum_{k=0}^{n} \binom{2n-k}{k} a_{n-k} \Longleftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} c_{n,k} b_k$$

where the generating function of $(c_{n,k})_{n\in N}$ is $x^k c^{2k+1}(x)$, $k = 0, 1, 2, \ldots$.

By [9], the general term of Riordan array $\left(\frac{1}{\sqrt{1-4x}}, xc^2(x)\right)$ is $\binom{2n}{n-k}$. Note that

$$x^k c^{2k+1}(x) = \frac{1}{\sqrt{1-4x}} \cdot x^k c^{2k}(x) - \frac{1}{\sqrt{1-4x}} \cdot x^{k+1} c^{2k+2}(x).$$

Then we have the general term of Riordan array $(c(x), xc^2(x))$ is

$$\binom{2n}{n-k} - \binom{2n}{n-k-1} = \frac{2k+1}{n+k+1}\binom{2n}{n+k}.$$

Therefore

$$c_{n,k} = \frac{2k+1}{n+k+1}\binom{2n}{n+k}. \quad \blacksquare$$

Corollary 3. *The Catalan numbers are related with the Schröder numbers by*

$$c_n = \sum_{k=0}^{n} (-1)^{n-k} c_{n,k} s_k. \tag{5}$$

From the proof of Theorem 2, we see the general term of Riordan array $(c(x), xc^2(x))$ is $c_{n,k}$. In the case of $k = 1$, it is the sequence A000245 in [8]. We further have the following properties for general k.

Theorem 4. *For $n \geq 0$, the sequence $c_{n,k}$ satisfies*

$$\sum_{k=0}^{n} c_{n,k} = \binom{2n}{n}.$$

Proof. For the Riordan array $(c(x), xc^2(x))$, consider the sequence $(1, 1, 1, \ldots)$ whose generating function is $a(x) = \frac{1}{1-x}$, then

$$b(x) = c(x)a(xc^2(x)) = \frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

So the result follows by the Riordan array theory. $\blacksquare$

Theorem 5. *For $n \geq 0$, the sequence $c_{n,k}$ satisfies*

$$\sum_{k=0}^{n}(2k+1)c_{n,k} = 4^n.$$

Proof. For the Riordan array $(c(x), xc^2(x))$, consider the sequence $(1, 3, 5, \ldots)$ whose generating function is $a(x) = \frac{1+x}{(1-x)^2}$, then

$$b(x) = c(x)a(xc^2(x)) = \frac{1}{1-4x} = \sum_{n=0}^{\infty} 4^n x^n.$$

So the result follows by the Riordan array theory. ■

Theorem 6. *For the given sequences a_n, b_n, we have the transformation*

$$b_n = \sum_{k=0}^{n} s_{n,k} a_k \Longleftrightarrow a_n = \sum_{k=0}^{n} (-1)^{n-k} t_{n,k} b_k$$

where the generating function of $(t_{n,k})_{n\in N}$ *is* $\frac{1-x-\sqrt{1-6x+5x^2}}{2x} \cdot \left(\frac{1-3x-\sqrt{1-6x+5x^2}}{2x}\right)^k$, $k = 0, 1, 2, \ldots$.

Proof. Let the Riordan array

$$F = C^{-1} = \left(\frac{1}{1-x}, \frac{x}{1-3x+x^2}\right)^{-1} = \left(\frac{1+x-\sqrt{1+6x+5x^2}}{-2x}, \frac{1+3x-\sqrt{1+6x+5x^2}}{-2x}\right).$$

Then we have the desired result. ■

Corollary 7. *The Motzkin numbers are related with the Schröder numbers by*

$$m_n = \sum_{k=0}^{n} (-1)^{n-k} t_{n,k} s_{k+1}, \tag{6}$$

3. Combinatorial proof of identity (1)

In this section, we give a combinatorial proof for the identity (1). For convenience, we denote the up step of a lattice path as u, the down step d and the (double) horizontal step h. We define the height of a step is the larger y-coordinate of the step, the height of a path is the largest height of its steps. The mountain of a lattice path is the subpath from an up step of height one to the following down step of height one.

Note that the mountain of height less than three in a Dyck path is ud or $u\,\underbrace{ud\cdots ud}_{ud \text{ pairs}}\,d$, we easily have the following property.

Proposition 8. *The set of all Dyck paths of semilength n which have k mountains and are of height less than three is enumerated by $\binom{n}{k}$.*

In order to prove the identity (1), we proceed to establish a bijection, and its domain and codomain denoted as:

$$\phi : \mathcal{D}_{n,0} \times \mathcal{M}_0 \bigcup \mathcal{D}_{n,1} \times \mathcal{M}_1 \bigcup \mathcal{D}_{n,2} \times \mathcal{M}_2 \bigcup \cdots \bigcup \mathcal{D}_{n,n} \times \mathcal{M}_n \Longrightarrow \mathcal{D}_n,$$

where $\mathcal{D}_n$ is the set of Dyck paths of semilength n, $\mathcal{D}_{n,k}$ the set of Dyck paths of semilength n with k mountains of height at most two, and $\mathcal{M}_n$ the set of Motzkin paths of length n ending with horizontal step, which is enumerated by m_{k-1}. We define the set products appearing in the domain as follows:

For a path $D \in \mathcal{D}_{n,k}$, we decompose it into k segments $D = D_1 D_2 \cdots D_k$, where the segment D_j is the j-th mountain of D. For a path $M \in \mathcal{M}_k$, write as $M = M_1 M_2 \cdots M_k$, where M_j is the j-th step of M. We map the j-th segment D_j and the j-th step M_j into the j-th part P_j of a Dyck path $P \in \mathcal{D}_n$, that is, $\phi(D_j \times M_j) = P_j$. Then we construct the corresponding Dyck path P by

$$\phi(D \times M) = \phi(D_1 \times M_1) \cdots \phi(D_k \times M_k) = P.$$

For convenience, we write $D_j = uQd$, where u is the j-th up step of height 1, d is the j-th down step of height 1, and Q is the subpath between u and d (in fact, Q is of form $\underbrace{ud \cdots ud}_{i \text{ pairs },i \geq 0}$). Denote Q^{-1} as the path changing u to d, and d to u in Q (that is, Q^{-1} is of form $\underbrace{du \cdots du}_{i \text{ pairs },i \geq 0}$). Now we consider each approach $\phi(D_j \times M_j)$ $(1 \leq j \leq k)$ in the following cases.

1. If $M_j = u$
   (a) If $j = 1$, then $\phi(D_1 \times M_1) = uQu$.
   (b) If $M_{j-1} = u$, then $\phi(D_j \times M_j) = uQ^{-1}u$.
   (c) If $M_{j-1} = d$, then $\phi(D_j \times M_j) = dQu$.
   (d) If $M_{j-1} = h$, then $\phi(D_j \times M_j) = uQu$, if M_{j-1} is of height 0; $\phi(D_j \times M_j) = dQ^{-1}u$, otherwise.
2. If $M_j = d$
   (a) If $M_{j-1} = u$, then $\phi(D_j \times M_j) = uQ^{-1}d$.
   (b) If $M_{j-1} = d$, then $\phi(D_j \times M_j) = dQ^{-1}d$.
   (c) If $M_{j-1} = h$, then $\phi(D_j \times M_j) = dQ^{-1}d$.
3. If $M_j = h$
   (a) If $j = 1$, then $\phi(D_1 \times M_1) = uQd$.
   (b) If $M_{j-1} = u$, then $\phi(D_j \times M_j) = uQ^{-1}u$.
   (c) If $M_{j-1} = d$, then $\phi(D_j \times M_j) = dQd$, if M_j is of height 0; $\phi(D_j \times M_j) = dQ^{-1}u$, otherwise.
   (d) If $M_{j-1} = h$, then $\phi(D_j \times M_j) = uQd$, if M_j is of height 0; $\phi(D_j \times M_j) = dQ^{-1}u$, otherwise.

From the construction, we easily see that P is a Dyck path of semilength n. In order to show that ϕ is a bijection, we construct the reverse map ϕ^{-1}.

For a Dyck path P of semilength n, we will use the induction to decompose P into k segments, and each segment P_j is corresponding to a mountain D_j of height less than 3 and a step M_j, denoted as $\phi^{-1}(P_j) = D_j \times M_j$ $(1 \leq j \leq k)$. For convenience, we denote $Q = \underbrace{ud \cdots ud}_{i \text{ pairs}, i \geq 0}$ and $Q^{-1} = \underbrace{du \cdots du}_{i \text{ pairs}, i \geq 0}$. Each approach is constructed in the following cases:

1. We first construct P_1, and the corresponding D_1, M_1.
   (a) If the height of the first mountain of P is less than 3, then P_1 is the first mountain, and $\phi^{-1}(P_1) = P_1 \times h$.
   (b) If the height of the first mountain of P is larger than 3, then P_1 consists of the steps before the first up step of height 3, that is, $P_1 = uQu$, and $\phi^{-1}(P_1) = uQd \times u$.
2. Suppose we have constructed $P_1, \ldots, P_{j-1}$ and the corresponding $D_1, M_1, \ldots, D_{j-1}, M_{j-1}$, then we construct P_j, D_j, M_j as follows.
   (a) If $M_{j-1} = u$,
      (i) If the steps after P_{j-1} are $uQ^{-1}uu \cdots$ (in the following text, we simply write as $uQ^{-1}uu$), then $P_j = uQ^{-1}u$, and $\phi^{-1}(P_j) = uQd \times u$.
      (ii) If the steps after P_{j-1} are $uQ^{-1}dd$. then $P_j = uQ^{-1}d$, and $\phi^{-1}(P_j) = uQd \times d$,
      (iii) If the steps after P_{j-1} are $uQ^{-1}ud$, then $P_j = uQ^{-1}u$, and $\phi^{-1}(P_j) = uQd \times h$.
   (b) If $M_{j-1} = d$,
      (i) If the steps after P_{j-1} are $dQuu$, then $P_j = dQu$, and $\phi^{-1}(P_j) = uQd \times u$.
      (ii) If the steps after P_{j-1} are $dQ^{-1}dd$, then $P_j = dQ^{-1}d$, and $\phi^{-1}(P_j) = uQd \times d$.
      (iii) If the steps after P_{j-1} are $dQdu$, then $P_j = dQd$, and $\phi^{-1}(P_j) = uQd \times h$.
      (iv) If the steps after P_{j-1} are $dQ^{-1}ud$, then $P_j = dQ^{-1}u$, and $\phi^{-1}(P_j) = uQd \times h$.
   (c) If $M_{j-1} = h$,

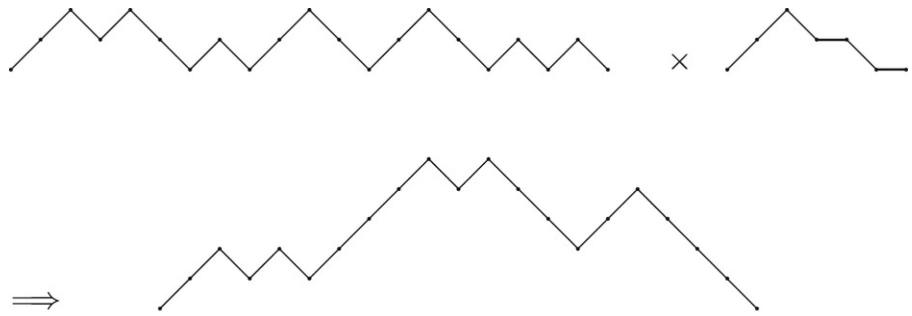

Fig. 1. Correspondence between $D \times M$ and P.

(i) If the steps after P_{j-1} are $uQuu$, then $P_j = uQu$, and $\phi^{-1}(P_j) = uQd \times u$.
(ii) If the steps after P_{j-1} are $uQ^{-1}uu$, then $P_j = uQ^{-1}u$, and $\phi^{-1}(P_j) = uQd \times u$.
(iii) If the steps after P_{j-1} are $dQ^{-1}dd$, then $P_j = dQ^{-1}d$, and $\phi^{-1}(P_j) = uQd \times d$.
(iv) If the steps after P_{j-1} are $uQdu$, then $P_j = uQd$, and $\phi^{-1}(P_j) = uQd \times h$.
(v) If the steps after P_{j-1} are $dQ^{-1}ud$, then $P_j = dQ^{-1}u$, and $\phi^{-1}(P_j) = uQd \times h$.

Then the map $\phi^{-1} : P \mapsto D \times M$ satisfying

$$D = D_1 D_2 \cdots D_k \quad \text{and} \quad M = M_1 M_2 \cdots M_k$$

An example is given in Fig. 1, given a paths of $\mathcal{D}_{10,6}$

$$D = u\ u\ d\ u\ d\ d\ u\ d\ u\ u\ d\ d\ u\ u\ d\ d\ u\ d\ u\ d,$$

where $D_1 = u\ u\ d\ u\ d\ d$, $D_2 = u\ d$, $D_3 = u\ u\ d\ d$, $D_4 = u\ u\ d\ d$, $D_5 = u\ d$, $D_6 = u\ d$, and a path of $\mathcal{M}_6$

$$M = u\ u\ d\ h\ d\ h,$$

where $M_1 = u$, $M_2 = u$, $M_3 = d$, $M_4 = h$, $M_5 = d$, $M_6 = h$. From the construction of ϕ, we get a Dyck path of semilength 10

$$P = u\ u\ d\ u\ d\ u\ u\ u\ u\ d\ u\ d\ d\ d\ u\ u\ d\ d\ d\ d,$$

where $P_1 = u\ u\ d\ u\ d\ u$, $P_2 = u\ u$, $P_3 = u\ d\ u\ d$, $P_4 = d\ d\ u\ u$, $P_5 = d\ d$, $P_6 = d\ d$.

4. Combinatorial proof of identity (3)

In this section, we give a combinatorial proof for the identity (3). Firstly, we give a combinatorial representation of the sequence $s_{n,k}$.

Proposition 9. *The set of all Schröder paths of semilength n which have k mountains and are of height less than three is enumerated by $s_{n,k}$.*

Proof. We let $P(x, z)$, $D(x, z)$ and $T(x, z)$ be the ordinary functions for the numbers of Schröder paths of semilength n with k mountains of height at most 0, 1 and 2, respectively. Here, x counts half the number of steps (the semilength) and z counts the number of mountains.

The equations for the generating functions $P(x, z)$, $D(x, z)$ and $T(x, z)$ are obtained from the "first return decomposition" of Schröder path. For a Schröder path S of height at most 2 (or 1), we have $S = uS_1dS_2$ or $S = hS_3$, where S_1 is a Schröder path of height at most 1 (or 0), S_2 and S_3 is a Schröder path of height at most 2 (or 1). Similarly, for a Schröder path S of height at most 0, then $S = hS_1$, where S_1 is a Schröder path of height at most 0. Hence

$$\begin{cases} P(x,z) &=& 1 + xP(x,z); \\ D(x,z) &=& 1 + xD(x,z) + xzP(x,1)D(x,z); \\ T(x,z) &=& 1 + xT(x,z) + xzD(x,1)T(x,z). \end{cases}$$

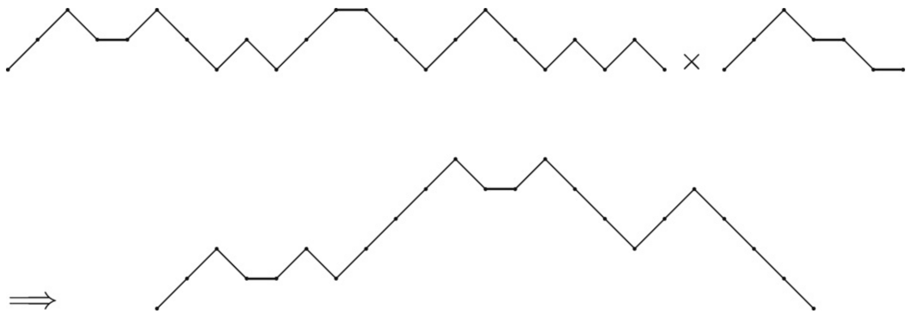

Fig. 2. Correspondence between $S \times M$ and P.

We arrive at

$$T(x, z) = \frac{1 - 3x + x^2}{1 - 4x + 4x^2 - x^3 + z(x - x^2)} = \frac{1/(1 - x)}{1 - zx/(1 - 3x + x^2)}.$$

Note that the generating function of $(s_{n,k})_{n \in N}$ is $\frac{1}{1-x}\left(\frac{x}{1-3x+x^2}\right)^k$. Finding the coefficient of $x^n z^k$ in the generating function $T(x, z)$, we obtain that the number of Schröder paths of semilength n with k mountains of height less than three is $s_{n,k}$. ■

In order to prove the identity (3), we process to established a bijection ψ.

$$\psi : \mathcal{S}_{n,0} \times \mathcal{M}_0 \bigcup \mathcal{S}_{n,1} \times \mathcal{M}_1 \bigcup \mathcal{S}_{n,2} \times \mathcal{M}_2 \bigcup \cdots \bigcup \mathcal{S}_{n,n} \times \mathcal{M}_n \Longrightarrow \mathcal{S}_n,$$

where $\mathcal{S}_n$ is the set of Schröder paths of semilength n, $\mathcal{S}_{n,k}$ the set of Schröder paths of semilength n with k mountains of height at most two. We define the set products appearing in the domain as follows:

For a Schröder path $S \in \mathcal{S}_{n,k}$ and a Motzkin path $M \in \mathcal{M}_k$, we omit the horizontal steps of S and reduce to a Dyck path S' with k mountains of height at most two. Using the bijection ϕ in Section 3, we obtain a Dyck path P'. It leads to a Schröder path P by inserting horizontal steps into the corresponding place of P'. That is, if the i-th step of S is a horizontal step, then we insert a horizontal step into P' as the i-th step of P.

An example is given in Fig. 2, given a path of $\mathcal{S}_{12,6}$

$$S = u\ u\ d\ h\ u\ d\ d\ u\ d\ u\ u\ h\ d\ d\ u\ u\ d\ d\ u\ d\ u\ d,$$

and a path of $\mathcal{M}_6$

$$M = u\ u\ d\ h\ d\ h.$$

The corresponding Schröder path of semilength 12 is

$$P = u\ u\ d\ h\ u\ d\ u\ u\ u\ u\ d\ h\ u\ d\ d\ d\ u\ u\ d\ d\ d\ d.$$

Acknowledgments

The authors would like to thank L.W. Shapiro for helpful comments. This work was done under the auspices of the Youth Fundation of DUT (NO. 200637).

References

[1] F.R. Bernhart, Catalan, Motzkin, and Riordan numbers, Discrete Math. 204 (1999) 73–112.
[2] E. Deutsch, L.W. Shapiro, A bijection between ordered trees and 2-Motzkin paths and its many consequences, Discrete Math. 256 (2002) 655–670.
[3] R. Donaghey, L.W. Shapiro, Motzkin numbers, J. Combin. Theory (A) 23 (1977) 291–301.
[4] H.W. Gould, L.C. Hsu, Some new inverse series relations, Duke Math. J. 40 (1973) 885–891.
[5] J. Riordan, Inverse relations and combinatorial identitities, Amer. Math. Monthly 71 (1964) 485–498.
[6] D. Rogers, Pascal triangles, Catalan numbers and renewal arrays, Discrete Math. 22 (1978) 301–310.

[7] L.W. Shapiro, S. Getu, W.-J. Woan, L.C. Woodson, The Riordan group, Discrete Appl. Math. 34 (1991) 229–239.
[8] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences.
[9] R. Sprugnoli, Riordan arrays and combinatorial sums, Discrete Math. 132 (1994) 267–290.
[10] R. Sprugnoli, Riordan arrays and the Abel-Gould identity, Discrete Math. 142 (1995) 213–233.
[11] R.P. Stanley, Enumerative Combinatorics, vol. 2, Cambridge University Press, Cambridge, 1999.
[12] Robert A. Sulanke, Bijective Recurrences concerning Schröoder paths, Electron. J. Combin. 5 (1998) #R47.