GENERALIZATIONS OF MODIFIED MORGAN-VOYCE POLYNOMIALS

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1. INTRODUCTION

In two recent articles [2] and [3], Ferri et al. introduced and studied the properties of two numerical triangles, which they called DFF and DFZ triangles. However, in a subsequent article, André-Jeannin [1] showed that the polynomials generated by the rows of these triangles are indeed the Morgan-Voyce polynomials $B_n(x)$ and $b_n(x)$, whose properties are well known [10] and [11]; in fact, the polynomials $B_n(x)$ and $b_n(x)$ have been used in the study of electrical networks since the 1960s (see, e.g., [8] and [9]). In the same article, André-Jeannin introduced a generalization of the Morgan-Voyce polynomials by defining the sequence of polynomials $\{P_n^{(r)}(x)\}$ by the relation

$$P_n^{(r)}(x) = (x+2)P_{n-1}^{(r)}(x) - P_{n-2}^{(r)}(x), \quad (n \geq 2), \tag{1a}$$

with

$$P_0^{(r)}(x) = 1 \text{ and } P_1^{(r)}(x) = x+r+1. \tag{1b}$$

Subsequently, Horadam [6] defined a closely related sequence of polynomials $\{Q_n^{(r)}(x)\}$ by the relation

$$Q_n^{(r)}(x) = (x+2)Q_{n-1}^{(r)}(x) - Q_{n-2}^{(r)}(x), \quad (n \geq 2), \tag{2a}$$

with

$$Q_0^{(r)}(x) = 2 \text{ and } Q_1^{(r)}(x) = x+r+2, \tag{2b}$$

and studied some of its properties.

The purpose of this article is first to generalize the two sequences of polynomials $\{P_n^{(r)}(x)\}$ and $\{Q_n^{(r)}(x)\}$, and to study some of their properties by first relating them to the parameters of electrical one-ports and then using the properties of such one-ports. Later, following Horadam [7], we will construct and study some of the properties of a composite polynomial which includes the two sets of generalized polynomials introduced in this article.

2. POLYNOMIALS $\tilde{P}_n^{(r)}(x)$ AND $\tilde{Q}_n^{(r)}(x)$

Consider the generalized polynomial $w_n(a, b; x)$ defined by

$$w_n(x) = (x+p)w_{n-1}(x) - w_{n-2}(x), \quad (n \geq 2), \tag{3a}$$

with

$$w_0(x) = a \text{ and } w_1(x) = b. \tag{3b}$$

We know that the solution of (3a) and (3b) is given by [5]:

$$w_n(x) = w_1(x)U_n(x) - w_0(x)U_{n-1}(x), \tag{4}$$

where

$$U_n(x) = w_n(0, 1; x). \tag{5}$$

Hence, we may observe that the modified Morgan-Voyce polynomials, $\widetilde{B}_n(x)$, $\widetilde{b}_n(x)$, $\widetilde{C}_n(x)$, and $\widetilde{c}_n(x)$, defined in [12], may be written as

$$\widetilde{B}_n(x) = w_n(1, x+p; x) = U_{n+1}(x), \tag{6a}$$
$$\widetilde{b}_n(x) = w_n(1, x+p-1; x) = U_{n+1}(x) - U_n(x) = \widetilde{B}_n(x) - \widetilde{B}_{n-1}(x), \tag{6b}$$
$$\widetilde{C}_n(x) = w_n(2, x+p; x) = U_{n+1}(x) - U_{n-1}(x) = \widetilde{B}_n(x) - \widetilde{B}_{n-2}(x), \tag{6c}$$
$$\widetilde{c}_n(x) = w_n(1, x+p+1; x) = U_{n+1}(x) + U_n(x) = \widetilde{B}_n(x) + \widetilde{B}_{n-1}(x). \tag{6d}$$

From (6b), (6c), and (6d), we see that

$$\widetilde{C}_n(x) = \widetilde{b}_n(x) + \widetilde{b}_{n-1}(x) = \widetilde{c}_n(x) - \widetilde{c}_{n-1}(x). \tag{7}$$

Let us now define the following two sets of generalized polynomials $\widetilde{P}_n^{(r)}(x)$ and $\widetilde{Q}_n^{(r)}(x)$ as

$$\widetilde{P}_n^{(r)}(x) = w_n(1, x+p+r-1; x) \tag{8a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = w_n(2, x+p+r; x). \tag{8b}$$

Hence, from (4), we have

$$\widetilde{P}_n^{(r)}(x) = U_{n+1}(x) + (r-1)U_n(x) \tag{9a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = U_{n+1}(x) - U_{n-1}(x) + rU_n(x). \tag{9b}$$

Using the relations given in (6a)-(6d), the above may be written as

$$\widetilde{P}_n^{(r)}(x) = \widetilde{b}_n(x) + r\widetilde{B}_{n-1}(x) \tag{10a}$$

and

$$\widetilde{Q}_n^{(r)}(x) = \widetilde{C}_n(x) + r\widetilde{B}_{n-1}(x). \tag{10b}$$

As a consequence of (10a), (10b), and (7), we also have the relation

$$\widetilde{Q}_n^{(r)}(x) = \widetilde{P}_n(x) + \widetilde{b}_{n-1}(x). \tag{10c}$$

It is readily seen that

$$\widetilde{P}_n^{(0)}(x) = \widetilde{b}_n(x), \tag{11a}$$
$$\widetilde{P}_n^{(1)}(x) = \widetilde{B}_n(x), \tag{11b}$$
$$\widetilde{P}_n^{(2)}(x) = \widetilde{c}_n(x), \tag{11c}$$
$$\widetilde{Q}_n^{(0)}(x) = \widetilde{C}_n(x). \tag{11d}$$

It is clear that these results are generalizations of those contained in [1] and [6].

3. $\widetilde{P}_n^{(r)}(x)$, $\widetilde{Q}_n^{(r)}(x)$ AND LADDER ONE-PORTS

In this article we assume that $p \geq 2$ and $r \geq 0$. Consider now the ladder one-port network shown in Figure 1(a), which consists only of resistors and inductors, and thus is an RL-network (see Appendix A), where the series resistors $r_1 = r_2 = r_3 = \cdots = r_n = (p-2)\alpha$ Ohms, the inductors $L_1 = L_2 = L_3 = \cdots = L_n = \alpha$ Henries, and the shunt resistors $R_1 = R_2 = R_3 = \cdots = R_n = \alpha$ Ohms. For such a network, the impedance z_1 of any of the series branches is given by

$$z_1 = (s+p-2)\alpha, \tag{12}$$

where s is the complex frequency variable, while the impedance z_2 of any of the shunt branches is given by

$$z_2 = \alpha. \tag{13}$$

It is known [9] that the driving point impedance (DPI) Z_a of such a network is given by

$$Z_a = z_2 \frac{b_n(w)}{B_{n-1}(w)}, \tag{14}$$

where

$$w = \frac{z_1}{z_2}, \tag{15}$$

and $B_n(w)$ and $b_n(w)$ are the Morgan-Voyce polynomials [8]. Hence,

$$Z_a = \alpha \frac{b_n(s+p-2)}{B_{n-1}(s+p-2)}.$$

However, $b_n(s+p-2) = \widetilde{b}_n(s)$ and $B_n(s+p-2) = \widetilde{B}_n(s)$. Hence, the DPI of the RL-ladder network of Figure 1(a) is given by

$$Z_a = \alpha \frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}. \tag{16}$$

Now consider the rational function $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$, where $k > 0$. Then

$$\frac{\widetilde{P}_n^{(r+k)}(s)}{\widetilde{P}_n^{(r)}(s)} = \frac{\widetilde{b}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{1}{\dfrac{r}{k} + \dfrac{1}{k}\dfrac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{17}$$

Using (16) and (17), we see that $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$ may be realized as the driving point admittance (DPA) Y_b of the network shown in Figure 1(b). It is observed that this network also is composed only of resistors and inductors. Thus, $\widetilde{P}_n^{(r+k)}(s) / \widetilde{P}_n^{(r)}(s)$ can be realized as the DPA of an RL-network.

Now consider the rational function $\widetilde{Q}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$, where again $k > 0$. Then

$$\frac{\widetilde{Q}_n^{(r+k)}(s)}{\widetilde{Q}_n^{(r)}(s)} = \frac{\widetilde{C}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = 1 + \frac{1}{\dfrac{r}{k} + \dfrac{1}{k}\dfrac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{18}$$

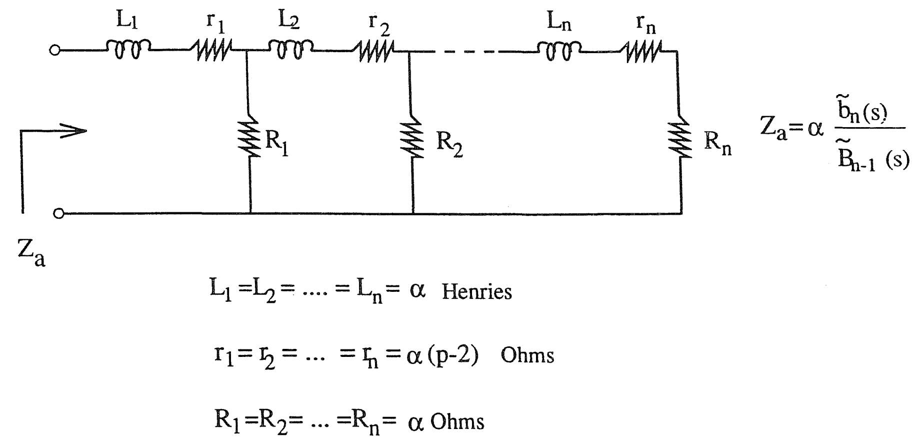
From the results given in [9], it is known that the function

$$\alpha \frac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}$$

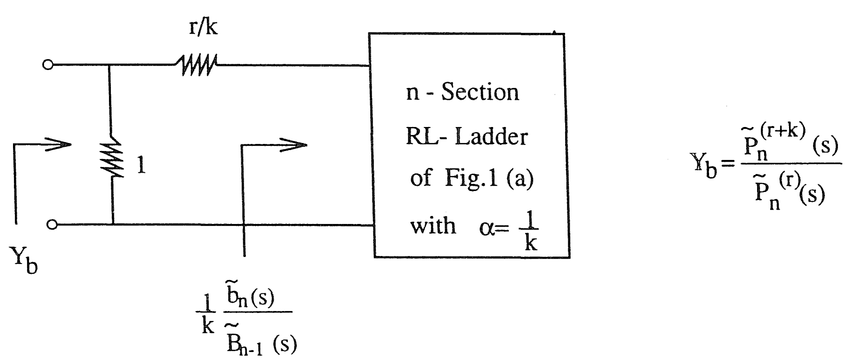
can be realized as the DPI of the RL-ladder network shown in Figure 2(a). Hence, from (18), we see that $\widetilde{Q}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$ can be realized as the DPA of an RL-network.

Now consider $\widetilde{P}_n^{(r+k)}(s) / \widetilde{Q}_n^{(r)}(s)$, $k \geq 0$. This may be expressed as

$$\frac{\widetilde{P}_n^{(r+k)}(s)}{\widetilde{Q}_n^{(r)}(s)} = \frac{\widetilde{b}_n(s) + (r+k)\widetilde{B}_{n-1}(s)}{\widetilde{C}_n(s) + r\widetilde{B}_{n-1}(s)} = \frac{(r+k) + \dfrac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}}{r + \dfrac{\widetilde{C}_n(s)}{\widetilde{B}_{n-1}(s)}}. \tag{19}$$



(a)



(b)

FIGURE 1

Since both $\widetilde{b}_n(s)/\widetilde{B}_{n-1}(s)$ and $\widetilde{C}_n(s)/\widetilde{B}_{n-1}(s)$ are RL-impedance functions, we see from (19) that $\widetilde{P}_n^{(r+k)}(s)/\widetilde{Q}_n^{(r)}(s)$ is a ratio of two RL-impedance functions. Therefore, in general, it is only a positive real function (see Appendix B) and thus need R, L, and C (capacitors) for its realization [13].

Using the properties of RL-networks (see Appendix A), we may now draw some conclusions regarding the locations of the zeros of $\widetilde{P}_n^{(r)}(s)$ and $\widetilde{Q}_n^{(r)}(s)$. Since $\widetilde{P}_n^{(r+k)}(s)/\widetilde{P}_n^{(r)}(s)$ $(k>0)$ is realizable as the DPA of an RL-network, we see that the zeros of $\widetilde{P}_n^{(r)}(s)$ are real, simple, and negative; further, they interlace with those of $\widetilde{P}_n^{(r+k)}(s)$, the zero closest to the origin being that of $\widetilde{P}_n^{(r)}(s)$. Similar statements hold with regard to the zeros of $\widetilde{Q}_n^{(r)}(s)$ and $\widetilde{Q}_n^{(r+k)}(s)$ $(k>0)$, since we have shown that $\widetilde{Q}_n^{(r+k)}(s)/\widetilde{Q}_n^{(r)}(s)$ is also a DPA of an RL-network. In addition, since $\widetilde{P}_n^{(r+k)}(s)/\widetilde{Q}_n^{(r)}(s)$ $(k\geq 0)$ is a ratio of two RL-admittance functions, the zeros of $\widetilde{P}_n^{(r+k)}(s)$ and $\widetilde{Q}_n^{(r)}(s)$ need not interlace; however, their zeros have a very interesting relationship on the negative real axis [4]. In this connection, it may be mentioned that the only known result is the one regarding the zeros of $\widetilde{P}_n^{(0)}(s)$, $\widetilde{P}_n^{(1)}(s)$, $\widetilde{P}_n^{(2)}(s)$, and $\widetilde{Q}_n^{(0)}(s)$, since these are the zeros of $\widetilde{b}_n(s)$, $\widetilde{B}_n(s)$, $\widetilde{c}_n(s)$, and $\widetilde{C}_n(s)$, respectively.

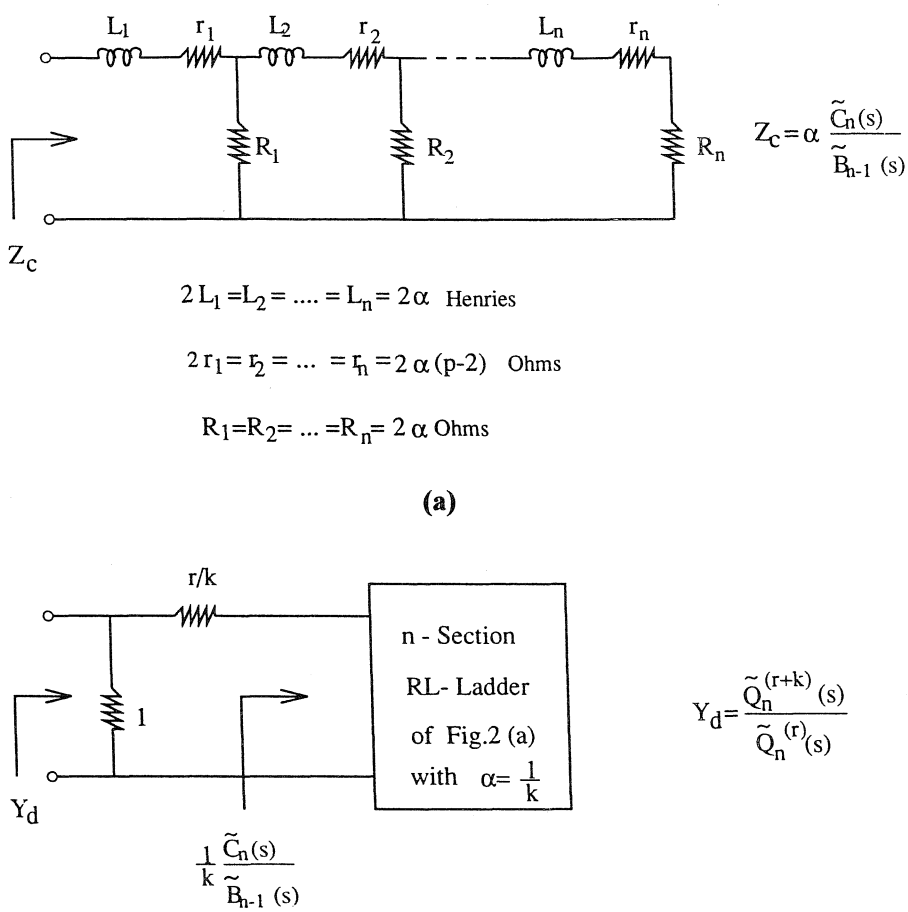


FIGURE 2

4. THE COMPOSITE POLYNOMIAL $\widetilde{R}_n^{(r,u)}(x)$

Following Horadam [7], we now define the composite polynomial $\widetilde{R}_n^{(r,u)}(x)$ by the relation

$$\widetilde{R}_n^{(r,u)}(x) = (x+p)\widetilde{R}_{n-1}^{(r,u)}(x) - \widetilde{R}_{n-2}^{(r,u)}(x), \quad (n \geq 2), \tag{20a}$$

with

$$\widetilde{R}_0^{(r,u)}(x) = u \quad \text{and} \quad \widetilde{R}_1^{(r,u)}(x) = x+p+r+u-2, \tag{20b}$$

where r and u are real numbers. It is clear that

$$\begin{aligned} \widetilde{R}_n^{(r,1)}(x) &= \widetilde{P}_n^{(r)}(x), \\ \widetilde{R}_n^{(r,2)}(x) &= \widetilde{Q}_n^{(r)}(x). \end{aligned} \tag{21}$$

Using the results of (3a), (3b), (4), and (5), we see that

$$\begin{aligned}\widetilde{R}_n^{(r,u)}(x) &= (x+p+r+u-2)U_n(x) - uU_{n-1}(x)\\ &= U_{n+1}(x) + (r-1)U_n(x) + (u-1)\{U_n(x) - U_{n-1}(x)\}.\end{aligned}$$

Using (9a) and (6b), the above relation may be rewritten as

$$\widetilde{R}_n^{(r,u)}(x) = \widetilde{P}_n^{(r)}(x) + (u-1)\widetilde{b}_{n-1}(x). \tag{22}$$

Substituting for $\widetilde{b}_{n-1}(x)$ from (10c), equation (22) reduces to

$$\widetilde{R}_n^{(r,u)}(x) = (u-1)\widetilde{Q}_n^{(r)}(x) - (u-2)\widetilde{P}_n^{(r)}(x). \tag{23a}$$

Now using (21), equation (23a) may also be rewritten as

$$\widetilde{R}_n^{(r,u)}(x) = (u-1)\widetilde{R}_n^{(r,2)}(x) - (u-2)\widetilde{R}_n^{(r,1)}(x). \tag{23b}$$

Let us now find the locations of the zeros of $\widetilde{R}_n^{(r,u)}(x)$ for $r \ge 0$ and $u \ge 1$. For this purpose, we first consider the function $\widetilde{R}_n^{(r+k,u)}(s)/\widetilde{R}_n^{(r,u)}(s)$ for $k > 0$. Using (22), we may write

$$\widetilde{R}_n^{(r+k,u)}(s) - \widetilde{R}_n^{(r,u)}(s) = \widetilde{P}_n^{(r+k)}(s) - \widetilde{P}_n^{(r)}(s) = k\widetilde{B}_{n-1}(s), \text{ using (10a).}$$

Using (22) and (10a), we get

$$\begin{aligned}\frac{\widetilde{R}_n^{(r+k,u)}(s)}{\widetilde{R}_n^{(r,u)}(s)} &= 1 + \frac{k\widetilde{B}_{n-1}(s)}{\widetilde{b}_n(s) + r\widetilde{B}_{n-1}(s) + (u-1)\widetilde{b}_{n-1}(s)}\\ &= 1 + \cfrac{1}{\frac{r}{k} + \frac{1}{k}\frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)} + \frac{u-1}{k}\frac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}}.\end{aligned} \tag{24}$$

From the results given in [9], it is known that the function

$$\frac{u-1}{k}\frac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}$$

may be realized as the DPI of the RL-ladder network shown in Figure 3(a), with $\alpha = (u-1)/k$. Further, as already mentioned in Section 3,

$$\frac{1}{k}\frac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}$$

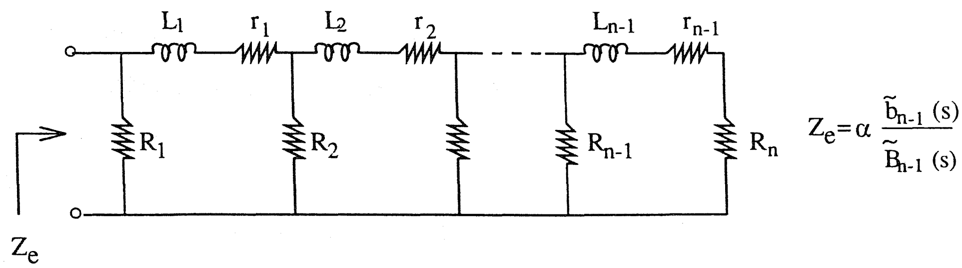
can be realized as the DPI of the RL-ladder network shown in Figure 1(a), with $\alpha = 1/k$. Hence, $\widetilde{R}_n^{(r+k,u)}(s)/\widetilde{R}_n^{(r,u)}(s)$ $(k > 0)$ may be realized as the DPA of the RL-network shown in Figure 3(b).

Again using the properties of RL-networks, we can state that the zeros of $\widetilde{R}_n^{(r,u)}(s)$ are real, simple, and negative; further, the zeros of $\widetilde{R}_n^{(r,u)}(s)$ interlace with those of $\widetilde{R}_n^{(r+k,u)}(s)$, the zero closest to the origin being that of $\widetilde{R}_n^{(r,u)}(s)$.

Now we consider the function $\widetilde{R}_n^{(r+k,u+t)}(s)/\widetilde{R}_n^{(r,u)}(s)$, where $k \ge 0$ and $t > 0$. From (22) and (10a), we have

$$\begin{aligned}\frac{\widetilde{R}_n^{(r+k,\,u+t)}(s)}{\widetilde{R}_n^{(r,\,u)}(s)} &= \frac{\widetilde{P}_n^{(r+k)}(s)+(u+t-1)\widetilde{b}_{n-1}(s)}{\widetilde{P}_n^{(r)}(s)+(u-1)\widetilde{b}_{n-1}(s)} \\ &= \frac{\widetilde{b}_n(s)+(r+k)\widetilde{B}_{n-1}(s)+(u+t-1)\widetilde{b}_{n-1}(s)}{\widetilde{b}_n(s)+r\widetilde{B}_{n-1}(s)+(u-1)\widetilde{b}_{n-1}(s)} \\ &= \frac{(r+k)+\dfrac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}+(u+t-1)\dfrac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}}{r+\dfrac{\widetilde{b}_n(s)}{\widetilde{B}_{n-1}(s)}+(u-1)\dfrac{\widetilde{b}_{n-1}(s)}{\widetilde{B}_{n-1}(s)}}.\end{aligned} \tag{25}$$

Since $\widetilde{b}_n(s)/\widetilde{B}_{n-1}(s)$ and $\widetilde{b}_{n-1}(s)/\widetilde{B}_{n-1}(s)$ are both RL-impedance functions, we see from (25) that $\widetilde{R}_n^{(r+k,\,u+t)}(s)/\widetilde{R}_n^{(r,\,u)}(s)$ $(k\geq 0, t>0)$ is a ratio of two RL-impedance functions. In view of this, as mentioned earlier in Section 3, the zeros of $\widetilde{R}_n^{(r,\,u)}(s)$ and those of $\widetilde{R}_n^{(r+k,\,u+t)}(s)$ $(k\geq 0,$ $t>0)$ need not interlace on the negative real axis [4].

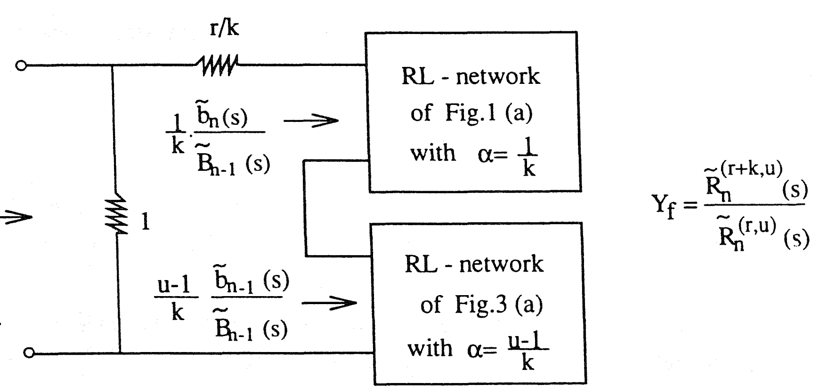


$L_1=L_2=\ldots=L_{n-1}=\alpha$ Henries

$r_1=r_2=\ldots=r_{n-1}=\alpha(p-2)$ Ohms

$R_1=R_2=\ldots=R_n=\alpha$ Ohms

(a)



(b)

FIGURE 3

5. CONCLUDING REMARKS

In this article we have generalized the results of André-Jeannin [1] and Horadam [6] and [7] concerning the sequences $P_n^{(r)}(x)$, $Q_n^{(r)}(x)$, and $R_n^{(r,u)}(x)$. We have also shown that there exist close relationships between these generalized sequences and RL-networks or certain types of RLC-networks. Using these relationships and the properties of such networks, results concerning the locations of the zeros of these generalized sequences have been derived. In view of similar results recently obtained for another pair of polynomials, it is worthwhile exploring such relationships between polynomial sequences and network functions to derive properties of such sequences using the well-known properties of RL, RC, LC, and RLC network functions, and vice-versa.

APPENDIX A

Properties of RL One-Port Networks [14]

A one-port electrical network is a two-terminal network consisting only of two kinds of elements, namely, resistors and inductors.

The driving point impedance $Z(s)$ of such an RL network satisfies the following properties:

(a) All poles and zeros are simple, and are located on the negative real axis of the s-plane.

(b) Poles and zeros interlace.

(c) The lowest critical frequency is a zero which may be located at $s=0$.

(d) The highest critical frequency is a pole which may be at infinity.

(e) $Z(0)<Z(\infty)$.

Also, the driving point admittance of an RL network satisfies the following properties:

(a) All poles and zeros are simple, and are located on the negative real axis of the s-plane.

(b) Poles and zeros interlace.

(c) The lowest critical frequency is a pole which may be located at $s=0$.

(d) The highest critical frequency is a zero which may be at infinity.

(e) $Y(0)>Y(\infty)$.

APPENDIX B

Positive Real Functions [14]

A function $F(s)$, s being a complex variable, is said to be a positive real function if it satisfies the following two conditions:

$$\operatorname{Re} F(s) \geq 0 \ \text{ for } \ \operatorname{Re} s \geq 0$$

and

$$F(s) \text{ is real when } s \text{ is real,}$$

where $\operatorname{Re} T$ denotes the real part of T.

A positive real function $F(s)$ can always be realized as the driving point impedance or admittance of a one-port RLC network, that is, a two-terminal network consisting only of resistors, inductors, and capacitors. Conversely, the driving point impedance and admittance functions of an RLC one-port network are always positive real.

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