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Motzkin Numbers

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In this paper Motzkin numbers M_n (which are related to Catalan numbers) are studied. The (known) connection to Tchebychev polynomials is discussed with applications to the Hankel matrices of Motzkin numbers. It is shown that the sequence M_n is logarithmically concave with lim $M_{n+1}/M_n = 3$. Finally, two ballot-number type sequences for M_n are derived, with an application to directed animals.

1. INTRODUCTION

In his forthcoming book [9] R. Stanley lists some 70 examples of enumeration problems which are counted by the *Catalan numbers* $C_n = \frac{1}{n+1}\binom{2n}{n}$. In addition, he shows that many of these settings give rise to closely related instances counted by the *Motzkin numbers* M_n, drawing mostly from material in the survey by Donaghey and Shapiro [2]. In the present paper we want to study several aspects of the Motzkin numbers. In Section 2 we consider a combinatorial setting particularly suited to our purposes. In Section 3 we demonstrate the close (and known) connection of Motzkin numbers to Tchebychev polynomials $U_n(x)$, giving several applications. In Section 4 we show that the sequence $M_1, M_2, M_3, \ldots$ is logarithmically concave and prove lim $M_{n+1}/M_n = 3$. In the last section two ballot-number type sequences are derived, illustrating the results with a few examples.

For background on the combinatorial coefficients involved, the reader is referred to any of the standard texts, e.g., [6, 9, 10].

2. A COMBINATORIAL SETTING

For convenience we list the defining recursions for Catalan and Motzkin numbers:

$$C_0 = 1,\, C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \qquad (n \geq 0) \tag{1}$$

$$M_0 = 1,\, M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k} \qquad (n \geq 0). \tag{2}$$

The first numbers are thus

n	0	1	2	3	4	5	6
C_n	1	1	2	5	14	42	132
M_n	1	1	2	4	9	21	51

Let $\mathcal{C}_n$ be the family of all $2 \times n$-arrays $\begin{matrix} a_1 a_2 \ldots a_n \\ b_1 b_2 \ldots b_n \end{matrix}$ with all a_i, b_i equal to 0 or 1 such that

(i) $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i \qquad (1 \leq k \leq n)$
(ii) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$.

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By verifying (1) one finds $|\mathcal{C}_n| = C_{n+1}$. Now let $\mathcal{M}_n$ be the subfamily of all arrays without $\genfrac{}{}{0pt}{}{1}{1}$ columns. Verifying (2) yields $|\mathcal{M}_n| = M_n$ where $|\mathcal{M}_0| = 1$ by definition. As examples we have

$$\mathcal{C}_2: \begin{matrix} 00 & 01 & 10 & 10 & 11 \\ 00 & 01 & 01 & 10 & 11 \end{matrix}$$

$$\mathcal{M}_3: \begin{matrix} 000 & 010 & 100 & 100 \\ 000 & 001 & 001 & 010 \end{matrix}$$

By interpreting $\genfrac{}{}{0pt}{}{1}{0}$ as a step to the right, $\genfrac{}{}{0pt}{}{0}{1}$ as a step to the left, and $\genfrac{}{}{0pt}{}{0}{0}$ as a loop, we obtain precisely the first example in [2]: M_n counts the number of n-step walks on the non-negative integers, starting and returning to 0 with steps $1, -1, 0$. All the standard formulae for the numbers M_n can be easily derived from this setting, e.g., $M_n = \sum_{k\geq 0} \binom{n}{2k} C_k$ and $C_{n+1} = \sum_{k=0}^{n} \binom{n}{k} M_k$.

3. Motzkin Numbers and Tchebychev Polynomials

It is well known that there is a connection between Motzkin numbers M_n and the Tchebychev polynomials $U_n(x)$. This connection (Proposition 1) can also be derived via the Riordan group (see [7]). We have chosen the present approach because it leads directly to the application concerning the Hankel matrices (Proposition 2).

Let $\mathcal{M}_h$ be the Motzkin family of the last section, and denote by $s_{h,n}$ the number of arrays with $\genfrac{}{}{0pt}{}{1}{0}$ in the first n columns, where $n \leq h$. Thus $s_{h,0} = M_h$ and $s_{h,1} = M_h - M_{h-1}$.

Lemma 1. *We have* $s_{h,n} = s_{h,n-1} - s_{h-1,n-1} - s_{h-2,n-2}$ *for* $1 \leq n \leq h$.

Proof. To compute $s_{h,n}$ we have to subtract from $s_{h,n-1}$ the number of arrays with $\genfrac{}{}{0pt}{}{1}{0}$ in the first $n-1$ columns and $\genfrac{}{}{0pt}{}{0}{0}$ resp. $\genfrac{}{}{0pt}{}{0}{1}$ in the n-th column. But these numbers are clearly $s_{h-1,n-1}$ resp. $s_{h-2,n-2}$. □

It follows from Lemma 1 by induction that

$$s_{h,n} = a_n M_h + a_{n-1} M_{h-1} + \cdots + a_0 M_{h-n} \tag{3}$$

with integer coefficients a_i. Note that the coefficients $a_0, \ldots, a_n$ are independent of h.

Consider now the *Motzkin polynomial* $S_n(x) = a_n x^n + \cdots + a_1 x + a_0$, with the a_is as in (3). Applying Lemma 1 and grouping the coefficients we arrive at the recursion

$$S_n(x) = (x-1)S_{n-1}(x) - S_{n-2}(x), \quad S_0(x) = 1, S_1(x) = x - 1. \tag{4}$$

In particular, $S_n(x)$ has degree n with leading coefficient $a_n = 1$.

The polynomials $S_n(x)$ will be our main tool in the study of the Motzkin numbers, using the following idea (usually called the symbolic method). Let $p(x) = \sum_{k=0}^{m} p_k x^k$ be any polynomial. Then we denote by $[p(x)]_{x=M}$ or simply $[p(x)]$ the number which results from the substitution $x^k \to M_k$ for all k, thus $[p(x)] = \sum_{k=0}^{m} p_k M_k$. Clearly, $[p(x) + q(x)] = [p(x)] + [q(x)]$ and $[cp(x)] = c[p(x)]$.

As our main example we note by our set-up

$$[x^k S_n(x)] = s_{n+k,n}. \tag{5}$$

In particular, we have

$$\begin{aligned} [x^k S_n(x)] &= 0 \qquad \text{for } k < n \\ [x^n S_n(x)] &= 1, \end{aligned} \tag{6}$$

because for $k < n$ there clearly is no array in $\mathcal{M}_{k+n}$ with n leading $\begin{smallmatrix}1\\0\end{smallmatrix}$ columns, whereas for $k = n$ there is exactly one such array in $\mathcal{M}_{2n}$, namely n $\begin{smallmatrix}1\\0\end{smallmatrix}$ columns followed by n $\begin{smallmatrix}0\\1\end{smallmatrix}$ columns. Note that $[S_0(x)] = 1$ and $[S_n(x)] = 0$ for all $n \geq 1$.

The following result summarizes the properties of the polynomials $S_n(x)$.

PROPOSITION 1. *We have*

(i) $S_n(x) = U_n(\frac{x-1}{2})$*, where* $U_n(x)$ *is the Tchebychev polynomial* $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$.
(ii) $S_n(x) = \sum_{k\geq 0}(-1)^k\binom{n-k}{k}(x-1)^{n-2k}$.
(iii) $(x-1)^n = \sum_{k\geq 0}(\binom{n}{k} - \binom{n}{k-1})S_{n-2k}(x)$.
(iv) $S_k(x)S_\ell(x) = S_{k+\ell}(x) + S_{k+\ell-2}(x) + S_{k+\ell-4}(x) + \cdots + S_{\ell-k}(x) \quad (0 \leq k \leq \ell)$.
(v) The roots of $S_n(x)$ *are* $2\cos\frac{k\pi}{n+1} + 1$ $(k = 1, \ldots, n)$.

PROOF. (i) The polynomials $U_n(x)$ satisfy the recursion

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad (n \geq 1)$$

(see [6]). Since $U_0(x) = S_0(x) = 1$, the result follows from (4).

(ii) We have (see [6])

$$U_n\left(\frac{x}{2}\right) = \sum_{k\geq 0}(-1)^k\binom{n-k}{k}x^{n-2k},$$

and hence

$$S_n(x) = \sum_{k\geq 0}(-1)^k\binom{n-k}{k}(x-1)^{n-2k}.$$

Tchebychev inversion (see [5, p. 62]) now yields (iii).

(iv) For $k = 0$ we have $S_0(x)S_\ell(x) = S_\ell(x)$, and (iv) is readily established by induction on k and (4).

(v) The roots of $U_n(x)$ are $\cos\frac{k\pi}{n+1}$ (see [5]), which implies the assertion. □

COROLLARY 1. *We have*

(i) $\sum_{j\geq 0}(-1)^j\sum_{k\geq 0}(-1)^k\binom{n-k}{k}\binom{n-2k}{j}M_j = 0 \quad (n \geq 1)$
(ii) $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}M_k = \begin{cases} 0 & \textit{for } n \textit{ odd} \\ C_{n/2} & \textit{for } n \textit{ even.} \end{cases}$

PROOF. (i) We know $[S_n(x)] = 0$ for $n \geq 1$ by (6). Taking $[p(x)]$ for the right-hand side of (ii) in the Proposition yields

$$0 = \sum_{k\geq 0}(-1)^k\binom{n-k}{k}\sum_{j\geq 0}(-1)^{n-2k-j}\binom{n-2k}{j}M_j$$

which is precisely (i).

(ii) Looking at Proposition 1(iii) we obtain for the left-hand side

$$[(x-1)^n] = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} M_k.$$

On the other hand, applying (6) to the right-hand side we find

$$\sum_{k\geq 0} \left(\binom{n}{k} - \binom{n}{k-1} \right) [S_{n-2k}(x)],$$

which is 0 for n odd and $\binom{n}{n/2}) - \binom{n}{n/2-1} = C_{n/2}$ for n even. $\square$

REMARK. Let $b_{n,j}$ be the coefficient of M_j in Corollary 1(i). Using the Zeilberger–Petkovšek algorithm [4] it can be shown that the $b_{n,j}$ admit no closed form (in hypergeometric terms), but satisfy the three term recursions:

$$(n+2-j)b_{n+2,j} - (n+2)b_{n+1,j} + (n+2+j)b_{n,j} = 0 \quad (j \text{ fixed})$$

$$3(j+1)(j+2)b_{n,j+2} + (2j+3)(j+1)b_{n,j+1} + (n+2+j)b_{n,j} = 0 \quad (n \text{ fixed}).$$

As our first application we look at the Hankel matrices

$$\widetilde{A}_n = \begin{pmatrix} M_0 & M_1 & \dots & M_n \\ M_1 & M_2 & \dots & M_{n+1} \\ \vdots & \vdots & & \vdots \\ M_n & M_{n+1} & \dots & M_{2n} \end{pmatrix}, \quad \widetilde{B}_n = \begin{pmatrix} M_1 & M_2 & \dots & M_n \\ M_2 & M_3 & \dots & M_{n+1} \\ \vdots & \vdots & & \vdots \\ M_n & M_{n+1} & \dots & M_{2n+1} \end{pmatrix}.$$

PROPOSITION 2. *We have*

(i) $\det \widetilde{A}_n = 1$ *for all* n,
(ii) $\det \widetilde{B}_n = 1, 0, -1$ *for* $n \equiv 0, 1 \pmod 6$, $n \equiv 2, 5 \pmod 6$, *and* $n \equiv 3, 4 \pmod 6$, *respectively.*

PROOF. Consider the $(n+1) \times (n+1)$-matrix $A_n = (a_{k\ell})$ where the $a_{k\ell}$ are the coefficients of $S_k(x)$, i.e., $S_k(x) = a_{k0} + a_{k1}x + \dots + a_{kk}x^k$, $0 \leq k \leq n$. A_n is thus a lower triangular matrix with $\det A_n = 1$ because all $a_{kk} = 1$. The k-th row of the product $A_n \widetilde{A}_n$ $(0 \leq k \leq n)$ is therefore

$$([S_k(x)], [xS_k(x)], \dots, [x^n S_k(x)]).$$

Next we compute $A_n \widetilde{A}_n A_n^T$. For the (k, ℓ)-entry we obtain by Proposition 1(iv)

$$\begin{aligned} a_{\ell 0}[S_k(x)] + a_{\ell 1}[xS_k(x)] + \dots + a_{\ell\ell}[x^\ell S_k(x)] &= [S_k(x)S_\ell(x)] \\ &= [S_{k+\ell}(x)] + [S_{k+\ell-2}(x)] \\ &\quad + \dots + [S_{|k-\ell|}(x)] \end{aligned}$$

which is 1 for $k = \ell$ and 0 for $k \neq \ell$ by (6). Thus $A_n \widetilde{A}_n A_n^T = I_n$ and therefore $\det \widetilde{A}_n = 1$.

Let us now consider $\widetilde{B}_n$. In this case the k-th row of $A_{n-1}\widetilde{B}_n$ $(0 \leq k \leq n-1)$ is

$$([xS_k(x)], [x^2 S_k(x)], \dots, [x^n S_k(x)]),$$

and we obtain for the (k, ℓ)-entry of $A_{n-1}\widetilde{B}_n A_{n-1}^T$

$$\begin{aligned} a_{\ell 0}[xS_k(x)] + a_{\ell 1}[x^2 S_k(x)] + \cdots + a_{\ell\ell}[x^{\ell+1}S_k(x)] &= [xS_k(x)S_\ell(x)] \\ &= [xS_{k+\ell}(x)] + \cdots + [xS_{|k-\ell|}(x)]. \end{aligned}$$

Using (6) again, we obtain for $k = \ell$ $[xS_0(x)] = M_1 = 1$, for $|k - \ell| = 1$ $[xS_1(x)] = M_2 - M_1 = 1$, and 0 for $|k - \ell| \geq 2$. The matrix $D_n = A_{n-1}\widetilde{B}_n A_{n-1}^T$ is therefore of the following form:

$$D_n = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & & 0 & \\ & 1 & 1 & & & \\ & & & \ddots & & 1 \\ & 0 & & & 1 & 1 \end{pmatrix}$$

with $\det \widetilde{B}_n = \det D_n$. The determinant of D_n is now easily evaluated by induction as stated in the proposition. □

The Motzkin numbers are thus the unique sequence of real numbers such that the determinants are alternately equal to $\det \widetilde{A}_n$ and $\det \widetilde{B}_n$, starting with $\widetilde{A}_0$, $\widetilde{B}_1$.

4. Logarithmic Concavity and Limit

The purpose of this section is to discuss the following three results:

(a) $\frac{M_n}{M_{n-1}} \leq \frac{M_{n+1}}{M_n}$ $(n \geq 1)$

(b) $\frac{M_n}{M_{n-1}} < 3$ $(n \geq 1)$

(c) $\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = 3$.

No combinatorial proof of (a), that is an injection of $\mathcal{M}(n)^2$ into $\mathcal{M}(n-1) \times \mathcal{M}(n+1)$ is known to me, similarly for (b).

Let us first look at (a). If n is odd, then Proposition 2 applies. Indeed, as all principal submatrices of $\widetilde{A}_n$ have determinant 1, $\widetilde{A}_n$ is positive definite which implies that all submatrices $\begin{pmatrix} M_{2m-2} & M_{2m-1} \\ M_{2m-1} & M_{2m} \end{pmatrix}$ have positive determinant. But this is precisely statement (a) for $n = 2m - 1$. However, since $\widetilde{B}_n$ is indefinite, for even n we have to proceed differently.

Let A_n be the matrix of the last section. We proved there $\widetilde{A}_n = A_n^{-1} A_n^{T^{-1}}$, so let us determine A_n^{-1}.

Lemma 2. *(i) We have* $(A_n^{-1})_{ij} = [x^i S_j(x)]$.
(ii) Let $A_n^{-1} = (b_{k,\ell})$, *then*

$$\begin{aligned} b_{k,\ell} &= b_{k-1,\ell-1} + b_{k-1,\ell} + b_{k-1,\ell+1} \\ b_{0,0} &= 1, b_{k,\ell} = 0 \text{ for } k < \ell. \end{aligned} \tag{7}$$

Proof. Setting $B_n = ([x^i S_j(x)])$, we find

$$\begin{aligned} (A_n B_n))_{k,\ell} &= \sum_{i=0}^{k} a_{ki}[x^i S_\ell(x)] = [S_k(x)S_\ell(x)] \\ &= [S_{k+\ell}(x)] + \cdots + [S_{|k-\ell|}(x)] = \delta_{k\ell}, \end{aligned}$$

and thus $B_n = A_n^{-1}$ as claimed. A_n^{-1} is therefore a lower triangular matrix by (6).

To prove (ii) we see by the recursion (4)

$$\begin{aligned} b_{k,\ell} = [x^k S_\ell(x)] &= [x^{k-1} x S_\ell(x)] \\ &= [x^{k-1} S_{\ell-1}(x)] + [x^{k-1} S_\ell(x)] + [x^{k-1} S_{\ell+1}(x)] \\ &= b_{k-1,\ell-1} + b_{k-1,\ell} + b_{k-1,\ell+1}. \end{aligned} \qquad \square$$

Consider the infinite lower triangular matrix $B = (b_{k,\ell})$ with rows $r_0, r_1, r_2, \ldots$ observing (7). As $\widetilde{A}_n = B_n B_n^T$ we obtain the useful result

$$r_k \cdot r_\ell = M_{k+\ell} \quad \text{for all } k, \ell, \tag{8}$$

where $r_k \cdot r_\ell$ denotes the usual inner product.

The recursion (7) together with (8) yields therefore another representation of the Motzkin numbers.

As an illustration let us consider the first rows of B according to the recursion (7):

$$B = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 2 & 2 & 1 & & \\ 4 & 5 & 3 & 1 & \\ 9 & 12 & 9 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

We obtain e.g. $r_2 \cdot r_4 = 2 \cdot 9 + 2 \cdot 12 + 1 \cdot 9 = 51 = M_6$, and similarly $r_3 \cdot r_3 = 4 \cdot 4 + 5 \cdot 5 + 3 \cdot 3 + 1 \cdot 1 = 51$.

REMARK. It was pointed out by the referee that the matrix B and its inverse A can also be determined by the elegant approach via the Riordan group (see [7]).

Let us rewrite (7) in the following compact form. For any vector $a = (a_0, a_1, a_2, \ldots)$ we set $a^+ = (0, a_0, a_1, \ldots)$ and $a^- = (a_1, a_2, \ldots)$. Then (7) reads

$$r_k = r_{k-1}^+ + r_{k-1} + r_{k-1}^-. \tag{9}$$

PROPOSITION 3. *We have* $M_n^2 \leq M_{n-1} M_{n+1}$ *for all* $n \geq 1$.

PROOF. By our remark above it suffices to consider n even, but let us prove the case n odd anyway. Set $n = 2m - 1$, then by (8), $M_{2m-1} = r_{m-1} \cdot r_m$, $M_{2m-2} = r_{m-1}^2$, $M_{2m} = r_m^2$, and $M_{2m-1}^2 \leq M_{2m-2} \cdot M_{2m}$ is equivalent to $(r_{m-1} \cdot r_m)^2 \leq r_{m-1}^2 \cdot r_m^2$, which is just Cauchy's inequality.

Now let $n = 2m$. Setting $a = r_{m-1}$, $b = r_m$, $c = r_{m+1}$, we have to prove by (8)

$$(a \cdot b)(b \cdot c) \geq (a \cdot c)(b \cdot b), \tag{10}$$

for any three consecutive rows of B. For $a = r_0$, $b = r_1$, $c = r_2$ this is true and we proceed by induction. Let $a = (a_i)$, $b = (b_i)$, $c = (c_i)$. We need the following lemma whose proof is omitted.

LEMMA 3. *Let* $0 \leq i \leq j$, *then*

(i) $a_i b_j \geq a_j b_i$

(ii) $a_i(b_{j-1}+b_{j+1}) \geq a_j(b_{i-1}+b_{i+1})$
(iii) $b_i(b_{j-1}+b_{j+1}) \geq b_j(b_{i-1}+b_{i+1})$.

To complete the proof set $c = b^+ + b + b^-$ as in (9). Then by (10) we have to show

$$(a \cdot b)(b \cdot b^+ + b \cdot b + b \cdot b^-) \geq (a \cdot b^+ + a \cdot b + a \cdot b^-)b^2$$

or

$$(a \cdot b)(b \cdot b^+ + b \cdot b^-) \geq (a \cdot b^+ + a \cdot b^-)b^2,$$

that is

$$\sum_{i,j}(a_ib_i)(b_jb_{j-1}) + (a_ib_i)(b_jb_{j+1}) \geq \sum_{i,j}(a_ib_{i-1})b_j^2 + (a_ib_{i+1})b_j^2.$$

For $i = j$ both contributions are identical. For $i < j$ we group together the contributions corresponding to the index pairs (i, j) and (j, i). For the left-hand side we obtain

$$L = a_ib_ib_jb_{j-1} + a_ib_ib_jb_{j+1} + a_jb_jb_ib_{i-1} + a_jb_jb_ib_{i+1}$$

and for the right-hand side

$$R = a_ib_{i-1}b_jb_j + a_ib_{i+1}b_jb_j + a_jb_{j-1}b_ib_i + a_jb_{j+1}b_ib_i.$$

This yields

$$L - R = (a_ib_j - a_jb_i)[b_i(b_{j-1}+b_{j+1}) - b_j(b_{i-1}+b_{i+1})],$$

and thus $L \geq R$ by the lemma. As this holds for any pair $i \leq j$, we are done. □

In a similar way the following result can be proved.

PROPOSITION 4. *We have* $\frac{M_n}{M_{n-1}} < 3$ *for all* $n \geq 1$.

PROPOSITION 5. *We have* $\lim_{n\to\infty} \frac{M_n}{M_{n-1}} = 3$.

PROOF. By Propositions 3 and 4, $\alpha = \lim \frac{M_n}{M_{n-1}}$ exists with $\alpha \leq 3$. To prove $\alpha = 3$ we use the notation $s_{h,n}$, $S_n(x) = x^n + a_{n-1}x^{n-1} + + \cdots + a_1x + a_0$ as in the previous section. As $s_{h,n} = M_h + a_{n-1}M_{h-1} + \cdots + a_0M_{h-n}$ counts the number of arrays with n leading $\frac{1}{0}$-columns, we have $s_{h,n} \geq 0$, and hence

$$\frac{M_h}{M_{h-1}} + a_{n-1} + a_{n-2}\frac{M_{h-2}}{M_{h-1}} + \cdots + a_0\frac{M_{h-n}}{M_{h-1}} \geq 0.$$

Going with h to infinity this implies

$$\alpha + a_{n-1} + a_{n-2}\alpha^{-1} + \cdots + a_0\alpha^{-(n-1)} \geq 0$$

or

$$\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 \geq 0 \ \text{ for all } \ n. \tag{11}$$

By Proposition 1(v), the second largest root of $S_n(x)$ is smaller than the largest root of $S_{n-1}(x)$, and we infer by induction and (11) that α is at least as large as the largest root of $S_n(x)$ for all n. But as shown in Proposition 1(v), the largest root of $S_n(x)$ equals $2\cos\frac{\pi}{n+1}+1$, and this goes to 3 with n to infinity, thus proving our claim. □

REMARK. Proposition 5 can also be proved directly using the ratio test (as pointed out by the referee) or by employing an asymptotic estimate as in [1].

5. Ballot Numbers

Recall that the Catalan numbers may be generated as follows (see [6]): Define $a_{n,k}$ $(0 \le k \le n)$ recursively by

$$\begin{aligned} a_{0,0} &= 1 \\ a_{n,k} &= a_{n-1,0} + \cdots + a_{n-1,k}, \quad n \ge 1, \quad 0 \le k \le n-1, \\ a_{n,n} &= 0 \quad n \ge 1. \end{aligned}$$

Then $C_n = \sum_{k=0}^{n} a_{n,k}$ and $a_{n+1,n} = C_n$. The $a_{n,k}$ are the *ballot numbers* with $a_{n,k} = \frac{n-k}{n+k}\binom{n+k}{n}$.

The aim of this section is to exhibit two ballot-number sequences for the Motzkin numbers which arise from two different combinatorial classifications.

Define the numbers $b_{n,k}$ $(0 \le k \le n+1)$ as follows:

$$\begin{aligned} b_{0,0} &= b_{0,1} = 1 \\ b_{n,k} &= b_{n-1,0} + \cdots + b_{n-1,k-2} + b_{n-1,k}, \quad n \ge 1, \quad 0 \le k \le n, \\ b_{n,n+1} &= b_{n,n}. \end{aligned} \tag{12}$$

Note that in the recursion the term $b_{n-1,k-1}$ is missing. The following table gives the first rows:

$n \backslash k$	0	1	2	3	4	5	6
0	1	1					
1	1	1	1				
2	1	1	2	2			
3	1	1	3	4	4		
4	1	1	4	6	9	9	
5	1	1	5	8	15	21	21

Proposition 6. *We have*

(i) $\sum_{k=0}^{n} b_{n,k} = M_{n+1}$

(ii) $b_{n,n} = M_n$

(iii) $b_{n,k} = \sum_{i\ge 0}\binom{n}{i}[\binom{k-i}{i} - \binom{k-i}{i+2}]$.

Proof. Define $b_{0,k} = 1 - \binom{k}{2}$ $(k \ge 0)$, and extend the recursion (12) to all n and k. Note that $b_{0,k}$ agrees with $b_{0,0} = b_{0,1} = 1$. Let $B_n(x) = \sum_{k\ge 0} b_{n,k}x^k$ be the generating function of the n-th row. By (12),

$$B_n(x) = \frac{x^2}{1-x}B_{n-1}(x) + B_{n-1}(x) \quad (n \ge 1)$$

and thus

$$B_n(x) = \left(\frac{1-x+x^2}{1-x}\right)^n B_0(x),$$

where

$$B_0(x) = \sum_{k\ge 0}(1 - \binom{k}{2})x^k = \frac{1}{1-x} - \sum_{k\ge 0}\binom{k}{2}x^k.$$

We therefore find for the coefficient of x^k in $B_n(x)$:

$$(x^k)B_n(x) = (x^k)\left(\frac{1-x+x^2}{1-x}\right)^n - (x^k)\left(\frac{1-x+x^2}{1-x}\right)^n \sum_{i\geq 0}\binom{i}{2}x^i.$$

Let us look at the first summand:

$$\left(\frac{1-x+x^2}{1-x}\right)^n \frac{1}{1-x} = (1+\frac{x^2}{1-x})^n \frac{1}{1-x} = \sum_{i=0}^{n}\binom{n}{i}x^{2i}\frac{1}{(1-x)^{i+1}}$$

$$= \sum_{i=0}^{n}\binom{n}{i}x^{2i}\sum_{j\geq 0}\binom{i+j}{i}x^j = \sum_{k\geq 0}(\sum_{i\geq 0}\binom{n}{i}\binom{k-i}{i})x^k.$$

Hence

$$(x^k)\left(\frac{1-x+x^2}{1-x}\right)^n \frac{1}{1-x} = \sum_{i\geq 0}\binom{n}{i}\binom{k-i}{i}.$$

For the second summand we similarly obtain

$$(x^k)\left(\frac{1-x+x^2}{1-x}\right)^n \sum_{i\geq 0}\binom{i}{2}x^i = \sum_{i\geq 0}\binom{n}{i}\binom{k-i}{i+2}.$$

Using the identity $\sum_{i\geq 0}\binom{n}{i}\binom{n-i}{i} = \sum_{i\geq 0}\binom{n}{2i}\binom{2i}{i}$ one easily finds

$$b_{n,n} = b_{n,n+1} = \sum_{i\geq 0}\binom{n}{2i}C_i = M_n.$$

Hence we have

$$\sum_{k=0}^{n} b_{n,k} = \sum_{k=0}^{n-1} b_{n,k} + b_{n,n+1} = b_{n+1,n+1} = M_{n+1},$$

and we are finished. □

EXAMPLE. A well-known instance counted by the Catalan numbers C_n are the non-crossing partitions of $\{1, 2, \ldots, n\}$ (see [8,9]). Call a partition $\pi = \{A_1, \ldots, A_t\}$ *non-crossing* if $a < b < c < d$ and $a, c \in A_i$, $b, d \in A_j$ imply $i = j$. Let us say, π is *strongly* non-crossing if, in addition, $A_i = \{a\}$, $b, c \in A_j$ and $b < a < c$ never occurs. We want to show that $|\Pi_n| = M_n$ where Π_n is the family of strongly non-crossing partitions.

As an example, we have for $n = 4$ the following $9 = M_4$ partitions:

$$\begin{array}{llll}
1\,2\,3\,4 & & & \\
1\,2\,3-4 & & & \\
1\,2-3\,4, & 1\,2-3-4, & 1\,4-2\,3 & \\
1-2\,3\,4, & 1-2\,3-4, & 1-2-3\,4, & 1-2-3-4,
\end{array}$$

where we have arranged the partitions according to $n - |A_1|$ with $1 \in A_1$. Note that the numbers 1, 1, 3, 4 in this classification are precisely the ballot-numbers $b_{3,k}$ $(0 \leq k \leq 3)$. In the light of Proposition 6, it remains to show that the numbers

$$m_{n,k} = \{\pi \in \Pi_{n+1} : n + 1 - |A_1| = k\} \qquad (0 \leq k \leq n)$$

satisfy (12), where we set $m_{n,n+1} = m_{n,n}$.

We trivially have $m_{0,0} = 1$. Let $\Pi_{n+1,k}$ be the subfamily with $|A_1| = n + 1 - k$, let $\pi \in \Pi_{n+1,k}$ with $1 \in A_1, 2 \in A_2$, and assume $k \le n-1$. We associate to π the following partition $\pi' \in \Pi_n$: Delete 1 from A_1 and merge $(A_1 \backslash 1) \cup A_2$, keeping all other blocks unchanged. It is easy to see that π' is strongly non-crossing. Now if $A_1 = A_2$, then $n - |A_1 \cup A_2 \backslash 1| = k$, and if $A_1 \neq A_2$, then $n - |A_1 \cup A_2 \backslash 1| \le k - 2$ because $|A_1| \ge 2$ and therefore $|A_2| \ge 2$ by the definition of strongly non-crossing. It is straightforward to check that $\pi \to \pi'$ is a bijection from $\Pi_{n+1,k}$ onto $\bigcup_{i=0}^{k-2} \Pi_{n,i} \cup \Pi_{n,k}$. In the case $k = n$, $A_1 = \{1\}$ is a singleton, and we clearly have $m_{n,n} = \sum_{k=0}^{n-1} m_{n-1,k} = \sum_{k=0}^{n-2} m_{n-1,k} + m_{n-1,n}$, thus proving our result.

REMARK. The same classification for the ordinary non-crossing partitions yields the ballot numbers $a_{n,k}$ for C_n.

REMARK. It may be interesting to see how the involution of non-crossing partitions described in [8] acts on the set of strongly non-crossing partitions.

Let us turn to the second ballot-number sequence. Let $c_{n,k}$ $(0 \le k \le n)$ be defined as follows:

$$\begin{aligned} c_{0,0} &= c_{1,0} = 1 \\ c_{n,k} &= (c_{n-2,0} + \cdots + c_{n-2,k}) + (c_{n-3,0} + \cdots + c_{n-3,k-1}), \quad n \ge 2, 0 \le k \le n-2, \\ c_{n,n-1} &= c_{n-2,0} + \cdots + c_{n-2,n-2} \quad (n \ge 2) \\ c_{n,n} &= 0 \quad (n \ge 1). \end{aligned} \tag{13}$$

The first values are given in the following table:

$n \backslash k$	0	1	2	3	4	5	6
0	1						
1	1	0					
2	1	1	0				
3	1	2	1	0			
4	1	3	3	2	0		
5	1	4	6	6	4	0	
6	1	5	10	13	13	9	0

PROPOSITION 7. *We have*

(i) $\sum_{k=0}^{n} c_{n,k} = M_n \qquad (n \ge 0)$
(ii) $c_{n,n-1} = M_{n-2} \qquad (n \ge 2)$.

PROOF. This time we give a combinatorial proof and determine the generating functions later. Let $\mathcal{M}_{n,k}$ be the subfamily of all arrays $\begin{smallmatrix} a_1 \ldots a_n \\ b_1 \ldots b_n \end{smallmatrix}$ in $\mathcal{M}_n$ with the leading 1 among the b_is appearing in place $n - k + 1$. The array has thus the following form:

$$\begin{matrix} a_1 a_2 \ldots 0 \ldots a_n \\ 00 \ldots \underbrace{1 \ldots b_n}_{k} \end{matrix} \qquad (0 \le k \le n-1).$$

Note that $\mathcal{M}_{n,0}$ contains just the all-zero array. We want to show that the numbers $p_{n,k} = |\mathcal{M}_{n,k}|$ satisfy the recursion (13). This will prove part (i) and also (ii) as $p_{n,n-1}$ clearly equals M_{n-2}.

We have $p_{1,0} = p_{2,0} = p_{2,1} = 1$ in agreement with (13), and $p_{0,0} = 1$ by definition. Suppose $n \geq 3$, $A \in \mathcal{M}_{n,k}$, and assume that there are i $\begin{smallmatrix}0\\0\end{smallmatrix}$ columns between the last 1 in the first row of A before the leading 1 in the second row:

$$\begin{array}{r|c|l} a_1 \ldots a_h 1 & \overbrace{\begin{matrix} 0 & & 0 \\ & \cdots & \\ 0 & & 0 \end{matrix}}^{i} & 0 \ldots \\ 0 \ldots 00 & & 1 \ldots \\ c & Z & c' \end{array}$$

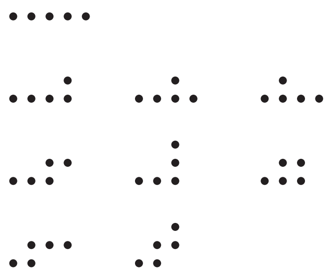
We now map A into $A' \in \mathcal{M}_{n-2} \cup \mathcal{M}_{n-3}$ as follows:

(a) If $i \geq 2$, remove two $\begin{smallmatrix}0\\0\end{smallmatrix}$ columns from Z,
(b) If $i = 0$, remove the columns c and c',
(c) If $i = 1$, remove c, Z, c'.

Clearly, A' is a Motzkin array. In case (a) we have $A' \in \mathcal{M}_{n-2,k}$, in case (b) $A' \in \mathcal{M}_{n-2,\leq k-1}$, and in case (c), $A' \in \mathcal{M}_{n-3,\leq k-1}$. Note that for $k = n - 1$, the cases (a) and (c) cannot occur. The map $A \to A'$ is easily seen to be a bijection, and we are finished. □

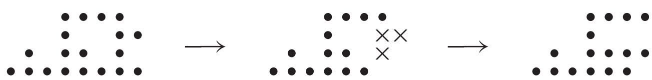
Again we note that the corresponding classification for $\mathcal{C}_n$ yields the ordinary ballot numbers.

EXAMPLE. A beautiful combinatorial setting counted by the Motzkin numbers was discovered by Gouyou-Beauchamps and Viennot [3]. M_n is the number of subsets $S \subseteq \mathbb{N} \times \mathbb{N}$ in the first octant, $0 \leq y \leq x$, of size $n + 1$ satisfying the following property: If $p \in S$, then there is a lattice path from $(0, 0)$ to p with steps $(1, 0)$ and $(0, 1)$ all of whose vertices lie in S. For $n = 4$ we obtain the following nine configurations:



If we classify these nine configurations according to the number of elements above the base line, we obtain precisely the ballot numbers $c_{4,k}$, $0 \leq k \leq 3$. The following operation shows that this holds in general, thereby providing an alternate proof of their result. Let P be an admissible configuration of size $n + 1$. Remove the two right–most points on the base line.

The new configuration will, in general, not be admissible anymore; let Q be the set of 'hanging' points, that is the set of those points which cannot be reached from $(0, 0)$ by an admissible path. Now slide Q one step down, diagonally to the left. There may arise one duplicate point which we only keep once. (Note, that this accounts for $n + 1$ to drop down to $n - 2$.) This yields the desired bijection $P \longrightarrow P'$. To illustrate the map $P \longrightarrow P'$ look at the following example, where the points denoted by $\times$ form the set Q.

$P =$  $= P'.$

Finally, we want to compute the generating functions $C_n(x)$ for the n-th row $(c_{n,0}, c_{n,1}, \ldots)$, extending the recursion (13) to all n and k. Using (13) backwards it is not hard to see that $c_{0,n} = c_{1,n}$ for all n, and

$$c_{0,0} = 1, c_{0,1} = 0, \text{ and } c_{0,n+2} = -M_0 + M_1 - \cdots + (-1)^{n-1} M_n \qquad (n \geq 0).$$

With $\sum_{n\geq 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$, we thus obtain

$$\begin{aligned} C_0(x) &= C_1(x) = 1 - \tfrac{1+x-\sqrt{1+2x-3x^2}}{2(1-x)} \\ &= \tfrac{1-3x+\sqrt{1+2x-3x^2}}{2(1-x)}. \end{aligned}$$

Furthermore, (13) implies

$$C_n(x) = \frac{C_{n-2}(x)}{1-x} + \frac{xC_{n-3}(x)}{1-x} + C_0(x)([n=0]+[n=1]).$$

Setting $F(x,t) = \sum_{n\geq 0} C_n(x) t^n$, we find

$$F(x,t) = \frac{t^2 F(x,t)}{1-x} + \frac{xt^3 F(x,t)}{1-x} + (1+t)C_0(x),$$

and hence

$$F(x,t) = \frac{(1+t)C_0(x)}{1 - \frac{t^2}{1-x} - \frac{xt^3}{1-x}} = \frac{C_0(x)}{1 - t - \frac{x}{1-x}t^2}.$$

Using partial fractions, this yields

$$C_n(x) = \frac{C_0(x)}{\sqrt{\frac{1+3x}{1-x}}} \left[\left(\frac{1+\sqrt{\frac{1+3x}{1-x}}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{\frac{1+3x}{1-x}}}{2} \right)^{n+1} \right],$$

and thus our final formula

$$C_n(x) = \frac{1-3x+\sqrt{1+2x-3x^2}}{1-x} \frac{1}{2^{n+1}} \sum_{i\geq 0} \binom{n+1}{2i+1} \left(\frac{1+3x}{1-x} \right)^i.$$

REMARK. It was pointed out by the referee that the two ballot-number sequences can, after suitable rearrangement, be recovered in the context of Riordan matrices. For the first sequence we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 1 & 0 & 0 & \ldots \\ 4 & 3 & 1 & 1 & 0 & \ldots \\ 9 & 6 & 4 & 1 & 1 & \ldots \\ & & \ldots & & & \\ & & \ldots & & & \end{pmatrix} = \left(M(x),\ \frac{1-\sqrt{\frac{1-3x}{1+x}}}{2x} \right),$$

and for the second

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 2 & 3 & 3 & 1 & 0 & \dots \\ 4 & 6 & 6 & 4 & 1 & \dots \\ & & \dots & & & \end{pmatrix} = (1 + xM(x), \quad x(1 + xM(x)),$$

where $M(x)$ is the generating function of the Motzkin numbers.

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