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Motzkin Numbers, Central Trinomial Coefficients and Hybrid Polynomials

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Abstract

We show that the formalism of hybrid polynomials, interpolating between Hermite and Laguerre polynomials, is very useful in the study of Motzkin numbers and central trinomial coefficients. These sequences are identified as special values of hybrid polynomials, a fact which we use to derive their generalized forms and new identities satisfied by them.

1 Introduction

The central trinomial coefficients (CTC) c_n are defined as the coefficients of x^n in the expansion of $(1+x+x^2)^n$. Various expressions have been given for these coefficients (see, for example, [2, 11]); here we will refer to the following form, see A002426 and A001006 in [13]:

$$c_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{(k!)^2\,(n-2k)!}, \tag{1}$$

which is the most useful for our purposes. An alternative approach according to which one can define the central trinomial coefficients is to follow [5] and to consider the Laurent polynomial

$$(1+x+x^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j}_2 x^j, \tag{2}$$

where the appropriate trinomial coefficients $\binom{n}{j}_2$ are given by:

$$\binom{n}{m}_2 = \sum_{j\geq 0} \frac{n!}{j!(m+j)!(n-2j-m)!}. \tag{3}$$

Comparing Eqs. (1) and (3) one immediately derives $c_n = \binom{n}{0}_2$.

The Motzkin numbers (MN) are connected to the number of planar paths associated with the combinatorial interpretation of c_n. They are defined as follows (see [2, 11]):

$$m_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{k!(k+1)!(n-2k)!}. \tag{4}$$

Similarly to the central trinomial coefficients also the Motzkin numbers can be expressed in terms of the coefficients $\binom{n}{m}_2$ simply as follows:

$$m_n = \frac{1}{n+1}\binom{n+1}{1}_2. \tag{5}$$

In the next sections we shall demonstrate that the Motzkin numbers m_n and the central trinomial coefficients c_n can be treated on the same footing and framed within the context of the theory of the hybrid polynomials, (see [8]). Recalling basic properties of the hybrid polynomials interpolating between standard two-variable Hermite and Laguerre polynomials we shall show that the central trinomial coefficients and the Motzkin numbers satisfy a simple recurrence which relates c_{n+1}, c_n and m_{n-1}. Moreover, the methods developed on the base on the hybrid polynomials formalism allow natural generalization of the notions of the central trinomial coefficients and the Motzkin numbers which is useful for the investigation of their properties.

Definition 1. The Hermite-Kampé de Fériét (HKdF) polynomials are defined by the following expression

$$H_n(x,y) = n!\sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2k}y^k}{k!(n-2k)!} \quad , \tag{6}$$

where $x, y \in C$. For special values of x and y the HKdF polynomials reduce to the well known ordinary Hermite polynomials [1]

$$H_n(x, -\frac{1}{2}) = He_n(x), \quad H_n(2x,-1) = H_n(x), \tag{7}$$

$H_n(x) = 2^{\frac{n}{2}} He_n(\sqrt{2}x)$.

Remark 2. The HKdF polynomials can be also defined through the following operational rules:

$$H_n(x,y) = \exp(y\frac{\partial^2}{\partial x^2}) \cdot x^n, \tag{8}$$

$$H_n(x,y) = (x + 2y\frac{\partial}{\partial x})^n \cdot \mathbf{1}, \tag{9}$$

and the relevant exponential generating function:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x,y) = \exp(xt + yt^2). \tag{10}$$

Other properties of the HKdF polynomials can be found in the review [7].

Definition 3. The two-variable Laguerre polynomials are defined as follows (see [6]):

$$L_n(x,y) = n! \sum_{k=0}^{n} \frac{(-1)^k y^{n-k} x^k}{(k!)^2 (n-k)!}. \tag{11}$$

They reduce to the ordinary Laguerre polynomials for the value of the argument $y = 1$.

Remark 4. The two-variable Laguerre polynomials (11) are also defined by the operational rule

$$L_n(x,y) = (y - \widehat{D}_x^{-1})^n \mathbf{1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k y^{n-k} \widehat{D}_x^{-k} \mathbf{1}, \tag{12}$$

where $\widehat{D}_x^{-1}$ is the inverse derivative operator whose action on the unity is given as follows:

$$\widehat{D}_x^{-k} \mathbf{1} = \frac{x^k}{k!}. \tag{13}$$

Indeed, substituting Eq. (13) into Eq. (12) we immediately recover Eq. (11) in the following form:

$$L_n(x,y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k y^{n-k} x^k}{k!}. \tag{14}$$

Hereby, we note that according to [8] the inverse derivative operator action on a function $f(x)$ is specified as follows:

$$\widehat{D}_x^{-k} f(x) = \frac{1}{(k-1)!} \int_0^x (x-\xi)^{k-1} f(\xi)\, d\xi, \quad (k = 1, 2, 3, \ldots), \tag{15}$$

and we specify its zeroth order action on the function $f(x)$ by the function itself:

$$\widehat{D}_x^0 \cdot f(x) = f(x). \tag{16}$$

Next we will introduce the hybrid Hermite-Laguerre polynomials combining the individual characteristics of both Laguerre and Hermite polynomials and explore their properties in the context of the central trinomial coefficients and Motzkin numbers.

Definition 5. The hybrid Hermite-Laguerre polynomials $\Pi_n(x,y)$ are defined by the following expression:

$$\Pi_n(x,y) = H_n(y, \widehat{D}_x^{-1}) \mathbf{1}. \tag{17}$$

Proposition 6. *The central trinomial coefficients are the particular case of the hybrid Hermite-Laguerre polynomials:*

$$c_n = \Pi_n(1,1). \tag{18}$$

Proof. Note that from the definition of HKdF and from Eq. (13), we find

$$\Pi_n(x,y) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^{n-2k}\widehat{D}_x^{-k}}{k!(n-2k)!}\mathbf{1} = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{y^{n-2k}x^k}{(k!)^2\,(n-2k)!} \tag{19}$$

and therefore the comparison of Eq. (19) with Eq. (1) yields Eq. (18). □

2 Central trinomial coefficients and special functions

In this Section we will focus our attention on some properties of the central trinomial coefficients and the calculation of their generating function.

Definition 7. I_0 denotes the zeroth order modified Bessel function of the first kind. $I_n(x)$ is defined as (see [4]):

$$I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{n+2r}}{r!(n+r)!}, \tag{20}$$

which is a particular case of the Tricomi function of α^{th} order where the parameter α is not necessarily an integer:

$$C_\alpha(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!\Gamma(r+\alpha+1)} = x^{-\frac{\alpha}{2}} I_\alpha(2\sqrt{x}). \tag{21}$$

Proposition 8. *The exponential generating function for the CTC is given by:*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} c_n = \exp(t) I_0(2t). \tag{22}$$

Proof. Using the definition of Eq. (17) and the generating function (10) of the HKdF polynomials we obtain:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\Pi_n(x,y) = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(y, \widehat{D}_x^{-1})\mathbf{1} = \exp(yt + \widehat{D}_x^{-1}t^2)\mathbf{1}. \tag{23}$$

The exponential on the r.h.s of Eq. (23) can be disentangled because y and $\widehat{D}_x^{-1}$ commute. Thus we get:

$$\exp(yt)\exp(\widehat{D}_x^{-1}t^2)\mathbf{1} = \exp(yt)\sum_{r=0}^{\infty} \frac{\widehat{D}_x^{-r}t^{2r}}{r!}\mathbf{1} = \exp(yt)\sum_{r=0}^{\infty} \frac{x^r t^{2r}}{(r!)^2}. \tag{24}$$

hen Eq. (22) follows from Eqs. (23), (24), (20) and (18) and the proposition is proved. □

Proposition 9. *The central trinomial coefficient can be expressed in terms of Legendre polynomials $P_n(x)$:*

$$c_n = i^n\sqrt{3^n}P_n(-\frac{i}{\sqrt{3}}). \tag{25}$$

Proof. As it has been shown in [6], hybrid polynomials $\Pi_n(x,y)$ have the following ordinary generating function:

$$\sum_{n=0}^{\infty} t^n \Pi_n(x,y) = \frac{1}{\sqrt{1-2yt+(y^2-4x)t^2}}, \quad \left|\sqrt{y^2-4x}t\right| < 1. \tag{26}$$

Since Legendre polynomials satisfy the analogous relation (see [9]) written below:

$$\sum_{n=0}^{\infty} t^n P_n(x) = \frac{1}{\sqrt{1-2xt+t^2}}, \quad |t| < 1, \tag{27}$$

we can easily rearrange the summation in (26) to obtain

$$\Pi_n(x,y) = (y^2-4x)^{\frac{n}{2}} P_n\left(\frac{y}{\sqrt{y^2-4x}}\right), \tag{28}$$

which, on account of Eq. (18), yields Eq. (25). □

Corollary 10. *The central trinomial coefficients satisfy the following recurrence [3]*

$$(n+1)c_{n+1} = (2n+1)c_n + 3nc_{n-1}. \tag{29}$$

Proof. Eq. (29) follows from Eq. (25) and from the well known recurrence for the Legendre polynomials [9]:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x). \tag{30}$$

□

So far, we have shown that the central trinomial coefficients can be written in terms of Legendre polynomials.

For an alternative derivation of the results of this section see [10].

In the next section we will demonstrate that analogous relations can be obtained for the Motzkin numbers too.

3 Motzkin numbers and special functions

In this section we concentrate on the calculation of the generating function for the associated hybrid polynomials, which will be defined below, and we study their properties related to the Motzkin numbers.

Definition 11. Associated CTC are defined by

$$c_n^\alpha = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n!}{(n-2k)!k!\Gamma(k+\alpha+1)}. \tag{31}$$

and the Motzkin numbers can be identified as a particular case of the associated CTC:

$$m_n = c_n^1. \tag{32}$$

Definition 12.
Recall the operator $\widehat{D}_{x,\alpha}^{-1}$ defined in [8] via the following rule for its action on the unity:

$$\widehat{D}_{x,\alpha}^{-n}\mathbf{1} = \frac{x^n}{\Gamma(n+\alpha+1)}. \tag{33}$$

Definition 13. The associated hybrid Hermite-Laguerre polynomials $\Pi_n^{(\alpha)}(x,y)$ are defined as follows:

$$\Pi_n^\alpha(x,y) = H_n(y, \widehat{D}_{x,\alpha}^{-1})\mathbf{1} = n!\sum_{kr=0}^{\left[\frac{n}{2}\right]} \frac{x^k y^{n-2k}}{(n-2k)!k!\Gamma(k+\alpha+1)}. \tag{34}$$

Proposition 14. *The associated hybrid polynomials $\Pi_n^{(\alpha)}(x,y)$ possess the following generating function:*

$$\sum_{n=0}^\infty \frac{t^n}{n!}\Pi_n^\alpha(x,y) = \exp(yt)(xt^2)^{-\frac{\alpha}{2}} I_\alpha(2t\sqrt{x}). \tag{35}$$

Proof. Using Eq. (34) and the generating function for the HKdF polynomials Eq. (10) we find that

$$\sum_{n=0}^\infty \frac{t^n}{n!}\Pi_n^\alpha(x,y) = \exp(yt + \widehat{D}_{x,\alpha}^{-1}t^2)\mathbf{1} = \exp(yt)\sum_{r=0}^\infty \frac{\widehat{D}_{x,\alpha}^{-r}t^{2r}}{r!}\mathbf{1}, \tag{36}$$

which yields Eq. (35) with account of Eq. (33). □

Corollary 15. *The MN can be identified as the particular case of the associated hybrid Hermite-Laguerre polynomials $\Pi_n^{(\alpha)}(x,y)$*

$$m_n = \Pi_n^1(1,1), \tag{37}$$

and satisfy the following identity:

$$\sum_{n=0}^\infty \frac{t^n}{n!}\Pi_n^1(1,1) = \frac{\exp(t)}{t} I_1(2t). \tag{38}$$

It is now evident that many of the properties of the CTC and of the MN can be derived from those of the hybrid polynomials.

Theorem 16. *The MN and the CTC are linked by the recurrence [3]*

$$c_{n+1} = c_n + 2n \cdot m_{n-1} \quad . \tag{39}$$

Proof. The HKdF polynomials satisfy the following recurrence relation [8]:

$$H_{n+1}(x,y) = H_n(x,y) + 2ynH_{n-1}(x,y). \tag{40}$$

The same recurrence, written in operational form for the hybrid case, reads as follows:

$$H_{n+1}(y, \widehat{D}_x^{-1})\mathbf{1} = \left[H_n(y, \widehat{D}_x^{-1}) + 2\widehat{D}_x^{-1} n H_{n-1}(y, \widehat{D}_x^{-1})\right]\mathbf{1}. \tag{41}$$

Then, employing the result of the action of the inverse derivative on the $H_n(y, \widehat{D}_x^{-1})\mathbf{1}$ as written below

$$\widehat{D}_x^{-1} H_n(y, \widehat{D}_x^{-1})\mathbf{1} = x\Pi_n^1(x,y), \tag{42}$$

we find from (40) the following recurrence:

$$\Pi_{n+1}(x,y) = \Pi_n(x,y) + 2nx\Pi_{n-1}^1(x,y). \tag{43}$$

Hence, we have proved also the particular case of this identity, given by Eq. (39). □

Corollary 17. *The MN can be expressed in terms of the central trinomial coefficients as follows:*

$$m_n = \frac{c_{n+2} - c_{n+1}}{2(n+1)}. \tag{44}$$

Corollary 18. *Define the p-associated CTC (p is an integer) in the following way:*

$$c_n^p = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{1}{(n-2k)!k!(k+p)!}. \tag{45}$$

Then, with help of identities Eqs. (41) and (43), we obtain the generalized form of the formula Eq. (44):

$$c_n^{p+1} = \frac{c_{n+2}^p - c_{n+1}^p}{2(n+1)}. \tag{46}$$

Note that for $p > 1$, the p-associated CTC c_n^p are not integers. For example, the first 11 c_n^p numbers ($n = 0 \dots 10$) for $p = 0, 1, 2$ are listed in Table 1.

n	c_n^0	c_n^1	$6 \cdot c_n^2$
0	1	1	3
1	1	1	3
2	3	2	5
3	7	4	9
4	19	9	18
5	51	21	38
6	141	51	84
7	393	127	192
8	1107	323	451
9	3139	835	1083
10	8953	2188	2649

Table 1. The p-associated CTC c_n^p for $n = 0, 1, 2, \ldots, 10$ and $p = 0, 1, 2$.
In the second column, *i.e.*, for $p = 1$we have the usual Motzkin numbers.

Before concluding this paper, we will add the following note on the further generalization of the CTC and MN as a consequence of the approach developed in the present work.

Definition 19. The m^{th} order p-associated CTC are defined as follows:

$$ {}_m c_n^p = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{1}{(n-mk)!k!(k+p)!}. \tag{47}$$

The above defined family of central trinomial coefficients is linked to the higher order hybrid polynomials. Their properties can be explored along the lines developed above. We just note, that they satisfy the following recurrence:

$$ {}_m c_{n+1}^p = {}_m c_n^p + m \frac{n!}{(n-m+1)!} {}_m c_{n-m+1}^p, \tag{48}$$

which is a straighforward generalization of Eq. (39). Observe that Eqs. (44), (46) and (48) are simple recurrences that clearly share common structure revealing inherent connection between c_n, c_n^p and ${}_m c_n^p$.

4 Discussion

In the present work we have reinterpreted the central trinomial coefficients and Motzkin numbers employing the general formalism, which underlies the theory of the hybrid polynomials. The analogous results could be achieved, using properties of the hypergeometric functions. In fact, using Eq. (6) and the definition of the hypergeometric function ${}_pF_q$, see [12], the following representation is valid

$$ H_n(x, y) = x^n \ {}_2F_0 \left(-\frac{n}{2}, \frac{1-n}{2}; \frac{4y}{x} \right), \tag{49}$$

where ${}_2F_0$ is the hypergeometric function. Most of the results of this paper may also be derived from this observation.

Even though we have referred to the coefficients ${}_m c_n^p, m > 2, p > 0$ as "central trinomial", they do not have the same interpretation as in the case[1] $p = 0, m = 2$. We have noted that for $p = 0, m = 1$, the CTC produce the Motzkin numbers. Thorough discussion of their combinatorial interpretation is intended for future investigations.

Since through Eq. (17) all the findings of this paper are related to the HKdF polynomials $H_n(x, y)$ it seems legitimate to look for their combinatorial interpretation. We just point that

[1] The coefficients of x^n of the expansion $(1 + x + x^m)^n$ are ${}_m d_n = n! \sum_{k=0}^{\left[\frac{n}{m}\right]} \frac{1}{k!((m-1)k)!(n-mk)!}$ and their properties can be also framed within the context of the properties of the hybrid polynomials.

for a large class of arguments x, y of $H_n(x, y)$ the resulting integer sequences can be given a precise representation which may be helpful in searching a combinatorial interpretation of CTC. We quote two examples of such a situation:

(a) For $x = 1$, $y = 1/2$ we have $H_n(1, 1/2) = {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; 2\right)$ which generates the sequence

$$1, 1, 2, 4, 10, 26, 76, 232, \ldots$$

for $n = 0, 1, 2, \ldots$. They are called involution numbers (see A000085 in [13]), whose classical combinatorial interpretation is the number of partitions of a set of n distinguishable objects into subsets of size one and two. This sequence counts also permutations consisting exclusively of fixed points and transpositions.

(b) Another example is supplied by the choice $x = y = 1/2$. Then the quantity $2^n H_n(1/2, 1/2) = {}_2F_0\left(-\frac{n}{2}, \frac{1-n}{2}; 8\right)$ furnishes the following integer sequence:

$$1, 1, 5, 13, 73, 281, 1741, \ldots$$

for $n = 0, 1, 2, \ldots$, see A115329 in [13]. It counts the number of partitions of a set into subsets of size one and two with the additional requirment that the constituents of a set of size two are assigned two colors.

Many other instances of such combinatorial interpretations may be given by judicious choices of parameters x and y in Eq. (49).

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