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11

The Mellin Transform

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11.1 Introduction

In contrast to Fourier and Laplace transformations that were introduced to solve physical problems, Mellin's transformation arose in a mathematical context. In fact, the first occurrence of the transformation is found in a memoir by Riemann in which he used it to study the famous Zeta function. References concerning this work and its further extension by M. Cahen are given in Reference 1. However, it is the Finnish mathematician, R. H. Mellin (1854-1933), who was the first to give a systematic formulation of the transformation and its inverse. Working in the theory of special functions, he developed applications to the solution of hypergeometric differential equations and to the derivation of asymptotic expansions. The Mellin contribution gives a prominent place to the theory of analytic functions and relies essentially on Cauchy's theorem and the method of residues. A biography of R. H. Mellin including a sketch of his works can be found in Reference 2. Actually, the Mellin transformation can also be placed in another framework, which in some respects conforms more closely to the original ideas of Riemann.

In this approach, the transformation is seen as a Fourier transformation on the multiplicative group of positive real numbers (i.e., group of dilations) and its development parallels the group-theoretical presentation of the usual Fourier transform.[3,4] One of the merits of this alternative presentation is to emphasize the fact that the Mellin transformation corresponds to an isometry between Hilbert spaces of functions. Besides its use in mathematics, Mellin's transformation has been applied in many different areas of physics and engineering.

Maybe the most famous application is the computation of the solution of a potential problem in a wedge-shaped region where the unknown function (e.g., temperature or electrostatic potential) is supposed to satisfy Laplace's equation with given boundary conditions on the edges. Another domain where Mellin's transformation has proved useful is the resolution of linear differential equations in $x(d/dx)$ arising in electrical engineering by a procedure analogous to Laplace's. More recently, traditional applications have been enlarged and new ones have emerged. A new impulse has been given to the computation of certain types of integrals by O. I. Marichev,[5] who has extended the Mellin method and devised a systematic procedure to make it practical.

The alternative approach to Mellin's transformation involving the group of dilations has specific applications in signal analysis and imaging techniques. Used in place of Fourier's transform when scale invariance is more relevant than shift invariance, Mellin's transform suggests new formal treatments. Moreover, a discretized form can be set up and allows the fast numerical computation of general expressions in which dilated functions appear, such as wavelet coefficients and time-frequency transforms.

The present chapter is divided into two parts that can be read independently. The first part (Section 11.2) deals with the introduction of the transformation as a holomorphic function in the complex plane, in a manner analogous to what is done with Laplace's transform. The definition of the transform is given in Section 11.2.1; its properties are described in detail and illustrated by examples. Emphasis is put in Section 11.2.1.6 on inversion procedures that are essential for a practical use of the transform. The applications considered in this first part (Section 11.2.2) are well-known for the most part: summation of series, computation of integrals depending on a parameter, solution of differential equations, and asymptotic expansion. The last example (Section 11.2.2.6), however, concerns a fairly recent application to the asymptotic analysis of harmonic sums arising in the analysis of algorithms.[6]

The second part (Section 11.3), which is especially oriented towards signal analysis and imaging, deals with the introduction of the Mellin transform from a systematic study of dilations. In Section 11.3.1, some notions of group theory are recalled in the special case of the group of positive numbers (dilation group) and Mellin's transformation is derived together with properties relevant to the present setting. The discretization of the transformation is performed in Section 11.3.2. A choice of practical applications is then presented in Section 11.3.3.

11.2 The Classical Approach and its Developments

11.2.1 Generalities on the Transformation

11.2.1.1 Definition and Relation to Other Transformations

Definition 11.2.1.1

Let $f(t)$ be a function defined on the positive real axis $0 < t < \infty$. The Mellin transformation $\mathcal{M}$ is the operation mapping the function f into the function F defined on the complex plane by the relation:

$$\mathcal{M}\left[f; s\right] \equiv F\left(s\right) = \int_0^\infty f\left(t\right) t^{s-1}\, dt \tag{11.1}$$

The function $F(s)$ is called the Mellin transform of f. In general, the integral does exist only for complex values of $s = a + jb$ such that $a < a_1 < a_2$, where a_1 and a_2 depend on the function $f(t)$ to transform. This introduces what is called the *strip of definition* of the Mellin transform that will be denoted by $S(a_1, a_2)$. In some cases, this strip may extend to a half-plane ($a_1 = -\infty$ or $a_2 = +\infty$) or to the whole complex s-plane ($a_1 = -\infty$ and $a_2 = +\infty$)

Example 11.2.1.1

Consider:

$$f\left(t\right) = H\left(t - t_0\right) t^z \tag{11.2}$$

where H is Heaviside's step function, t_0 is a positive number and z is complex. The Mellin transform of f is given by:

$$\mathcal{M}\left[f; s\right] = \int_{t_0}^\infty t^{z+s-1}\, dt = -\frac{t_0^{z+s}}{z+s} \tag{11.3}$$

provided s is such that $Re(s) < -Re(z)$. In this case the function $f(s)$ is holomorphic in a half-plane.

Example 11.2.1.2
The Mellin transform of the function:

$$f(t) = e^{-pt} \quad p > 0 \tag{11.4}$$

is equal, by definition, to:

$$\mathcal{M}[f; s] = \int_0^\infty e^{-pt}\, t^{s-1}\, dt \tag{11.5}$$

Using the definition (see Appendix A) of the Gamma function, we obtain

$$\mathcal{M}[f; s] = p^{-s}\,\Gamma(s) \tag{11.6}$$

Recalling that the Gamma function is analytic in the region $Re(s) > 0$, we conclude that the strip of holomorphy is a half-plane as in the first example.

Example 11.2.1.3
Consider the function:

$$f(t) = (1+t)^{-1} \tag{11.7}$$

Its Mellin transform can be computed directly using the calculus of residues. But another method consists in changing the variables in (11.1) from t to x defined by:

$$t+1 = \frac{1}{1-x}, \quad x = \frac{t}{t+1}, \quad dx = \frac{dt}{(t+1)^2} \tag{11.8}$$

The transform of (11.7) is then expressed by:

$$\mathcal{M}[f; s] = \int_0^1 x^{s-1}(1-x)^{-s}\, dx \tag{11.9}$$

with the condition $0 < Re(s) < 1$. This integral is known (Appendix A) to define the beta function $B(s, 1-s)$ which can also be written in terms of Gamma functions. The result is given by the expression:

$$\begin{aligned}\mathcal{M}[f; s] &= B(s, 1-s) \\ &= \Gamma(s)\,\Gamma(1-s)\end{aligned} \tag{11.10}$$

which is analytic in the strip of existence of (11.9). An equivalent formula is obtained using a property (Appendix A) of the Gamma function:

$$\mathcal{M}[f; s] = \frac{\pi}{\sin \pi s} \tag{11.11}$$

valid in the same strip.

Relation to Laplace and Fourier Transformations

Mellin's transformation is closely related to an extended form of Laplace's. The change of variables defined by:

$$t = e^{-x}, \quad dt = -e^{-x}\, dx \tag{11.12}$$

transforms the integral (11.1) into:

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x}) e^{-sx}\, dx \tag{11.13}$$

After the change of function:

$$g(x) \equiv f(e^{-x}) \tag{11.14}$$

one recognizes in (11.13) the *two-sided* Laplace transform of g usually defined by:

$$\mathfrak{L}[g;\, s] = \int_{-\infty}^{\infty} g(x)\, e^{-sx}\, dx \tag{11.15}$$

This can be written symbolically as:

$$\mathcal{M}[f(t);\, s] = \mathfrak{L}[f(e^{-x});\, s] \tag{11.16}$$

The occurrence of a strip of holomorphy for Mellin's transform can be deduced directly from this relation. The usual right-sided Laplace transform is analytic in a half-plane $Re(s) > \sigma_1$. In the same way, one can define a left-sided Laplace transform analytic in the region $Re(s) < \sigma_2$. If the two half-planes overlap, the region of holomorphy of the two-sided transform is thus the strip $\sigma_1 < Re(s) < \sigma_2$ obtained as their intersection.

To obtain Fourier's transform, write now $s = a + 2\pi j\beta$ in (11.13):

$$F(s) = \int_{-\infty}^{\infty} f(e^{-x})\, e^{-ax}\, e^{-j2\pi\beta x}\, dx \tag{11.17}$$

The result is

$$\mathcal{M}[f(t);\, a + j2\pi\beta] = \mathfrak{F}[f(e^{-x})\, e^{-ax};\, \beta] \tag{11.18}$$

where $\mathfrak{F}$ represents the Fourier transformation defined by:

$$\mathfrak{F}[f;\, \beta] = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\beta x}\, dx \tag{11.19}$$

Thus, for a given value of $Re(s) = a$ belonging to the definition strip, the Mellin transform of a function can be expressed as a Fourier transform.

11.2.1.2 Inversion Formula

A direct way to invert Mellin's transformation (11.1) is to start from Fourier's inversion theorem. As is well know, if $\check{f} = \mathfrak{F}[f;\beta]$ is the Fourier transform (11.19) of f, the original function is recovered by:

$$f(x) = \int_{-\infty}^{\infty} \check{f}(\beta)\, e^{j2\pi\beta x}\, d\beta \tag{11.20}$$

Applying this formula to (11.17) with $s = a + j2\pi\beta$ yields:

$$f(e^{-x})\, e^{-ax} = \int_{-\infty}^{\infty} F(s)\, e^{j2\pi\beta x}\, d\beta \tag{11.21}$$

Hence, going back to variables t and s:

$$f(t) = t^{-a} \int_{-\infty}^{\infty} F(s)\, t^{-j2\pi\beta}\, d\beta \tag{11.22}$$

The inversion formula finally reads:

$$f(t) = (1/2\pi j) \int_{a-j\infty}^{a+j\infty} F(s)\, t^{-s}\, ds \tag{11.23}$$

where the integration is along a vertical line through $Re(s) = a$. Here a few questions arise. What value of a has to be put into the formula? What happens when a is changed? Is the inverse unique? In what case is f a function defined for all t's?

It is clear that if F is holomorphic in the strip $S(a_1, a_2)$ and vanishes sufficiently fast when $Im(s) \to \pm\infty$, then by Cauchy's theorem, the path of integration can be translated sideways inside the strip without affecting the result of the integration. More precisely, the following theorem holds:[7,8]

Theorem 11.2.1.1

If, in the strip $S(a_1, a_2)$, $F(s)$ is holomorphic and satisfies the inequality:

$$|F(s)| \le K |s|^{-2} \tag{11.24}$$

for some constant K, then the function $f(t)$ obtained by formula (11.23) is a continuous function of the variable $t \in (0,\infty)$ and its Mellin transform is $F(s)$.

Remark that this result gives only a sufficient condition for the inversion formula to yield a continuous function.

From a practical point of view, it is important to note that the inversion formula applies to a function F, holomorphic in a given strip, and that the uniqueness of the result holds only with respect to that strip. In fact, a Mellin transform consists of a pair: a function $F(s)$ and a strip of holomorphy $S(a_1, a_2)$. A unique function $F(s)$ with several disjoint strips of holomorphy will in general have several reciprocals, one for each strip. Some examples will illustrate this point.

Example 11.2.1.4

The Mellin transform of the function:

$$f(t) = \left(H(t - t_0) - H(t)\right) t^{z} \tag{11.25}$$

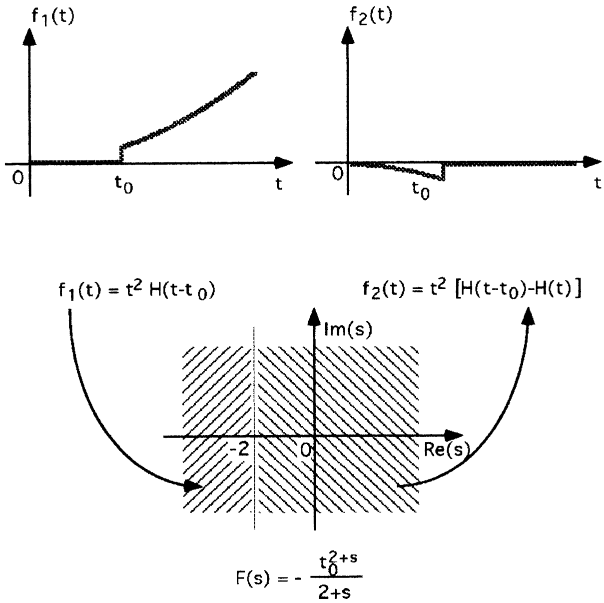

FIGURE 11.1 Examples of results when the regions of holomorphy are changed.

is given by:

$$\mathcal{M}[f;s] = -\frac{t_0^{z+s}}{(z+s)} \tag{11.26}$$

provided $Re(s) > -Re(z)$. Comparing (11.26) and (11.3), we see an example of two functions $F(s)$ having the same analytical expression but considered in two distinct regions of holomorphy: the inverse Mellin transforms, given respectively by (11.25) and (11.2) are indeed different (see Figure 11.1).

Example 11.2.1.5

Gamma function continuation. From the result of Example 11.2.1.2 considered for $p = 1$, the function $f(t) = e^{-t}$, $t > 0$ is known to be the inverse Mellin transform of $\Gamma(s)$, $Re(s) > 0$. Besides, it may be checked that $\Gamma(s)$ satisfies the hypotheses of Theorem 11.2.1.1; this is done by using Stirling's formula which implies the following behavior of the Gamma function:[5]

$$|\Gamma(a+ib)| \sim \sqrt{2\pi}\,|b|^{a-1/2}\, e^{-|b|\pi/2}, \quad |b| \to \infty \tag{11.27}$$

Thus, the inversion formula (11.23) can be applied here and gives an integral representation of e^{-t} as:

$$e^{-t} \equiv (1/2\pi j)\int_{a-j\infty}^{a+j\infty} \Gamma(s)\, t^{-s}\, ds, \qquad a > 0 \tag{11.28}$$

It is known that the Γ-function can be analytically continued in the left half-plane except for an infinite number of poles at the negative or zero integers. The inverse Mellin transform of the Gamma function for different strips of holomorphy will now be obtained by transforming the identity (11.28). The contour of integration can be shifted to the left and the integral will only pick up the values of the residues at each pole (Figure 11.2). Explicitly, if $a > 0$ and $-N < a' < -N$, N integer, we have:

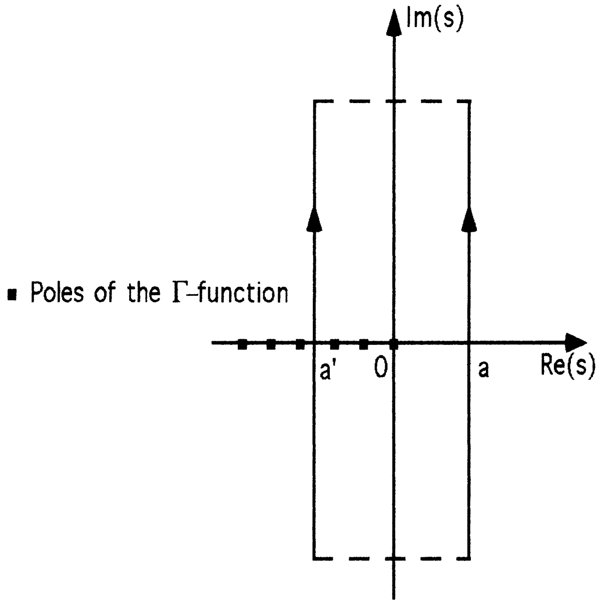

FIGURE 11.2 Different contours of integration for the inverse Mellin transform of the Gamma function. The contributions from the horizontal parts go to zero as Im(s) goes to infinity.

$$\left(1/2\pi j\right)\int_{a-j\infty}^{a+j\infty}\Gamma\left(s\right)t^{-s}\,ds=\sum_{n=0}^{N-1}\frac{\left(-1\right)^{n}}{n!}t^{n}+\left(1/2\pi j\right)\int_{a'-j\infty}^{a'+j\infty}\Gamma\left(s\right)t^{-s}\,ds \tag{11.29}$$

Hence, the inversion formula of the Γ-function in the strip $S(-N, -N + 1)$ gives the result:

$$\left(1/2\pi j\right)\int_{a'-j\infty}^{a'+j\infty}\Gamma\left(s\right)t^{-s}\,ds=e^{-t}-\sum_{n=0}^{N-1}\frac{\left(-1\right)^{n}}{n!}t^{n},\quad -N<a'<-N+1 \tag{11.30}$$

The integral term represents the remainder in the Taylor expansion of e^{-t} and can be shown to vanish in the limit $N \to \infty$ by applying Stirling formula.

As a corollary of the inversion formula, a Parseval relation can be established for suitable classes of functions.

Corollary Let $\mathcal{M}[f; s]$ and $\mathcal{M}[g; s]$ be the Mellin transforms of functions f and g with strips of holomorphy S_f and S_g, respectively, and suppose that some real number c exists such that $c \in S_f$ and $1 - c \in S_g$. Then Parseval's formula can be written as:

$$\int_0^{\infty} f\left(t\right)g\left(t\right)\,dt=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\mathcal{M}\left[f;\,s\right]\mathcal{M}\left[g;\,1-s\right]\,ds \tag{11.31}$$

This formula may be established formally by computing the right-hand side of (11.31) using definition (11.1):

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\mathcal{M}\left[f;\,s\right]\mathcal{M}\left[g;1-s\right]\,ds=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\mathcal{M}\left[g;1-s\right]\int_0^{\infty}f\left(t\right)t^{s-1}\,dt\,ds \tag{11.32}$$

Exchanging the two integrals:

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\mathcal{M}\left[f;s\right]\mathcal{M}\left[g;1-s\right]ds=\frac{1}{2\pi j}\int_{0}^{\infty}f\left(t\right)\int_{c-j\infty}^{c+j\infty}\mathcal{M}\left[g;1-s\right]t^{s-1}\,dt\,ds \tag{11.33}$$

and using the inverse formula (11.23) for g leads to (11.31).

Different sets of conditions ensuring the validity of this Parseval formula may be stated (see Reference 9 page 108). The crucial point is the interchange of integrals that cannot always be justified.

11.2.1.3 Transformation of Distributions

The extension of the correspondence (11.1) to distributions has to be considered to introduce a larger framework in which Dirac delta and other singular functions can be treated straightforwardly. The distributional setting of Mellin's transformation has been studied mainly by Fung Kang,[10] A. H. Zemanian,[7] and O. P. Misra and J. L. Lavoine.[8] We refer the interested reader to these works for a thorough treatment. As we will see, several approaches of the subject are possible as it was the case for Fourier's transformation.

It is possible to define the Mellin transform for all distributions belonging to the space $\mathcal{D}'_+$ of distributions on the half-line $(0, \infty)$ The procedure[10] is to start from the space $\mathcal{D}(0,\infty)$ of infinitely differentiable functions of compact support on $(0,\infty)$ and to consider the set Q of their Mellin transforms. It can be shown that it is a space of entire functions which is isomorphic, as a linear topological space, to the space Z of Gelfand and Shilov.[11] This space can be used as a space of test functions and the one-to-one correspondence thus defined between elements of spaces $\mathcal{D}(0,\infty)$ and Q can then be carried (i.e., transposed) to the dual spaces $\mathcal{D}'_+$ and Q'. In this operation, a Mellin transform is associated with any distribution in $\mathcal{D}'_+$ and the result belongs to a space Q' formed of analytic functionals (see Example 11.2.1.6 for an illustration). The situation is quite analogous to that encountered with the Fourier transformation where a correspondence between distributions spaces $\mathcal{D}'$ and Z' is established.

Actually, it may be efficient to restrict the class of distributions for which the Mellin transformation will be defined, as is usually done in Fourier analysis with the introduction of the space $\mathcal{S}'$ of tempered distributions.[11] In the present case, a similar approach can be based on the possibility to single out subspaces of $\mathcal{D}'_+$ whose elements are Mellin-transformed into functions which are analytic in a given strip. This construction will now be sketched.

The most practical way to proceed is to give a new interpretation of formula (11.1) by considering it as the application of a distribution f to a test function t^{s-1}:

$$F\left(s\right)=\left\langle f,\,t^{s-1}\right\rangle \tag{11.34}$$

A suitable space of test functions $\mathcal{T}(a_1, a_2)$ containing all functions t^{s-1} for s in the region $a_1 < Re(s) < a_2$ may be introduced as follows.[8] The space $\mathcal{T}(a_1, a_2)$ is composed of functions $\phi(t)$ defined on $(0,\infty)$ and with continuous derivatives of all orders going to zero as t approaches either zero or infinity. More precisely, there exists two positive numbers ζ_1, ζ_2, such that, for all integers k, the following conditions hold:

$$t^{k+1-a_1-\zeta_1}\,\phi^{(k)}\left(t\right)\to 0,\quad t\to 0 \tag{11.35}$$

$$t^{k+1-a_2-\zeta_2}\,\phi^{(k)}\left(t\right)\to 0,\quad t\to 0 \tag{11.36}$$

A topology on $\mathcal{T}$ is defined accordingly, it can be verified that all functions in $\mathcal{D}(0,\infty)$ belong to $\mathcal{T}(a_1, a_2)$.*
The space of distributions $\mathcal{T}'(a_1, a_2)$ is then introduced as a linear space of continuous linear functionals

*More precisely, one can show that $\mathcal{D}(0,\infty)$ is dense in $\mathcal{T}(a_1,a_2)$.

on $\mathcal{T}(a_1, a_2)$. It may be noticed that if α_1, α_2 are two real numbers such that $a_1 < \alpha_1 < \alpha_2 < a_2$, then $\mathcal{T}(\alpha_1, \alpha_2)$ is included in $\mathcal{T}(a_1, a_2)$. One may so define a whole collection of ascending spaces $\mathcal{T}(a_1, a_2)$ with compatible* topologies, thus ensuring the existence of limit spaces when $a_1 \to -\infty$ and/or $a_2 \to \infty$.

Hence, the dual spaces of distributions are such that $\mathcal{T}'(a_1, a_2) \subset \mathcal{T}'(\alpha_1, \alpha_2)$ and $\mathcal{T}'(-\infty, +\infty)$ is included in all of them. Moreover, as a consequence of the status of $\mathcal{D}(0,\infty)$ relatively to $\mathcal{T}(a_1, a_2)$, the space $\mathcal{T}'(a_1, a_2)$ is a subspace of distributions in $\mathcal{D}'_+$. The precise construction of these spaces is explained in Reference 8. A slightly different presentation is given in Reference 7 and leads to these same spaces denoted by $\mathcal{M}'(a_1, a_2)$.

With the above definitions, the Mellin transform of an element $f \in \mathcal{T}'(a_1, a_2)$ is defined by:

$$\mathcal{M}\left[f;\, s\right] \equiv F\left(s\right) = \left\langle f,\, t^{s-1}\right\rangle \tag{11.37}$$

The result is always a conventional function $F(s)$ holomorphic in the strip $a_1 < Re(s) < a_2$.

In summary, every distribution in $\mathcal{D}'_+$ has a Mellin transform which, as a rule, is an analytic functional. Besides, it is possible to define subspaces $\mathcal{T}'(a_1, a_2)$ of $\mathcal{D}'_+$ whose elements, f, are Mellin transformed by formula (11.37) into functions $F(s)$ holomorphic in the strip $S(a_1, a_2)$. Any space, $\mathcal{T}'$, contains in particular Dirac distributions and arbitrary distributions of bounded support. They are stable under derivation and multiplication by a smooth function. Their complete characterization is given by the following theorems.

Theorem 11.2.1.2

(Uniqueness theorem[7]) Let $\mathcal{M}[f;s] = F(s)$ and $\mathcal{M}[h;s] = H(s)$ be Mellin transforms with strips of holomorphy S_f and S_h, respectively. If the strips overlap and if $F(s) \equiv H(s)$ for $s \in S_f \cap S_h$, then $f \equiv h$ as distributions in $\mathcal{T}'(a_1, a_2)$ where the interval (a_1, a_2) is given by the intersection of $S_f \cap S_h$ with the real axis.

Theorem 11.2.1.3

(Characterization of the Mellin transform of a distribution in $\mathcal{T}'(a_1, a_2)$.[7,8] A necessary and sufficient condition for a function $F(s)$ to be the Mellin transform of a distribution $f \in \mathcal{T}'(a_1, a_2)$ is

- $F(s)$ is analytic in the strip $a_1 < Re\,(s) < a_2)$,
- For any closed substrip $\alpha_1 \le Re(s) \le \alpha_2$ with $a_1 < \alpha_1 < \alpha_2 < a_2$ there exists a polynomial P such that $|F(s)| \le P(|s|)$ for $\alpha_1 \le Re(s) \le \alpha_2$.

Example 11.2.1.6

(Example of analytic functional) The function t^z, z complex, defines a distribution in $\mathcal{D}'_+$ according to:

$$\left\langle t^z, \phi\right\rangle = \int_0^\infty t^z\, \phi\left(t\right) dt, \quad \phi \in \mathcal{D}\left(0, \infty\right) \tag{11.38}$$

But it may seem that this distribution does not belong to any of the spaces $\mathcal{T}'(a_1, a_2)$. Its Mellin transform may nevertheless be defined by the following formula:

$$\left\langle \mathcal{M}\left[t^z\right], \psi\right\rangle_M = \left\langle t^z, \phi\right\rangle \tag{11.39}$$

where $< , >_M$ denotes duality in the space of Mellin transforms, $\phi \equiv M^{-1}\psi$ is an element of $\mathcal{D}(0,\infty)$ and, consequently, ψ is an entire function. According to (11.39) and definition (11.2.1), we obtain:

$$\begin{aligned}\left\langle \mathcal{M}\left[t^z\right], \psi\right\rangle_M &= \mathcal{M}\left[\phi;\, z+1\right] \\ &= \psi\left(z+1\right)\end{aligned} \tag{11.40}$$

*In fact, $\mathcal{T}(-\infty, a_2)$, $\mathcal{T}(a_1, +\infty)$, and $\mathcal{T}(-\infty, +\infty)$ are defined as inductive limits.

Since distribution $\mathcal{M}[t^z]$ applied to ψ gives the value of ψ in a point of the complex plane, it can be symbolized by a delta function. To introduce the notation, we need the explicit form of duality $\langle\, ,\rangle_M$ which comes out of Parseval formula. According to (11.31), it is given for entire functions χ, ψ by:

$$\langle \chi, \psi \rangle_M = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \chi(s)\psi(1-s)\, ds \tag{11.41}$$

where c is any real number. A more usual form is obtained by setting:

$$\tilde{\psi}(s) \equiv \psi(1-s) \tag{11.42}$$

and

$$\langle \chi, \tilde{\psi} \rangle \equiv \langle \chi, \psi \rangle_M \tag{11.43}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \chi(s)\tilde{\psi}(s)\, ds \tag{11.44}$$

With these definitions, (11.40) can be written:

$$\langle \mathcal{M}[t^z], \tilde{\psi} \rangle = \tilde{\psi}(-z) \tag{11.45}$$

and the notation:

$$\mathcal{M}[t^z] = \delta(s+z) \tag{11.46}$$

can be proposed. Such Dirac distributions in the complex plane are defined in Reference 11.

Example 11.2.1.7

The Mellin transform of the Dirac distribution $\delta(t - t_0)$ is found by applying the general rule:

$$\langle \delta(t-t_0), \phi \rangle = \phi(t_0) \tag{11.47}$$

to the family of functions $\phi(t) = t^{s-1}$. One obtains:

$$\begin{aligned} \mathcal{M}[\delta(t-t_0); s] &= \langle \delta(t-t_0), t^{s-1} \rangle \\ &= t_0^{s-1} \end{aligned} \tag{11.48}$$

for any value of the positive number t_0. Moreover the result is holomorphic in the whole complex s-plane.

It is instructive to verify explicitly the inverse formula on this example. According to (11.23), the inverse Mellin transform $\mathcal{M}^{-1}[t_0^{s-1};\ t]$ can be written as:

$$\mathcal{M}^{-1}\left[t_0^{s-1};\, t\right] = \frac{1}{2\pi j t_0} \int_{-j\infty}^{j\infty} \left(\frac{t}{t_0}\right)^{-s} ds \tag{11.49}$$

since the choice $a = 0$ is allowed by the holomorphy property of the integrand in the whole plane. Setting $s = j\beta$ in (11.49) and performing the integration leads to the equivalent expressions:

$$\begin{aligned}\mathcal{M}^{-1}\left[t_0^{s-1};\, t\right] &= \frac{1}{2\pi t_0} \int_{-\infty}^{\infty} e^{-j\beta \ln\left(t/t_0\right)}\, d\beta \\ &= t_0^{-1}\, \delta\left(\ln t - \ln t_0\right)\end{aligned} \tag{11.50}$$

The expected result:

$$\mathcal{M}^{-1}\left[t_0^{s-1};\, t\right] = \delta\left(t - t_0\right) \tag{11.51}$$

comes out by using the classical formula:

$$\delta\left(f(t)\right) = \left|f'\left(t_0\right)\right|^{-1} \delta\left(t - t_0\right) \tag{11.52}$$

in which $f(t)$ is a function having a simple zero in $t = t_0$.

Example 11.2.1.8

Consider the distribution:

$$f = \sum_{n=1}^{\infty} \delta\left(t - pn\right), \qquad p > 0 \tag{11.53}$$

Its Mellin transform is given by:

$$\begin{aligned}\left\langle \sum_{n=1}^{\infty} \delta\left(t - pn\right),\, t^{s-1} \right\rangle &= \sum_{n=1}^{\infty} \left(pn\right)^{s-1} \\ &= p^{s-1} \sum_{n=1}^{\infty} n^{s-1}\end{aligned} \tag{11.54}$$

The sum converges uniformly for $Re(s) < 0$ and can be expressed in terms of Riemann's Zeta function[13] (see Appendix A). Explicitly, we have

$$\mathcal{M}\left[\sum_{n=1}^{\infty} \delta\left(t - pn\right);\, s\right] = p^{s-1}\, \zeta\left(1 - s\right), \qquad Re\left(s\right) < 0 \tag{11.55}$$

11.2.1.4 Some Properties of the Transformation

This paragraph describes the effect on the Mellin transform $\mathcal{M}[f;s]$ of some special operations performed on f. The resulting formulas are very useful for deducing new correspondences from a given one.

Let $F(s) = \mathcal{M}[f;s]$ be the Mellin transform of a distribution which is supposed to belong to $\mathcal{T}'(\sigma_1,\sigma_2)$ and denote by $S_f = \{s{:}\sigma_1 < Re(s) < \sigma_2\}$ its strip of holomorphy (σ_1 is either finite or $-\infty$, σ_2 is finite or ∞). Then the following formulas hold with the regions of holomorphy as indicated. The notation of functions will be used but this must not obscure the fact that f is a distribution and that all operations performed on f, especially differentiation, must be understood in the generalized sense of distributions.

- *Scaling of the original variable by a positive number:*

$$\mathcal{M}\left[f(rt);s\right] = r^{-s}F(s), \quad s\in S_f, \quad r>0 \tag{11.56}$$

- *Raising of the original variable to a real power:*

$$\mathcal{M}\left[f(t^r);s\right] = |r|^{-1}F(r^{-1}s), \quad r^{-1}s\in S_f, \quad r \text{ real } \neq 0 \tag{11.57}$$

- *Multiplication of the original function by* ln*t:*

$$\mathcal{M}\left[(\ln t)^k f(t);s\right] = \frac{d^k}{ds^k}F(s), \quad s\in S_f,\ k \text{ positive integer} \tag{11.58}$$

- *Multiplication of the original function by some power of t:*

$$\mathcal{M}\left[(t)^z f(t);s\right] = F(s+z), \quad s+z\in S_f, \quad z \text{ complex} \tag{11.59}$$

- *Derivation of the original function:*

$$\mathcal{M}\left[\frac{d^k}{dt^k}f(t);s\right] = (-1)^k(s-k)_k F(s-k), \quad s-k\in S_f,\ k \text{ positive integer} \tag{11.60}$$

where the symbol $(s-k)_k$ is defined for k integer by:

$$(s-k)_k \equiv (s-k)(s-k+1)\ldots(s-1) \tag{11.61}$$

Formulas (11.59) and (11.60) can be used in various ways to find the effect of linear combinations of differential operators such that $t^k(d/dt)^m$, k, m integers. The most remarkable result is

$$\mathcal{M}\left[\left(t\frac{d}{dt}\right)^k f(t);s\right] = (-1)^k s^k F(s) \tag{11.62}$$

Other combinations can be computed. We have, for example,

$$\mathcal{M}\left[\frac{d^k}{dt^k}t^k f(t);s\right] = (-1)^k(s-k)_k F(s) \tag{11.63}$$

$$\mathcal{M}\left[t^k \frac{d^k}{dt^k} f(t); s\right] = (-1)^k (s)_k F(s) \tag{11.64}$$

where $s \in S_f$, k a positive integer and

$$(s)_k \equiv s(s+1)\ldots(s+k-1) \tag{11.65}$$

These relations are easily verified on infinitely differentiable functions. It is important to stress that they are essentially true for distributions. That implies in particular that all derivatives occurring in the formulas are to be taken according to the distribution rules. An example dealing with a discontinuous function will make this manifest.

Example 11.2.1.9
Consider the function:

$$f(t) = H(t-t_0)t^z, \quad z \text{ complex} \tag{11.66}$$

According to the results of Example 11.2.1.1, the Mellin transform of f is given by

$$\mathcal{M}[f; s] \equiv F(s) = -\frac{t_0^{z+s}}{z+s} \tag{11.67}$$

Applying formula (11.60) for $k = 1$ yields:

$$\begin{aligned} \mathcal{M}\left[\frac{df}{dt}; s\right] &= -(s-1)\, F(s-1) \\ &= (s-1)\, \frac{t_0^{z+s-1}}{z+s-1} \end{aligned} \tag{11.68}$$

which can be rewritten as:

$$\mathcal{M}\left[\frac{df}{dt}; s\right] = -z\frac{t_0^{z+s-1}}{z+s-1} + t_0^{z+s-1} \tag{11.69}$$

or, recognizing the Mellin transforms obtained in Examples 11.2.1.1 and 11.2.1.7:

$$\mathcal{M}\left[\frac{df}{dt}; s\right] = \mathcal{M}\left[zH(t-t_0)t^{z+s-1}; s\right] + \mathcal{M}\left[t_0^z\,\delta(t-t_0); s\right] \tag{11.70}$$

This result shows explicitly that, in formula (11.60), f is differentiated as a distribution. The first term in (11.70) corresponds to the derivative of the function for $t \neq t_0$ and the second term is the Dirac distribution arising from the discontinuity at $t = t_0$.

Additional results on the Mellin transforms of primitives can be established for particular classes of functions. Namely, if $x > 1$, integration by parts leads to the result:

$$\mathcal{M}\left[\int_x^{\infty} f(t)\, dt;\, s\right] = \int_0^{\infty} s^{-1} x^s f(x)\, dx$$
$$= s^{-1} F(s+1) \tag{11.71}$$

provided the integrated part $s^{-1}x^s\int_x^{\infty} f(t)\ dt$ is equal to zero for $x = 0$ and $x = \infty$.

In the same way, but with different conditions on *f*, one establishes:

$$\mathcal{M}\left[\int_0^{x} f(t)\, dt;\, s\right] = -s^{-1} F(s+1) \tag{11.72}$$

11.2.1.5 Relation to Multiplicative Convolution

The usual convolution has the property of being changed into multiplication by either a Laplace or a Fourier transformation. In the present case, a multiplicative convolution,[10] also called Mellin-type convolution,[7,8] is defined which has a similar property with respects to Mellin's transformation. In the same way as the usual convolution of two distributions in $\mathcal{D}$(IR) does not necessarily exist, the multiplicative convolution of distributions in $\mathcal{D}'_+$ can fail to define a distribution. To avoid such problems, we shall restrict our considerations to spaces $\mathcal{T}'(a_1, a_2)$.

Definition 11.2.1.2

Let *f*, *g* be two distributions belonging to some space $\mathcal{T}'(a_1, a_2)$. The multiplicative convolution of *f* and *g* is a functional $(f \vee g)$ whose action on test functions $\theta \in \mathcal{T}(a_1, a_2)$ is given by:

$$\langle f \vee g, \theta \rangle = \langle f(t), \langle g(\tau), \theta(t\tau) \rangle \rangle \tag{11.73}$$

It can be shown that $f \vee g$ is a distribution which belongs to the space $\mathcal{T}'(a_1, a_2)$.

If the distributions *f* and *g* are represented by locally integrable functions, definition (11.73) can be written explicitly as:

$$\langle f \vee g, \theta \rangle = \int_0^{\infty}\int_0^{\infty} f(t) g(\tau) \theta(t\tau)\, dt\, d\tau \tag{11.74}$$

A change of variables then leads to the following expression for the multiplicative convolution of the functions *f* and *g*:

$$(f \vee g)(\tau) = \int_0^{\infty} f(t) g\left(\frac{\tau}{t}\right) \frac{dt}{t} \tag{11.75}$$

The so-called exchange formula for usual convolution has an analog for multiplicative convolution. It is expressed by the following theorem.[7,8]

Theorem 11.2.1.4 (exchange formula)

The Mellin transform of the convolution product $f \vee g$ of two distributions belonging to $\mathcal{T}'(a_1, a_2)$ is given by the formula:

$$\mathcal{M}[f \vee g;\, s] = F(s) G(s), \quad a_1 < Re(s) < a_2 \tag{11.76}$$

where $F(s)$ and $G(s)$ are the Mellin transforms of distributions *f* and *g*, respectively.

The proof is a simple application of the definitions. According to (11.37), the Mellin transform of distribution $f \vee g \in \mathcal{T}'(a_1, a_2)$ is given by:

$$\mathcal{M}\left[f \vee g; s\right] = \left\langle f \vee g, t^{s-1} \right\rangle, \quad a_1 < Re\left(s\right) < a_2 \tag{11.77}$$

or, using the definition (11.73) of convolution:

$$\mathcal{M}\left[f \vee g; s\right] = \left\langle f\left(t\right), \left\langle g\left(\tau\right), \left(t\tau\right)^{s-1} \right\rangle \right\rangle \tag{11.78}$$

which can be rewritten as:

$$\mathcal{M}\left[f \vee g; s\right] = \left\langle f\left(t\right), t^{s-1} \right\rangle \left\langle g\left(\tau\right), \tau^{s-1} \right\rangle \tag{11.79}$$

Formula (11.73) allows to consider the multiplicative convolution of general distributions not belonging to space $\mathcal{T}'(a_1, a_2)$. However, in that case, it is not ensured that the product exists as a distribution.

General Properties of the Multiplicative Convolution

In this paragraph, f and g are distributions which belong to $\mathcal{T}'(a_1, a_2)$ and k is a positive integer. The following properties are easy to verify. In fact, some of them are a direct consequence of the exchange formula.

1. *Commutativity*

$$f \vee g = g \vee f \tag{11.80}$$

2. *Associativity*

$$\left(f \vee g\right) \vee h = f \vee \left(g \vee h\right) \tag{11.81}$$

3. *Unit element*

$$f \vee \delta\left(t-1\right) = f \tag{11.82}$$

4. *Action of the operator $t(d/dt)$*

$$\begin{aligned} \left(t\frac{d}{dt}\right)^k \left(f \vee g\right) &= \left[\left(t\frac{d}{dt}\right)^k f\right] \vee g \\ &= f \vee \left[\left(t\frac{d}{dt}\right)^k g\right] \end{aligned} \tag{11.83}$$

i.e., it is sufficient to apply the operator to one of the factors.

5. *Multiplication by* $\ln t$.

$$\left(\ln t\right)\left(f \vee g\right) = \left[\left(\ln t\right) f\right] \vee g + f \vee \left[\left(\ln t\right) g\right] \tag{11.84}$$

6. *Convolution with Dirac distributions and their derivatives*

$$\delta(t-a) \vee f = a^{-1} f(a^{-1}t) \tag{11.85}$$

$$\delta(t-p) \vee \delta(t-p') = \delta(t-pp'), \quad p, p' > 0 \tag{11.86}$$

$$\delta^{(k)}(t-1) \vee f = (d/dt)^k (t^k f) \tag{11.87}$$

Proof of relation (11.87) — According to the definition of the k-derivative of the Dirac distribution, the multiplicative convolution of f with $\delta^{(k)}(t-1)$ is given by:

$$\begin{aligned}\left\langle f \vee \delta^{(k)}(t-1),\ \theta(t)\right\rangle &= \left\langle f(t), \left\langle \delta^{(k)}(\tau-1),\ \theta(t\tau)\right\rangle\right\rangle \\ &= \left\langle f(t), \left\langle \delta(\tau-1), (-1)^k \left(\frac{d}{d\tau}\right)^k \theta(t\tau)\right\rangle\right\rangle\end{aligned} \tag{11.88}$$

or, after performing an ordinary differentiation and applying the definition of δ:

$$\left\langle f \vee \delta^{(k)}(t-1),\ \theta(t)\right\rangle = \left\langle f(t), (-1)^k t^k \theta^{(k)}(t)\right\rangle \tag{11.89}$$

The usual rules of calculus with distributions and the commutativity of convolution yield:

$$\left\langle \delta^{(k)}(t-1) \vee f, \theta(t)\right\rangle = \left\langle \left(\frac{d}{dt}\right)^k \left(t^k f(t)\right),\ \theta(t)\right\rangle \tag{11.90}$$

Finally, identity (11.87) follows from the fact that (11.90) holds for any function θ belonging to $\mathcal{T}(a_1, a_2)$.

11.2.1.6 Hints for a Practical Inversion of the Mellin Transformation

In many applications, it is essential to be able to perform explicitly the Mellin inversion. This is often the most difficult part of the computation and we now give some indications on different ways to proceed.

Compute the inversion integral: This direct approach is not always the simplest. In some cases, however, the integral (11.23) can be computed by the method of residues.

Use rules of Section 11.2.1.4 to exploit the inversion formula: Property (11.62) in particular can be used to extend the domain of practical utility of the inversion formula (11.23). Indeed, in the case where the Mellin transform $F(s)$, holomorphic in the strip $S(a_1, a_2)$ with (a_1, a_2) finite, does not satisfy condition (11.24), suppose that a positive integer k can be found such that:

$$\left|s^{-k} F(s)\right| \le K |s|^{-2} \tag{11.91}$$

The inversion formula (11.23) can now be used on the function $G(s)$ defined by:

$$G(s) = (-1)^k s^{-k} F(s), \quad a_1 < Re(s) < a_2 \tag{11.92}$$

and yields a continuous function $g(t)$. Using rule (11.62) and the uniqueness of the Mellin transform, we conclude that the reciprocal of $F(s)$ is the distribution f defined by:

$$f = \left(t \frac{d}{dt} \right)^k g(t) \tag{11.93}$$

In spite of the fact that the continuous function $g(t)$ is not necessarily differentiable everywhere, formula (11.93) remains meaningful since derivatives are taken in the sense of distributions. In fact, the above procedure corresponds to generalizing Theorem 11.2.1.1 in the following form:

Theorem 11.2.1.5 [7,8]

Let $F(s)$ be a function holomorphic in the strip $S(a_1, a_2)$ with a_1, a_2 finite. If there exists an integer $k \geq 0$ such that $s^{2-k} F(s)$ is bounded as $|s|$ goes to infinity, then the inverse Mellin transform of $F(s)$ is the unique distribution f given by:

$$f = \left(t \frac{d}{dt} \right)^k g(t) \tag{11.94}$$

where $g(t)$ is a continuous function obtained by the formula:

$$g(t) = \frac{(-1)^k}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(s)\, s^{-k}\, t^{-s}\, ds \tag{11.95}$$

with $a \in S(a_1, a_2)$.

Other inversion formulas may be obtained in the same way,[8] by using rules (11.64) and (11.63), respectively. They are

$$f = t^k \left(\frac{d}{dt} \right)^k g(t), \quad \text{where} \quad g(t) = \frac{(-1)^k}{2\pi j} \int_{a-j\infty}^{a+j\infty} \frac{F(s)}{(s)_k} t^{-s}\, ds \tag{11.96}$$

$$f = \left(\frac{d}{dt} \right)^k t^k g(t), \quad \text{where} \quad g(t) = \frac{(-1)^k}{2\pi j} \int_{a-j\infty}^{a+j\infty} \frac{F(s)}{(s-k)_k} t^{-s}\, ds \tag{11.97}$$

Use the tables: In simple cases, exploitation of tables[14-16] and use of the rules of calculus exhibited in Section 11.2.1.4 are sufficient to obtain the result.

In more difficult cases, it may be rewarding to use the systematic approach developed by Marichev[5] and applicable to a large number of functions. Suppose we are given a function $F(s)$ holomorphic in the strip $S(\sigma_1, \sigma_2)$ and we want to find its inverse Mellin transform. The first step is to try and cast F into the form of a fraction involving only products of Γ-functions, the variable s appearing only with the coefficient ±1. This looks quite restrictive, but in fact many simple functions can be so rewritten using the properties of Γ-functions, recalled in Appendix A. Thus, $F(s)$ is brought to the form:

$$F(s)=C\prod_{i,j,k,l}\frac{\Gamma(a_i+s)\Gamma(b_j-s)}{\Gamma(c_k+s)\Gamma(d_l-s)} \tag{11.98}$$

where C, a_i, b_j, c_k, d_l are constants and where $Re(s)$ is restricted to the strip $S(\sigma_1,\sigma_2)$ now defined in terms of these.

For such functions, the explicit computation of the inversion integral (11.23) can be performed by the theory of residues and yields a precise formula given in Reference 5 as Slater's theorem. The result has the form of a function of hypergeometric type. The important point is that most special functions are included in this class. For a thorough description of the method, the reader is referred to Marichev's book[5] which contains simple explanations along with all the proofs and exhaustive tables.

Special forms related to the use of polar coordinates:[17] The analytical solution of some two-dimensional problems in polar coordinates (r,θ) is obtained by using a Mellin transformation with respect to the radial variable r. In this approach, one can be faced with the task of inverting expressions of the type $\cos(s\theta)\,F(s)$ or $\sin(s\theta)\,F(s)$. We will show that, for a large class of problems, the reciprocals of the products $\cos(s\theta)\,F(s)$ and $\sin(s\theta)\,F(s)$ can be obtained straightforwardly from the knowledge of the reciprocal of $F(s)$.

Let $f(r)$, f *real-valued*, be the inverse Mellin transform of $F(s)$ in strip $S(a_1, a_2)$ and suppose that f can be analytically continued into a function $f(z)$, $z \equiv re^{j\theta}$, in some sector $|\theta| < \beta$ of the complex plane. If the rule of scaling (11.56) can be extended to the complex numbers, we have

$$\mathcal{M}\left[f\left(re^{j\theta}\right);\,s\right]=e^{-j\theta s}\,\mathcal{M}\left[f(r);\,s\right] \tag{11.99}$$

where the Mellin transforms are with respect to r.

In fact, this formula can be established by contour integration in a sector $|\arg z| < \beta$ where the function f is such that:

$$z^s f(z)\to 0 \quad as \quad |z|\to 0 \text{ or } \infty \tag{11.100}$$

Remark that since f has a Mellin transform with strip of definition $S(a_1, a_2)$, this condition already holds on the real axis when $a_1 < Re\,(s) < a_2$.

Recalling that f is real-valued, we can take the real and imaginary parts of (11.99) and obtain, for real s, the formulas:

$$\mathcal{M}\left[Re\left(f\left(re^{j\theta}\right)\right);\,s\right]=\cos(s\theta)F(s) \tag{11.101}$$

$$\mathcal{M}\left[Im\left(f\left(re^{j\theta}\right)\right);\,s\right]=-\sin(s\theta)F(s) \tag{11.102}$$

These can be extended to complex s and yield the inverse Mellin transforms of $\cos(s\theta)\,F(s)$ and $\sin(s\theta)$ $F(s)$ for $a_1 < Re(s) < a_2$ and $|\theta| < \beta$.

Example 11.2.1.10[17]
To illustrate the use of the above rules, we shall perform explicitly the inversion of:

$$F(s)=\frac{\cos s\theta}{s\cos s\alpha},\quad \alpha \text{ real} \tag{11.103}$$

in the strip $0 < Re(s) < \pi/(2\alpha)$.

Using the result of Example 11.2.1.3 and rule (11.57), we obtain:

$$\mathcal{M}\left[\left(1+r^2\right)^{-1}; s\right] = \frac{\tau}{2\sin\left(\pi s/2\right)}, \quad 0 < Re\left(s\right) < 1 \tag{11.104}$$

Recalling that

$$\int_r^\infty \frac{dx}{1+s^2} = \pi/2 - \tan^{-1} r \tag{11.105}$$

and using rule (11.71) gives:

$$\mathcal{M}\left[\pi/2 - \tan^{-1} r; s\right] = \frac{\pi}{2s\cos \pi s/2}, \quad 0 < Re\left(s\right) < 1 \tag{11.106}$$

Using again property (11.57) but with $\nu = \pi/2\alpha$ finally gives:

$$\mathcal{M}\left[\pi/2 - \tan^{-1} r^\nu; s\right] = \frac{\pi}{2s\cos\left(\pi s/2\nu\right)}, \quad 0 < Re\left(s\right) < \nu \tag{11.107}$$

To find the domain in which function $f(z) = \pi/2 - \tan^{-1}z^\nu$, where $z = re^{j\theta}$, verifies the condition (11.100), we write it under the form:

$$f\left(z\right) = \left(1/2j\right)\ln\left(\frac{z^\nu + j}{z^\nu - j}\right) \tag{11.108}$$

subject to the choice of the determination for which $0 < \tan^{-1}r < \pi/2$. The result is $|\theta| \equiv |\arg(z)| < \pi/2$. Relation (11.101) yields the result:

$$\mathcal{M}^{-1}\left[\frac{\cos\left(s\theta\right)}{s\cos\left(s\alpha\right)}; s\right] = Re\left(\pi/2 - \tan^{-1} z^\nu\right), \;\; 0 < Re\left(s\right) < \pi/\left(2\alpha\right), \;\; \left|\theta\right| < \pi/2 \tag{11.109}$$

The real part of $f(re^{j\theta})$ is given explicitly by:

$$Re\left(\pi/2 - \tan^{-1} z^\nu\right) = \begin{cases} 1 - \pi^{-1}\tan^{-1}\dfrac{2r^\nu\cos\nu\theta}{1-r^{2\nu}} & 0 \le r < 1 \\ \pi^{-1}\tan^{-1}\dfrac{2r^\nu\cos\nu\theta}{r^{2\nu-1}} & r > 1 \end{cases}$$

11.2.2 Standard Applications

11.2.2.1 Summation of Series

Even if a numerical computation is intended, Mellin's transformation may be used with profit to transform slowly convergent series either into integrals that can be computed exactly or into more rapidly convergent series.

Let S represent a series of the form:

$$S = \sum_{n=1}^{\infty} f(n) \tag{11.110}$$

in which the terms are samples of a function $f(t)$ for integer values of the variable $t \in (0, \infty)$. If this function has a Mellin transform $F(s)$ with $S(a_1, a_2)$ as strip of holomorphy, it can be written:

$$f(t) = (2\pi j)^{-1} \int_{a-j\infty}^{a+j\infty} F(s)\, t^{-s}\, ds, \quad a_1 < a < a_2 \tag{11.111}$$

Substituting this identity in (11.110) yields:

$$S = (2\pi j)^{-1} \sum_{n=1}^{\infty} \int_{a-j\infty}^{a+j\infty} F(s) n^{-s}\, ds \tag{11.112}$$

Now, if $F(s)$ is such that sum and integral can be exchanged, an integral expression for S is obtained:

$$S = (2\pi j)^{-1} \int_{a-j\infty}^{a+j\infty} F(s)\zeta(s)\, ds \tag{11.113}$$

where $\zeta(s)$ is the Riemann zeta function defined by:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{11.114}$$

The integral (11.113) is then evaluated by the methods of Section 11.2.1.6. The calculus of residues may give the result as an infinite sum which, hopefully, will be more rapidly convergent than the original series.

Some care is necessary when going from (11.112) to (11.113). Actually, if the interchange of summation and integration is not justified, the expression (11.113) can fail to represent the original series (see Reference 17, page 216 for an example).

Example 11.2.2.1

Compute the sum:

$$S(y) = \sum_{n=1}^{\infty} \frac{\cos ny}{n^2} \tag{11.115}$$

From Table 11.3 and properties (11.56) and (11.59), one finds:

$$\mathcal{M}\left[\frac{\cos ty}{t^2};\ s\right] = -y^{2-s}\,\Gamma(s-2)\cos(\pi s/2), \quad 2 < Re(s) < 3 \tag{11.116}$$

Hence, the sum can be rewritten as:

$$S = -\left(1/2\pi j\right)\int_{a-j\infty}^{a+j\infty} y^{2-s}\,\Gamma\left(s-2\right)\cos\left(\pi s/2\right)\zeta\left(s\right)\,ds \tag{11.117}$$

where the interchange of summation and integration is justified by absolute convergence. The integral can be rearranged by using Riemann's functional relationship (see Reference 17 and Appendix A):

$$\pi^{s}\zeta\left(1-s\right) = 2^{1-s}\,\Gamma\left(s\right)\cos\left(\pi s/2\right)\zeta\left(s\right) \tag{11.118}$$

Then (11.117) becomes:

$$S = -\left(1/2\pi j\right)\int_{a-j\infty}^{a+j\infty} y^{2-s}\,2^{s-1}\,\pi^{s}\,\frac{\zeta\left(1-s\right)}{\left(s-1\right)\left(s-2\right)}\,ds \tag{11.119}$$

The integral is easily computed by the method of residues, closing the contour to the left where the integrand goes to zero. The function $\zeta(s)$ is analytic everywhere except at $s = 1$ where it has a simple pole with residue equal to 1. The result is

$$S = \frac{y^2}{4} - \frac{\pi y}{2} + \frac{\pi^2}{6} \tag{11.120}$$

11.2.2.2 Computation of Integrals Depending on a Parameter

Essentially, the technique concerns integrals which can be brought to the form:

$$K\left(x\right) = \int_0^{\infty} K_0\left(t\right)\,K_1\left(x/t\right)\frac{dt}{t}, \qquad x > 0 \tag{11.121}$$

One recognizes the expression of a multiplicative convolution. Such an integral can be computed by performing the following steps:

- Mellin transform functions K_0 and K_1 to obtain $\mathcal{M}[K_0;s]$ and $\mathcal{M}[K_1;s]$.
- Multiply the transforms to obtain $\mathcal{M}[K;s] \equiv \mathcal{M}[K_0;s]\ \mathcal{M}[K_1;s]$.
- Find the inverse Mellin transform of $\mathcal{M}[K;s]$ using the tables. The result will in general be expressed as a combination of generalized hypergeometric series.

For the last operation, the book by O. I. Marichev[5] can be of great help as previously mentioned in Section 11.2.1.6. The method can be extended to allow the computation of integrals of the form:

$$K\left(x_1, \ldots, x_N\right) = \int_0^{\infty} K_0\left(t\right)\left[\prod_{i=1}^{n} K_i\left(\frac{x_i}{t}\right)\right]\frac{dt}{t} \tag{11.122}$$

where $x_1, \ldots, x_N$ are positive variables.

It can be verified that the multiple Mellin transform defined by:

$$\}\left[K; s_1, \ldots, s_N\right] = \int_0^{\infty}\cdots\int_0^{\infty} K\left(x_1, \ldots, x_N\right) x_1^{s_1-1}\cdots x_N^{s_N-1}\,dx_1\ldots dx_N \tag{11.123}$$

allows the expression (11.122) to be factored as:

$$\mathcal{M}\left[K; s_1, \ldots, s_N\right] = \mathcal{M}\left[K_0; s_1 + s_2 + \ldots + s_N\right] \prod_{i=1}^{N} \mathcal{M}\left[K_i; s_i\right] \tag{11.124}$$

Techniques of inversion for this expression are developed in a book by R. J. Sasiella[18] devoted to the propagation of electromagnetic waves in turbulent media.

11.2.2.3 Mellin's Convolution Equations

These are not always expressed with integrals of type (11.75) but also with differential operators which are polynomials in $(t(d/dt))$. Such equations are of the general form:

$$Lu(t) \equiv \left(a_n \left(t\frac{d}{dt} \right)^n + a_{n-1} \left(t\frac{d}{dt} \right)^{n-1} + \ldots + a_0 \right) u(t) = g(t) \tag{11.125}$$

By using the identity:

$$\left(t(d/dt)\right)^k u(t) \equiv \left[\left(t(d/dt)\right)^k \delta(t-1)\right] \vee u(t) \tag{11.126}$$

they can be written as a convolution:

$$\sum_{k=0}^{n} a_k \left(t(d/dt)\right)^k \delta(t-1) \vee u(t) = g(t) \tag{11.127}$$

The more usual Euler-Cauchy differential equation, which is written as:

$$\left(b_n t^n (d/dt)^n + b_{n-1} t^{n-1} (d/dt)^{n-1} + \ldots + b_0 \right) u(t) = g(t) \tag{11.128}$$

can be brought to the form (11.125) by using relations such that:

$$\left(t\frac{d}{dt} \right)^2 = t\frac{d}{dt} + t^2 \frac{d^2}{dt^2} \tag{11.129}$$

It can also be transformed directly into a convolution which reads:

$$\sum_{k=0}^{n} b_k t^k \delta^{(k)}(t-1) \vee u(t) = g(t) \tag{11.130}$$

The Mellin treatment of convolution equations will be explained in the case of Equation (11.127) since it is the most characteristic.

Suppose that the known function g has a Mellin transform $\mathcal{M}[g; s] = G(s)$ that is holomorphic in the strip $S(\sigma_l, \sigma_r)$. We shall seek solution u which admit a Mellin transform $U(s)$ holomorphic in the same strip or in some substrip. The Mellin transform of Equation (11.127) is obtained by using the convolution property and relation (11.62):

$$A(s)U(s) = G(s) \tag{11.131}$$

where

$$A(s) \equiv \sum_{k=0}^{\infty} a_k (-1)^k s^k \tag{11.132}$$

Two different situations may arise.

1. Either $A(s)$ has no zeros in the strip $S(\sigma_l, \sigma_r)$. In that case, $U(s)$ given by $G(s)/A(s)$ can be inverted in the strip. According to Theorems 11.2.1.2 and 11.2.1.3, the unique solution is a distribution belonging to $\mathcal{T}'(\sigma_1, \sigma_2)$.
2 Or $A(s)$ has m zeros in the strip. The main strip $S(\sigma_l, \sigma_r)$ can be decomposed into adjacent substrips

$$\sigma_l < Re(s) < \sigma_1,\ \sigma_1 < Re(s) < \sigma_2, \ldots \sigma_m < Re(s) < \sigma_r \tag{11.133}$$

The solution in the k-substrip is given by the Mellin inverse formula:

$$u(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \frac{G(s)\, t^{-s}}{A(s)}\, ds \tag{11.134}$$

where $\sigma_k < c < \sigma_{k+1}$. There is a different solution in each strip, two solutions differing by a solution of the homogeneous equation.

11.2.2.4 Solution of a Potential Problem in a Wedge[7,8,17]

The problem is to solve Laplace's equation in an infinite two-dimensional wedge with Dirichlet boundary conditions. Polar coordinates with origin at the apex of the wedge are used and the sides are located at $\theta = \pm\alpha$. The unknown function $u(r,\theta)$ is supposed to verify:

$$\Delta u = 0, \qquad 0 < r < \infty, -\alpha < \theta < \alpha \tag{11.135}$$

with the following boundary conditions:

1. On the sides of the wedge, if R is a given positive number:

$$u(r, \pm\alpha) = \begin{cases} 1 \text{ if } 0 < r < R \\ 0 \text{ if } r > R \end{cases} \tag{11.136}$$

   or, equivalently:

$$u(r, \pm\alpha) = H(R - r) \tag{11.137}$$

2. When r is finite, $u(r,\theta)$ is bounded.
3. When r tends to infinity, $u(r,\theta) \sim r^{-\beta}$, $\beta > 0$.

In polar coordinates, Equation (11.135) multiplied by r^2 yields:

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{11.138}$$

The above conditions on $u(r,\theta)$ ensure that its Mellin transform $U(s,\theta)$ with respect to r exists as a holomorphic function in some region $0 < Re(s) < \beta$. The equation satisfied by U is obtained from (11.138) by using property (11.59) of the Mellin transformation and reads:

$$\frac{d^2U}{d\theta^2}(s, \theta) + s^2 U(s, \theta) = 0 \tag{11.139}$$

The general solution of this equation can be written as:

$$U(s, \theta) = A(s) e^{js\theta} + B(s) e^{-js\theta} \tag{11.140}$$

Functions A, B are to be determined by the boundary condition (11.137) which leads to the following requirement on U:

$$U(s, \pm\alpha) = R^s s^{-1} \quad \text{for} \quad Re(s) > 0 \tag{11.141}$$

Explicitly, this is written as:

$$A(s) e^{js\alpha} + B(s) e^{-js\alpha} = a^s s^{-1} \tag{11.142}$$

$$A(s) e^{-js\alpha} + B(s) e^{js\alpha} = a^s s^{-1} \tag{11.143}$$

and leads to the solution:

$$A(s) = B(s) = \frac{R^s}{2s \cos(s\alpha)} \tag{11.144}$$

The solution of the form (11.140) which verifies (11.141) is given by:

$$U(s, \theta) = \frac{R^s \cos(s\theta)}{s \cos(s\alpha)} \tag{11.145}$$

This function U is holomorphic in the strip $0 < Re(s) < \pi/(2\alpha)$. Its inverse Mellin transform is a function $u(r,\theta)$ that is obtained from the result of Example 11.2.1.10.

11.2.2.5 Asymptotic Expansion of Integrals

The Laplace transform $I[f;\lambda]$ defined by:

$$I[f; \lambda] = \int_0^\infty e^{-\lambda t} f(t)\, dt \tag{11.146}$$

has an asymptotic expansion as λ goes to infinity which is characterized by the behavior of the function f when $t \to 0+$.[9,14,19] With the help of Mellin's transformation, one can extend this type of study to other transforms of the form:

$$I\left[f;\lambda\right]=\int_0^{\infty} h\left(\lambda t\right) f\left(t\right) dt \tag{11.147}$$

where h is a general kernel.

Examples of such *h-transforms*[9] are

- Fourier transform: $h(\lambda t) = e^{j\lambda t}$
- Cosine and Sine transforms: $h(\lambda t) = \cos(\lambda t)$ or $\sin(\lambda t)$
- Laplace transform: $h(\lambda t) = e^{-\lambda t}$
- Hankel transform: $h(\lambda t) = J_v(\lambda t)(\lambda t)^{1/2}$ where J_v is the Bessel function of the first kind
- Generalized Stieltjes transform: $h(\lambda t) = \lambda^v \int_o^{\infty} f(t)/(1+\lambda t)^v\, dt$

A short formal overview of the procedure will be given below. The theory is exposed in full generality in Reference 9. It includes the study of asymptotic expansions when $\lambda \to 0+$ in relation with the behavior of f at infinity and the extension to complex values of λ. The case of oscillatory h-kernels is given special attention.

Suppose from now on that f and h are locally integrable functions such that the transform $I[f;\lambda]$ exists for the large λ. The different steps leading to an asymptotic expansion of $I[f;\lambda]$ in the limit $\lambda \to +\infty$ are the following:

1. *Mellin transform the functions h and f and apply Parseval's formula.* The Mellin transforms $\mathcal{M}[f;s]$ and $\mathcal{M}[h;s]$ are supposed to be holomorphic in the strips $\eta_1 < Re(s) < \eta_2$ and $< \alpha_1 < Re(s) < \alpha_2$, respectively. Assuming that Parseval's formula may be applied and using property (11.56), one can write (11.146) as

$$I\left[f;\lambda\right]=\frac{1}{2\pi j}\int_{r-j\infty}^{r+j\infty} \lambda^{-s}\,\mathcal{M}\left[h;s\right]\mathcal{M}\left[f;1-s\right] ds \tag{11.148}$$

   where r is any real number in the strip of analyticity of the function G defined by

$$G\left(s\right)=\mathcal{M}\left[h;s\right]\mathcal{M}\left[f;1-s\right], \quad \max\left(\alpha_1, 1-\eta_2\right) < Re\left(s\right) < \min\left(\alpha_2, 1-\eta_1\right) \tag{11.149}$$

2. *Shift of the contour of integration to the right and use of Cauchy's formula.* Suppose $G(s)$ can be analytically continued in the right half-plane $Re(s) \geq \min(\alpha_2, 1-\eta_1)$ as a meromorphic function. Remark that this assumption implies that $M[f;s]$ may be continued to the right half-plane $Re(s) > \alpha_2$ and $M[h;s]$ to the left $Re(s) < \eta_1$. Suppose moreover that the contour of integration in (11.148) can be displaced to the right as far as the line $Re(s) = R > r$. A sufficient condition ensuring this property is that

$$\lim_{|b|\to\infty} G\left(a+jb\right)=0 \tag{11.150}$$

   for all a in the interval $[r, R]$.

   Under these conditions, Cauchy's formula may be applied and yields:

$$I\left[f;\lambda\right]=-\sum_{r<Re\left(s\right)<R} \mathrm{Res}\left(\lambda^{-s} G\left(s\right)\right)+\frac{1}{2\pi j}\int_{R-j\infty}^{R+j\infty} \lambda^{-s} G\left(s\right) ds \tag{11.151}$$

   where the discrete summation involves the residues (denoted Res) of function $\lambda^{-s}G(s)$ at the poles lying inside the region $r < Re(s) < R$.

3. *The asymptotic expansion.* The relation (11.151) is an asymptotic expansion provided the error bounds hold. A sufficient condition to ensure that the integral term is of order $O(\lambda^{-R})$ is that G satisfy:

$$\int_{-\infty}^{\infty} \left| G(R + jb) \right| db < \infty \tag{11.152}$$

The above operations can be justified step by step when treating a particular case. The general theory gives a precise description of the final form of the asymptotic expansion, when it exists, in terms of the asymptotic properties of h when $t \to +\infty$ and of f when $t \to 0+$.

The above procedure is easily adapted to give the asymptotic expansion of $I[f;\lambda]$ when $\lambda \to 0+$.

Example 11.2.2.2

Consider the case where the kernel, h, is given by

$$h(t) = \frac{1}{1+t} \tag{11.153}$$

The integral under consideration is thus:

$$I[f;\lambda] = \int_0^{\infty} \frac{f(t)}{1+\lambda t}\, dt \tag{11.154}$$

The function f must be such that the integral exists. In addition, it is supposed to have a Mellin transform holomorphic in the strip $\sigma_1 < Re(s) < \sigma_2$ and to have an asymptotic development as $t \to 0$ of the form:

$$f \sim \sum_{m=0}^{\infty} t^{a_m} p_m \tag{11.155}$$

where the numbers $Re(a_m)$ increase monotonically to $+\infty$ as $m \to +\infty$ and the numbers p_m may be arbitrary.

To apply the method, we first compute the Mellin transform of h which is given by:

$$\mathcal{M}\left[(1+t)^{-1}; s\right] = \frac{\pi}{\sin \pi s}, \quad 0 < Re(s) < 1 \tag{11.156}$$

It can be continued in the half-plane $Re(s) > 0$ where it has simple poles at $s = 1,2, \ldots$ and decays along imaginary lines as follows:

$$\frac{\pi}{\sin \pi(a+jb)} = O\left(e^{-\pi|b|}\right) \quad \text{for all } a \tag{11.157}$$

As for function f, its behavior given by (11.155) ensures[9] that the Mellin transform $\mathcal{M}[f; s]$ has an analytic continuation in the half-plane $Re(s) \le \sigma_1 = -Re(a_0)$ which is a meromorphic function with poles at the points $s = -a_m$. Moreover, one finds the following behavior at infinity for the continued Mellin transform:

$$\lim_{|b|\to\infty} \mathcal{M}\left[f; a+jb\right] = 0, \quad \text{for all } a < \sigma_2 \tag{11.158}$$

In this situation, the method will lead to an asymptotic expansion which can be written explicitly. For example, in the case where $a_m \neq 0,1,2, \ldots$, the poles of $\mathcal{M}[f;1-s]$, which occur at $1-s=-a_m$ are distinct from those of $\mathcal{M}[h;s]$ at $s = m+1$ and the expansion of I is given by:

$$I\left[f;\,\lambda\right] \sim \sum_{m=0}^{\infty} \lambda^{-1-a_m} \frac{\pi}{\sin\left(\pi a_m\right)} \operatorname{Res}_{s=1+a_m} \left\{\mathcal{M}\left[f;\,1-s\right]\right\} + \sum_{m=0}^{\infty} \lambda^{-1-m}\, \mathcal{M}\left[f;\,-m\right] \operatorname{Res}_{s=m+1} \left\{\frac{\pi}{\sin\left(\pi s\right)}\right\} \tag{11.159a}$$

Hence,

$$I\left[f;\,\lambda\right] \sim \sum_{m=0}^{\infty} \lambda^{-1-a_m p_m} \frac{\pi}{\sin\left(\pi a_m\right)} + \sum_{m=0}^{\infty} \left(-1\right)^m \lambda^{-1-m}\, \mathcal{M}\left[f;\,-m\right] \tag{11.159b}$$

In particular, if $f(t) = (1/t)e^{-(1/t)}$, all the p_m are equal to zero and the expansion is just:

$$I\left[f;\,\lambda\right] \sim \sum_{n=0}^{\infty} \left(-1\right)^n \left(\lambda\right)^{-n-1} \Gamma\left(n+1\right) \tag{11.160}$$

11.2.2.6 Asymptotic Behavior of Harmonic Sums

A more recent application of Mellin's transformation concerns the study of functions defined by series of the following type:

$$g\left(x\right) \equiv \sum_k \lambda_k\, f\left(\mu_k x\right), \quad k \text{ integer},\ x > 0 \tag{11.161}$$

where λ_k and μ_k are real parameters.[20,21] Such functions, which can be interpreted as a superposition of generalized harmonics associated with a base function $f(x)$, are referred to as harmonic sums,[6] the parameters λ_k, μ_k being interpreted as amplitude and frequency, respectively. Those expressions arise in applications of combinatorial theory, especially in the evaluation of algorithms where the problem is to find the behavior of $g(x)$ when x tends to 0 or infinity.[22] Mellin's transformation comes in as an essential tool to obtain asymptotic expansions of this type of function. In the following, a brief account of the method will be given and an example will be treated. For further details, the reader should refer to the review paper by P. Flajolet et al.[6] which contains precise theorems and numerous examples of application.

The first step in establishing an asymptotic expansion of the function $g(x)$ defined by (11.161) is to find the expression of its Mellin transform $\mathcal{M}[g;s] \equiv G(s)$. When the sum on k is finite, the linearity of the transformation and the property (11.56) relative to scaling allow one to write:

$$G\left(s\right) \equiv \mathcal{M}\left[g;\,s\right] = \Lambda\left(s\right) F\left(s\right) \tag{11.162}$$

where

$$F\left(s\right) \equiv \mathcal{M}\left[f;\,s\right] \tag{11.163}$$

and the function $\Lambda(s)$ is defined by:

$$\Lambda(s) \equiv \sum_k \lambda_k \mu_k^{-s} \tag{11.164}$$

Thus, the Mellin transformation performs a separation between the base function $f(x)$ and the parameters λ_k, μ_k. In the more general case of an infinite sum, the validity of the procedure, which involves interchanging sum and integral, will depend on the properties of the functions $f(x)$ and $\Lambda(s)$; the latter has the form of a generalized Dirichlet series for which convergence theorems exist. From now on, we suppose that the Mellin transform of $g(x)$ has the form (11.162) and, in addition, is holomorphic in a strip $S(\sigma_1,\sigma_2)$ and satisfies the conditions of Theorem 11.2.1.1. In that case, the inversion formula (11.23) allows to recover $g(x)$ by an integration in the complex s plane along an imaginary line in the strip of holomorphy according to

$$g(x) = (1/2\pi j)\int_{a-j\infty}^{a+j\infty} \Lambda(s)F(s)x^{-s}\,ds \tag{11.165}$$

with $\sigma_1 < a < \sigma_2$.

The second step relies on the possibility to continue the function $G(s)$ in a half plane as a meromorphic function of sufficient decrease at infinity. The asymptotic development of $g(x)$ as $x \to 0$ is then given by the approximate computation of (11.165) using the method of residues in the left half-plane P_l:

$$g(x) \sim \sum_{P_l} Res\left[G(s)x^{-s}\right] \tag{11.166}$$

The development for $x \to \infty$ is likewise obtained by considering the right half-plane P_r:

$$g(x) \sim -\sum_{P_r} Res\left[G(s)x^{-s}\right] \tag{11.167}$$

This formal procedure is valid under general conditions involving the separate behaviors of the functions $\Lambda(s)$, associated with the parameters, and $F(s)$, characterizing the base function. The result is a systematic correspondence between the properties of the poles of $G(s)$ in the complex plane and the terms of the asymptotic series of $g(x)$.[6,21] A simple example will illustrate this procedure.

Example 11.2.2.3
Consider the sum

$$g(x) = \sum_{k=1}^{\infty} d(k)\, e^{-kx} \tag{11.168}$$

where $d(k)$ represents the number of divisors of k. The Mellin transform is given by:

$$\mathcal{M}[g;s] \equiv G(s) = \sum_{k=1}^{\infty} d(k)k^{-s}\Gamma(s) \tag{11.169}$$

The identity[12]

$$\sum_{k=1}^{\infty} d(k) k^{-s} = \zeta^2(s) \tag{11.170}$$

allows us to write

$$G(s) = \zeta^2(s)\Gamma(s) \tag{11.171}$$

The holomorphy domain of $G(s)$ is $Re(s) > 1$. Recall that the Zeta function has a simple pole at $s = 1$ and simple zeros at the pair negative integers. Hence, taking also into account the properties of the Gamma function, we find that $G(s)$ has a double pole in $s = 1$ and simple poles at $s = 0$ and $s = -2n-1$, n integer. The singular terms of the Laurent expansion of $G(s)$ are

At $s = 1$

$$G(s) \sim \frac{1}{(s-1)^2} + \frac{\gamma}{s-1}, \qquad \gamma = \text{Euler constant}$$

At $s = 0$

$$G(s) \sim \frac{1}{4s} \tag{11.172}$$

At $s = -2n - 1$

$$G(s) \sim -\frac{\zeta^2(-2n-1)}{(2n+1)!\,(s+2n+1)}$$

In addition, the decreasing properties of $G(s)$ when $|s| \to +\infty$ allow to write:

$$G(s) \sim \sum_{Re(s)\le 1} Res\left(G(s)\, x^{-s}\right) \tag{11.173}$$

Computation of the residue at $s = 1$ gives:

$$Res_{s=1}\left(G(s)x^{-s}\right) = \frac{d}{ds}\left[(s-1)^2 G(s)x^{-s}\right]_{s=1} \tag{11.174}$$

or using (11.172):

$$Res_{s=1}\left(G(s)x^{-s}\right) = (\gamma - \ln x)x^{-1} \tag{11.175}$$

The other residues are at simple poles and the asymptotic expansion of $g(x)$ for $x \to 0$ finally reads:

$$g(x) = (\gamma - \ln x)\frac{1}{x} + \frac{1}{4} + \sum_{k=0}^{\infty} \frac{\zeta^2(-2n-1)}{(2n+1)!} x^{2n+1} \tag{11.176}$$

The value of $\zeta(-2n-1)$ can be found in tables in Reference 16.

In this example, the use of Mellin's transformation has led easily to the $x = 0$ behavior of $g(x)$ which was not obvious on the definition (11.168). Other types of series involving poles in the complex domain of s can be handled in an analogous way.[6]

11.3 Alternative Approach Related to the Dilation Group and its Representations

11.3.1 Theoretical Aspects

11.3.1.1 Construction of the Transformation

Rather than start directly by giving the explicit formula of the Mellin transformation, we will construct it explicitly as a tool especially devoted to the computation of functionals involving scalings of a variable. Such an introduction of the transform may be found, for example, in the book by N. Ya.Vilenkin[3] or, in a more applied context, in articles.[23-25]

If $Z(v)$ is a function defined on the positive half-axis $(0, \infty)$, a scaling of the variable v by a positive number a leads to a new function $Z'(v)$ which is related to $Z(v)$ by the change of variable:

$$v \to av \tag{11.177}$$

The set of such transformations forms a group which is isomorphic to the multiplicative group of positive real numbers. In practice, the scaled function is often defined by the transformation:

$$Z(v) \to a^{1/2} Z(av) \tag{11.178}$$

which does not change the value of the usual scalar product:

$$(Z_1, Z_2) \equiv \int_0^{\infty} Z_1(v)\, Z_2^*(v)\, dv \tag{11.179}$$

in which the symbol $*$ denotes complex conjugation. However, there are serious physical reasons to consider more general transformations of the form:

$$\mathcal{D}_a: \; Z(v) \to (\mathcal{D}_a Z)(v) \equiv a^{r+1} Z(av) \tag{11.180}$$

where r is a given real number. The general correspondence $a \leftrightarrow \mathcal{D}_a$ is such that:

$$\begin{gathered} \mathcal{D}_a \mathcal{D}_{a'} = \mathcal{D}_{aa'} \\ \mathcal{D}_1 \mathcal{D}_a = \mathcal{D}_a \mathcal{D}_1 = \mathcal{D}_a \\ (\mathcal{D}_a)^{-1} = \mathcal{D}_{a^{-1}} \end{gathered} \tag{11.181}$$

Thus, for any value of r, the set of $\mathcal{D}_a$ operations constitutes a representation of a group. These transformations preserve the following scalar product:

$$\left(Z_1, Z_2\right) \equiv \int_0^{\infty} Z_1\left(v\right) Z_2^*\left(v\right) v^{2r+1}\, dv \qquad (11.182)$$

that is to say, we have

$$\left(\mathcal{D}_a Z_1, \mathcal{D}_a Z_2\right) = \left(Z_1, Z_2\right) \qquad (11.183)$$

The scalar product (11.182) defines a norm for the functions $Z(v)$ on the positive axis IR^+. We have

$$\left\|Z\right\|^2 \equiv \int_0^{\infty} \left|Z\left(v\right)\right|^2 v^{2r+1}\, dv \qquad (11.184)$$

The corresponding Hilbert space will be denoted by $L^2(IR^+, v^{2r+1}\, dv)$. It is handled in the same manner as an ordinary L^2 space with the measure dv replaced by $v^{2r+1}\, dv$ in all formulas. In this space, the meaning of (11.183) is that the set of operations $\mathcal{D}_a$ constitutes a unitary representation of the multiplicative group of positive numbers.

The value of r will be determined by the specific applications to be dealt with. Examples of adjustments of this parameter are given in Reference 26 where the occurrence of dilations in radar imaging is analyzed.

When confronted with expressions involving functions modified by dilations of the form (11.180), it may be advantageous to use a Hilbert space basis in which the operators $\mathcal{D}_a$ have a diagonal expression. This leads to a decomposition of functions Z into simpler elements on which the scaling operation breaks down to a mere multiplication by a complex number. Such a procedure is familiar when considering the operation which translates a function $f(t)$, $t \in \mathrm{IR}$ according to:

$$f\left(t\right) \rightarrow f\left(t - t_0\right) \qquad (11.185)$$

In that case, the exponentials $e^{\lambda t}$, λ complex, are functions that are multiplied by a number $e^{\lambda t_0}$ in a translation. If $\lambda = j\alpha$, α real, these functions are unitary representations of the translation group in $L^2(\mathrm{IR})$ and provide a generalized* orthonormal basis for functions in this space. The coefficients of the development of function $f(t)$ on this basis are obtained by scalar product with the basis elements and make up the Fourier transform. In the present case, analogous developments will connect the Mellin transformation to the unitary representations of the dilation group in $L^2(\mathrm{IR}^+, v^{2r+1}\, dv)$.

For simplicity and for future reference, the diagonalization of $\mathcal{D}_a$ will be performed on its infinitesimal form defined by the operator $\mathcal{B}$ whose action on function $Z(v)$ is given by:

$$\left(\mathcal{B}Z\right)\left(v\right) \equiv -\frac{1}{2\pi j} \frac{d}{da}\left[\left(\mathcal{D}_a Z\right)\left(v\right)\right]_{\left(a=1\right)} \qquad (11.186)$$

The computation yields:

$$\mathcal{B} = -\frac{1}{2\pi j}\left(v \frac{d}{dv} + r + 1\right) \qquad (11.187)$$

*Such families of functions which do not belong to the Hilbert space under consideration but are treated like bases by physicists are called *improper bases.* Their use can be rigorously justified.

The operator $\mathscr{B}$ is a self-adjoint operator and the unitary representation $\mathscr{D}_a$ is recovered from $\mathscr{B}$ by exponentiation:*

$$\mathscr{D}_a = e^{-2\pi ja\mathscr{B}} \tag{11.188}$$

where the exponential of the operator is defined formally by the infinite series:

$$e^{-2\pi ja\mathscr{B}} = \sum_{n=O}^{\infty} \left(-1\right)^n \frac{\left(2\pi ja\mathscr{B}\right)^n}{n!} \tag{11.189}$$

Here, we only need to find the eigenfunctions of $\mathscr{B}$, i.e., the solutions of the differential equation:

$$\mathscr{B}Z\left(\nu\right) = \beta Z\left(\nu\right) \tag{11.190}$$

with β real.

The solution is, up to an arbitrary factor:

$$E_\beta\left(\nu\right) = \nu^{-2\pi j\mathscr{B}-r-1} \tag{11.191}$$

As ensured by the construction, any member of this family of functions E_β is just multiplied by a phase when a scaling is performed:

$$\mathscr{D}_a:\ E_\beta\left(\nu\right) \to a^{-2\pi j\beta}\, E_\beta\left(\nu\right) \tag{11.192}$$

Moreover, the family $\{E_\beta\}$ is orthonormal and complete as will now be shown. The orthonormality is obtained by setting $\nu = e^{-x}$ in the expression:

$$\left(E_\beta, E_{\beta'}\right) = \int_0^\infty \nu^{-2\pi j\left(\beta-\beta'\right)-1}\, d\nu \tag{11.193}$$

which becomes:

$$\left(E_\beta, E_{\beta'}\right) = \int_{-\infty}^\infty e^{2\pi j\left(\beta-\beta'\right)x}\, dx \tag{11.194}$$

The result is

$$\left(E_\beta, E_{\beta'}\right) = \delta\left(\beta-\beta'\right) \tag{11.195}$$

To show completeness, we compute:

$$\int_{-\infty}^\infty E_\beta\left(\nu\right)E_\beta^*\left(\nu'\right)\, d\beta \equiv \int_{-\infty}^\infty e^{-2\pi j\beta\ln\left(\nu/\nu'\right)}\left(\nu\nu'\right)^{-r-1}\, d\beta \tag{11.196}$$

*This is known as Stone's theorem

$$= \left(\nu\nu'\right)^{-r-1} \delta\left(\ln\left(\nu\right) - \ln\left(\nu'\right)\right) \tag{11.197}$$

and, using the rule of calculus with delta functions recalled in (11.52), we obtain:

$$\int_{-\infty}^{\infty} E_\beta\left(\nu\right) E_\beta^*\left(\nu'\right) d\beta = \nu^{-2r-1} \delta\left(\nu - \nu'\right) \tag{11.198}$$

Any function Z in $L^2(\mathrm{IR}^+, \nu^{2r+1}\, d\nu)$ can thus be decomposed on the basis E_β with coefficients $\mathcal{M}[Z](\beta)$ given by:

$$\mathcal{M}\left[Z\right]\left(\beta\right) = \left(Z, E_\beta\right) \tag{11.199}$$

or explicitly:

$$\mathcal{M}\left[Z\right]\left(\beta\right) = \int_0^{\infty} Z\left(\nu\right) \nu^{2\pi j\beta + r}\, d\nu \tag{11.200}$$

The set of coefficients, considered as a function of β, constitutes what is called the Mellin transform of function Z. This definition coincides with the usual one (11.2.1.1) provided we set $s = r + 1 + 2\pi j\beta$. Thus,

$$\mathcal{M}\left[Z\right]\left(\beta\right) \equiv \mathcal{M}\left[Z;\, r + 1 + 2\pi j\beta\right] \tag{11.201}$$

But the viewpoint here is different. The value of $Re(s) = r + 1$ is fixed once and for all as it is forced upon us by the representation of dilations occurring in the physical problem under study. Thus, the situation is closer to the Fourier than to the Laplace case and an L^2 theory is developed naturally.

The property (11.198) of completeness for the basis implies that the Mellin transformation (11.199) from $Z(\nu)$ to $\mathcal{M}[Z](\beta)$ is norm-preserving:

$$\int_{-\infty}^{\infty} \left|\mathcal{M}\left[Z\right]\left(\beta\right)\right|^2 d\beta = \left(Z, Z\right) \tag{11.202}$$

A Parseval formula (also called unitarity property) follows immediately:

$$\int_{-\infty}^{\infty} \mathcal{M}\left[Z_1\right]\left(\beta\right) \mathcal{M}^*\left[Z_2\right]\left(\beta\right) d\beta = \left(Z_1, Z_2\right) \tag{11.203}$$

where

$$\mathcal{M}^*\left[Z\right]\left(\beta\right) \equiv \left[\mathcal{M}\left[Z\right]\left(\beta\right)\right]^* \tag{11.204}$$

The decomposition formula of function $Z(\nu)$ on basis $\{E_\beta(\nu)\} \equiv \nu^{-2\pi j\beta - r - 1}$ which can be obtained from (11.200) and (11.198) constitutes the inversion formula for the Mellin transformation:

$$Z\left(\nu\right) = \int_{-\infty}^{\infty} \mathcal{M}\left[Z\right]\left(\beta\right) \nu^{-2\pi j\beta - r - 1}\, d\beta \tag{11.205}$$

By construction, the Mellin transformation performs the diagonalization of the operators $\mathcal{B}$ and $\mathcal{D}_a$. Indeed, by definition (11.199), the Mellin transform of the function $(\mathcal{B}Z)(v)$ is given by:

$$\mathcal{M}\left[\mathcal{B}Z\right](\beta)=\left(\mathcal{B}Z,\, E_\beta\right) \tag{11.206}$$

or, using the fact that $\mathcal{B}$ is self-adjoint and that E_β is an eigenfunction of $\mathcal{B}$:

$$\mathcal{M}\left[\mathcal{B}Z\right](\beta)=\beta\left(Z,\, E_\beta\right) \tag{11.207}$$

Thus,

$$\mathcal{M}\left[\mathcal{B}Z\right](\beta)=\beta\,\mathcal{M}\left[Z\right](\beta) \tag{11.208}$$

In the same way, the Mellin transform of $\mathcal{D}_a Z$ is computed using the unitarity of $\mathcal{D}_a$:

$$\begin{aligned}\mathcal{M}\left[\mathcal{D}_a Z\right](\beta)&=\left(\mathcal{D}_a Z,\, E_\beta\right)\\&=\left(Z,\, \mathcal{D}_{a^{-1}}\; E_\beta\right)\end{aligned} \tag{11.209}$$

Thus,

$$\mathcal{M}\left[\mathcal{D}_a Z\right](\beta)=a^{-2\pi j\beta}\,\mathcal{M}\left[Z\right](\beta) \tag{11.210}$$

All these results can be summed up in the following proposition.

Theorem 11.3.1.1
Let Z(v) be a function in L^2 (IR^+, $v^{2r+1}\, dv$). Its Mellin transform defined by:

$$\mathcal{M}\left[Z\right](\beta)=\int_0^\infty Z(v)\,v^{2\pi j\beta+r}\, dv \tag{11.211}$$

belongs to L^2 (IR).

The inversion formula is given by:

$$Z(v)=\int_{-\infty}^{\infty}\mathcal{M}\left[Z\right](\beta)\,v^{-2\pi j\beta-r-1}\, d\beta \tag{11.212}$$

An analog of Parseval's formula (unitarity) holds as:

$$\int_{-\infty}^{\infty}\mathcal{M}\left[Z_1\right](\beta)\,\mathcal{M}^*\left[Z_2\right](\beta)\, d\beta=\int_0^\infty Z_1(v)\,Z_2^*(v)\,v^{2r+1}\, dv \tag{11.213}$$

For any function Z, the Mellin transform of the dilated function

$$\left(\mathcal{D}_a Z\right)(v)\equiv a^{r+1}\,Z(av) \tag{11.214}$$

is given by:

$$\mathcal{M}\left[\mathcal{D}_a Z\right](\beta) = a^{-2\pi j\beta}\,\mathcal{M}\left[Z\right](\beta) \tag{11.215}$$

11.3.1.2 Uncertainty Relations

As in the case of the Fourier transformation, there is a relation between the spread of a function and the spread of its Mellin transform. To find this relation, we will consider the first two moments of the density functions $|Z(\nu)|^2$ and $|\mathcal{M}[Z](\beta)|^2$ connected by (11.213). The mean value of ν with density $|Z(\nu)|^2$ is defined by the formula:

$$\bar{\nu} \equiv \frac{(Z, \nu Z)}{(Z, Z)} \tag{11.216}$$

The mean value of ν^2 is defined by an analogous formula. The mean square deviation σ_ν^2 of variable ν can then be computed according to:

$$\sigma_\nu^2 \equiv \overline{(\nu - \bar{\nu})^2} \tag{11.217}$$

In the same way, in the space of Mellin transforms, the mean value of β is defined by:

$$\bar{\beta} \equiv \frac{(\tilde{Z}, \beta\tilde{Z})}{(\tilde{Z}, \tilde{Z})} \tag{11.218}$$

where $\tilde{Z}$ denotes the Mellin transform of Z. Using Parseval formula (11.213) and property (11.208), one can also rewrite this mean value in terms of the original function $Z(\nu)$ as:

$$\bar{\beta} = \frac{(Z, \mathcal{B}Z)}{(Z, Z)} \tag{11.219}$$

where the operator $\mathcal{B}$ has been defined by formula (11.187). At this point, it is convenient to introduce the following notation for any operator $\mathcal{O}$ acting on Z:

$$\langle \mathcal{O} \rangle \equiv \frac{(Z, \mathcal{O}Z)}{(Z, Z)} \tag{11.220}$$

and to rewrite (11.219) as:

$$\bar{\beta} = \langle \mathcal{B} \rangle \tag{11.221}$$

The mean square deviation σ_β^2 of variable β can also be expressed in terms of the operator $\mathcal{B}$ according to:

$$\sigma_\beta^2 = \left\langle (\mathcal{B} - \bar{\beta})^2 \right\rangle \tag{11.222}$$

A simple way to obtain a lower bound on the product $\sigma_\beta \sigma_v$ is to introduce the operator $\mathcal{X}$ defined by:

$$\mathcal{X} \equiv \mathcal{B} - \overline{\beta} + j\lambda\left(v - \overline{v}\right) \tag{11.223}$$

where λ is a real parameter. The obvious requirement that the norm of $\mathcal{X}Z(v)$ must be positive or zero whatever the value of λ is expressed by the inequality:

$$\left\|\mathcal{X}Z\right\|^2 = \left(Z, \mathcal{X}^* \mathcal{X}Z\right) \geq 0 \tag{11.224}$$

where $\mathcal{X}^*$ denotes the adjoint of $\mathcal{X}$. This constraint implies the positivity of the expression:

$$\left\langle\left(\mathcal{B} - \overline{\beta} + j\lambda\left(v - \overline{v}\right)\right)\left(\mathcal{B} - \overline{\beta} - j\lambda\left(v - \overline{v}\right)\right)\right\rangle \tag{11.225}$$

Developing and using relations (11.217) and (11.222), we obtain:

$$\lambda^2 \sigma_v^2 + j\lambda\left\langle v\mathcal{B} - \mathcal{B}v\right\rangle + \sigma_\beta^2 \geq 0 \tag{11.226}$$

The computation of $v\mathcal{B} - \mathcal{B}v$ yields:

$$\left(v\mathcal{B} - \mathcal{B}v\right)Z\left(v\right) = \frac{-1}{2\pi j}\left(v^2 \frac{d}{dv} - v\frac{d}{dv}v\right)Z\left(v\right) \tag{11.227}$$

$$= \frac{1}{2\pi j} vZ\left(v\right) \tag{11.228}$$

With this result, condition (11.226) becomes:

$$\lambda^2 \sigma_v^2 + \left(\lambda/2\pi\right)\overline{v} + \sigma_\beta^2 \geq 0 \tag{11.229}$$

The left member is a quadratic expression of the parameter λ. Its positivity whatever the value of λ means that the coefficients of the expression verify:

$$\sigma_v \sigma_\beta \geq \overline{v}/4\pi \tag{11.230}$$

The functions for which this product is minimal are such that there is equality in (11.224). Hence, they are solutions of the equation:

$$\left[\mathcal{B} - \overline{\beta} - j\lambda\left(v - \overline{v}\right)\right]Z\left(v\right) = 0 \tag{11.231}$$

and are found to be

$$K\left(v\right) \equiv e^{-2\pi\lambda v} v^{2\pi\lambda\overline{v} - r - 1 - 2\pi j\overline{\beta}} \tag{11.232}$$

These functions, first introduced by Klauder,[27] are the analogs of Gaussians in Fourier theory.

11.3.1.3 Extension of the Mellin Transformation to Distributions

The definition of the transformation has to be extended to distributions to be able to treat generalized functions such as Dirac's which are currently used in electrical engineering. Section 11.2.1.3 can be read at this point for a general view of the possible approaches. Here we only give a succinct definition that will generally be sufficient and we show on explicit examples how computations can be performed.

First, a test function space $\mathcal{T}$ is constructed so as to contain the functions $\nu^{2\pi j\beta+r}$, $\nu > 0$, $\beta \in$ IR for a fixed value of r. Examples of such spaces are the spaces $\mathcal{T}(a_1, a_2)$ of Section 11.2.1.3 provided a_1, a_2 are chosen verifying the inequality $a_1 < r + 1 < a_2$.[7,8] Then the space of distributions $\mathcal{T}'$ is defined as usual as a linear space $\mathcal{T}'$ of continuous functionals on $\mathcal{T}$. It can be shown that the space $\mathcal{T}'$ contains the distributions of bounded support on the positive axis and, in particular, the Dirac distributions.

The Mellin transform of a distribution Z in a space $\mathcal{T}'$ can always be obtained as the result of the application of Z to the set of test functions $\nu^{2\pi j\beta+r}$, $\beta \in$ IR, i.e., as

$$\mathcal{M}[Z](\beta) \equiv \left\langle Z, \nu^{2\pi j\beta+r} \right\rangle \tag{11.233}$$

With this extended definition, it is easily verified that relations (11.208) and (11.215) still hold. One more property that will be useful, especially for discretization, is relative to the effect of translations on the Mellin variable. Computing $\mathcal{M}[Z]$ for the value $\beta+c$,c real, yields:

$$\begin{aligned}\mathcal{M}[Z](\beta+c) &= \left\langle Z, \nu^{2\pi j(\beta+c)+r} \right\rangle \\ &= \left\langle Z\nu^{2\pi jc}, \nu^{2\pi j\beta+r} \right\rangle\end{aligned} \tag{11.234}$$

and the result:

$$\mathcal{M}[Z](\beta+c) = \mathcal{M}\left[Z\nu^{2\pi jc}\right](\beta) \tag{11.235}$$

Example 11.3.1.1

The above formula (11.233) allows to compute the Mellin transform of $\delta(\nu - \nu_0)$ by applying the usual definition of the Dirac distribution:

$$\left\langle \delta(\nu-\nu_0), \phi \right\rangle = \phi(\nu_0) \tag{11.236}$$

to the function $\phi(\nu) = \nu^{2\pi j\beta+r}$, thus giving:

$$\mathcal{M}\left[\delta(\nu-\nu_0)\right](\beta) \equiv \left\langle \delta(\nu-\nu_0),\ \nu^{2\pi j\beta+r} \right\rangle = \nu_0^{2\pi j\beta+r} \tag{11.237}$$

Example 11.3.1.2: The geometric Dirac comb

In problems involving dilations, it is natural to introduce a special form of the Dirac comb defined by

$$\begin{aligned}\Delta_A^r(\nu) &\equiv \sum_{n=-\infty}^{+\infty} A^{-nr}\,\delta(\nu - A^n) \\ &\equiv \sum_{n=-\infty}^{+\infty} A^{nr}\,\delta(\nu - A^{-n})\end{aligned} \tag{11.238}$$

where A is a positive number. The values of ν which are picked out by this distribution form a geometric progression of ratio A. Moreover, the comb Δ_A^r is invariant in a dilation by an integer power of A. Indeed, using definition (11.214), we have

$$\mathscr{D}_A \Delta_A^r(\nu) \equiv A^{r+1} \Delta_A^r(A\nu) \tag{11.239}$$

$$= \sum_{n=-\infty}^{\infty} A^{-(n-1)r} \delta(\nu - A^{n-1}) \tag{11.240}$$

$$= \Delta_A^r(\nu) \tag{11.241}$$

The distribution Δ_A^r will be referred to as the geometric Dirac comb and is represented in Figure 11.3.

Distribution Δ_A^r does not belong to $\mathscr{T}'$ and, hence, its Mellin transform cannot be obtained by formula (11.233). However, the property of linearity of the Mellin transformation and result (11.237) allow to write:

$$\mathcal{M}[\Delta_A^r](\beta) = \sum_{n=-\infty}^{+\infty} A^{2j\pi\beta n} \tag{11.242}$$

The right-hand side of (11.242) is a Fourier series which can be summed by Poisson's formula:

$$\ln A \sum_{n=-\infty}^{\infty} e^{2j\pi n\beta \ln A} = \sum_{n=-\infty}^{\infty} \delta\left(\beta - \frac{n}{\ln A}\right) \tag{11.243}$$

This leads to:

$$\mathcal{M}[\Delta_A^r](\beta) = \frac{1}{\ln A} \sum_{n=-\infty}^{+\infty} \delta\left(\beta - \frac{n}{\ln A}\right) \tag{11.244}$$

Thus, the Mellin transform of a geometric Dirac comb Δ_A^r on IR$^+$ is an arithmetic Dirac comb on IR (Figure 11.3).

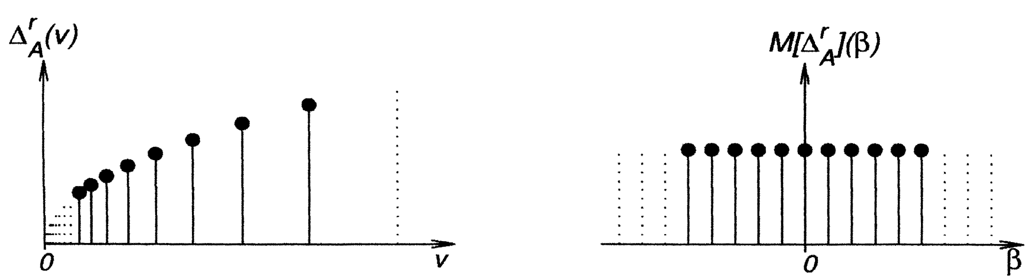

FIGURE 11.3 Geometrical Dirac comb in IR$^+$-space and corresponding arithmetical Dirac comb in the Mellin space (case $r = -1/2$).

11.3.1.4 Transformations of Products and Convolutions

The relations between product and convolution that are established by a Fourier transformation have analogs here. Classical convolution and usual product in the space of Mellin transforms correspond respectively to a special invariant product and a multiplicative convolution in the original space. The latter operations can also be defined directly by their transformation properties under a dilation as will now be explained.

Invariant product

The dilation-invariant product of the functions Z_1 and Z_2 which will be denoted by the symbol $\circ$ is defined as:

$$\left(Z_1 \circ Z_2\right)(v) \equiv v^{r+1} Z_1(v) Z_2(v) \tag{11.245}$$

It is nothing but the usual product of the functions multiplied by the $(r+1)^{\text{th}}$ power of the variable. Relation (11.245) defines an internal law on the set of functions that is stable by dilation since:

$$\mathcal{D}_a\left[Z_1\right] \circ \mathcal{D}_a\left[Z_2\right] = \mathcal{D}_a\left[Z_1 \circ Z_2\right] \tag{11.246}$$

where $\mathcal{D}_a$ is the operation (11.180).

Now, we shall compute the Mellin transform of the product $Z_1 \circ Z_2$. According to definition (11.211), this is given by:

$$\mathcal{M}\left[Z_1 \circ Z_2\right](\beta) = \int_0^{+\infty} v^{r+1} Z_1(v)\, Z_2(v)\, v^{2j\pi\beta + r}\, dv \tag{11.247}$$

Replacing Z_1 and Z_2 by their inverse Mellin transforms given by (11.212) and using the orthogonality relation (11.195) to perform the v-integration, we obtain:

$$\mathcal{M}\left[Z_1 \circ Z_2\right](\beta) = \int_{-\infty}^{\infty} d\beta_1 \int_{-\infty}^{\infty} \mathcal{M}\left[Z_1\right](\beta_1)\, \mathcal{M}\left[Z_2\right](\beta_2)\, d\beta_2\, \delta\left(\beta - \beta_1 - \beta_2\right) \tag{11.248a}$$

$$= \int_{-\infty}^{\infty} \mathcal{M}\left[Z_1\right](\beta_1)\, \mathcal{M}\left[Z_2\right](\beta - \beta_1)\, d\beta_1 \tag{11.248b}$$

where we recognize the classical convolution of the Mellin transforms.

Theorem 11.3.1.2.

The Mellin transform of the invariant product (11.245) of the two functions Z_1 and Z_2 is equal to the convolution of their Mellin transforms:

$$\mathcal{M}\left[Z_1 \circ Z_2\right](\beta) = \left(\mathcal{M}\left[Z_1\right] * \mathcal{M}\left[Z_2\right]\right)(\beta) \tag{11.249}$$

Multiplicative convolution: For a given function Z_1 (resp Z_2), the usual convolution $Z_1 * Z_2$ can be seen as the most general linear operation commuting with translations that can be performed on Z_1 (resp Z_2). By analogy, the *multiplicative convolution* of Z_1 and Z_2 is defined as the most general linear operation on Z_1 (resp Z_2) that commutes with dilations. More precisely, suppose that a linear operator $\mathcal{A}$ is defined in terms of a kernel function $A(v, v')$ according to:

$$\mathscr{A}\left[Z_1\right]\left(\nu\right)=\int_0^{+\infty} A\left(\nu,\nu'\right)Z_1\left(\nu'\right)\,d\nu' \tag{11.250}$$

Then the requirement that transformation $\mathscr{D}_a$ applied either on Z_1 or $\mathscr{A}[Z_1]$ yield the same results implies that:

$$a^{r+1}\ \mathscr{A}\left[Z_1\right]\left(a\nu\right)=a^{r+1}\int_0^{+\infty} A\left(\nu,\nu'\right)\,Z_1\left(a\nu'\right)\,d\nu' \tag{11.251}$$

must be true for any function Z_1. Comparing (11.251) to (11.250), we thus obtain the following constraint on the kernel $A(\nu, \nu')$:

$$A\left(\nu,\nu'\right)\equiv a\ A\left(a\nu,a\nu'\right) \tag{11.252}$$

valid for any a. For $a = \nu'^{-1}$, we obtain the identity:

$$A\left(\nu,\nu'\right)\equiv\frac{1}{\nu'}\ A\left(\frac{\nu}{\nu'},1\right) \tag{11.253}$$

which shows that the operator $\mathscr{A}$ can be expressed by using a function of a single variable. Thus, any linear transformation acting on function Z_1 and commuting with dilations can be written in the form:

$$\int_0^{+\infty} Z_1\left(\nu'\right)\,Z_2\left(\frac{\nu}{\nu'}\right)\frac{d\nu'}{\nu'} \tag{11.254}$$

where $Z_2(\nu)$ is an arbitrary function.

It can be verified, by changing variables, that the above expression is symmetrical with respect to the two functions Z_1 and Z_2. It defines the multiplicative convolution of these functions which is usually denoted by $Z_1 \vee Z_2$:

$$Z_1\vee Z_2\equiv\int_0^{+\infty} Z_1\left(\nu'\right)\,Z_2\left(\frac{\nu}{\nu'}\right)\frac{d\nu'}{\nu'} \tag{11.255}$$

On this definition, it can be observed that dilating one of the factors Z_1 or Z_2 of the multiplicative convolution is equivalent to dilating the result, i.e.,

$$\mathscr{D}_a\left[\left(Z_1\vee Z_2\right)\left(\nu\right)\right]\equiv\left[Z_1\vee\left(\mathscr{D}_a Z_2\right)\right]\left(\nu\right) \tag{11.256}$$

$$\equiv\left[\left(\mathscr{D}_a Z_1\right)\vee Z_2\right]\left(\nu\right) \tag{11.257}$$

where $\mathscr{D}_a$ is defined in (11.214).

For applications, an essential property of the multiplicative convolution is that it is converted into a classic product when a Mellin transformation is performed.

$$\mathscr{M}\left[Z_1\vee Z_2\right]\left(\beta\right)=\mathscr{M}\left[Z_1\right]\left(\beta\right)\,\mathscr{M}\left[Z_2\right]\left(\beta\right) \tag{11.258}$$

To prove this result, we write the definition of $\mathcal{M}[Z_1 \vee Z_2]\,(\beta)$ which is, according to (11.211) and (11.255):

$$\mathcal{M}\left[Z_1 \vee Z_2\right]\left(\beta\right) = \int_0^{\infty} \nu^{2\pi j\beta + r}\, Z_1\left(\nu'\right) Z_2\left(\frac{\nu}{\nu'}\right) \frac{d\nu'}{\nu'}\, d\nu \tag{11.259}$$

The change of variables from ν to $x = \nu/\nu'$ yields the result.

Theorem 11.3.1.3.

The Mellin transform of the multiplicative convolution (11.255) of functions Z_1 and Z_2 is equal to the product of their Mellin transforms:

$$\mathcal{M}\left[Z_1 \vee Z_2\right]\left(\beta\right) = \mathcal{M}\left[Z_1\right]\left(\beta\right) \mathcal{M}\left[Z_2\right]\left(\beta\right) \tag{11.260}$$

Remark

It can be easily verified that the above theorems remain true if Z_1, Z_2 are distributions provided the composition laws involved in the formulas may be applied.

11.3.2 Discretization and Fast Computation of the Transform

Discretization of the Mellin transform (11.211) is performed along the same lines as discretization of the Fourier transform. It concerns signals with support practically limited, both in ν-space and in β-space. The result is a discrete formula giving a linear relation between N geometrically spaced samples of $Z(\nu)$ and N arithmetically spaced samples of $\mathcal{M}[Z]\,(\beta)$.[24, 25, 28] The fast computation of this discretized transform involves the same algorithms as used in the Fast Fourier Transformation (FFT).

Before proceeding to the discretization itself, we introduce the special notions of sampling and periodizing that will be applied to the function $Z(\nu)$.

11.3.2.1 Sampling in Original and in Mellin Variables

Sampling and periodizing are operations that are well defined in the Mellin space of functions $\tilde{Z}(\beta)$ and can be expressed in terms of Dirac combs. We shall show that the corresponding operations in the space of original functions $Z(\nu)$ involve the geometrical Dirac combs introduced in Section 11.3.1.3.

Arithmetic Sampling in Mellin Space

Given a function $M(\beta) \equiv \mathcal{M}[Z](\beta)$, the arithmetically sampled function $M_S(\beta)$ with sample interval $1/\ln Q, Q$ real, is usually defined by:

$$M_S\left(\beta\right) \equiv \frac{1}{\ln Q} \sum_{n=-\infty}^{+\infty} \mathcal{M}\left[Z\right]\left(\beta\right) \delta\left(\beta - \frac{n}{\ln Q}\right) \tag{11.261}$$

Remark that besides sampling, this definition contains a factor $1/\ln Q$ that is a matter of convenience.

To compute the inverse Mellin transform of this function $M_S(\beta)$, we remark that, due to relation (11.244), it can also be written as a product of Mellin transforms in the form:

$$M_S\left(\beta\right) = \mathcal{M}\left[Z\right]\left(\beta\right) \mathcal{M}\left[\Delta_Q^r\right]\left(\beta\right) \tag{11.262}$$

where Δ_Q^r is the geometric Dirac comb (11.238). Applying now theorem 11.3.1.3, we write M_S as

$$M_S\left(\beta\right) = \mathcal{M}\left[Z \vee \Delta_Q^r\right]\left(\beta\right) \tag{11.263}$$

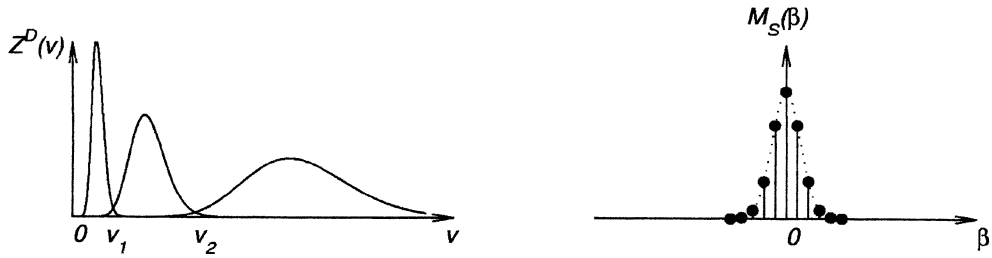

FIGURE 11.4 Correspondence between the dilatocycled form of a function and its Mellin transform.

This relation implies that the inverse Mellin transform of the impulse function $M_S(\beta)$ is the function $Z^D(v)$ given by:

$$Z^D(v) \equiv \left(Z \vee \Delta_Q^r\right)(v) \tag{11.264}$$

The definition of Z^D can be cast into a more explicit form by using the definition of the multiplicative convolution and the expression (11.238) of Δ_Q^r:

$$\left(Z \vee \Delta_Q^r\right)(v) = \int_0^{+\infty} Z\left(\frac{v}{v'}\right)\left[\sum_{n=-\infty}^{+\infty} Q^{nr}\,\delta\left(v' - Q^{-n}\right)\right]\frac{dv'}{v'} \tag{11.265}$$

The expression (11.264) finally becomes:

$$Z^D(v) = \sum_{n=-\infty}^{+\infty} Q^{n(r+1)}\,Z\left(Q^n v\right) \tag{11.266}$$

As seen on Figure 11.4, function Z^D is constructed by juxtaposing dilated replicas of Z. This operation will be referred to as *dilatocycling* and the function Z^D itself as the *dilatocycled form* of Z with ratio Q. In the special case where the support of function Z is the interval $[v_1, v_2]$ and the ratio Q verifies $Q \geq [v_2/v_1]$, the restriction of Z^D to the support $[v_1, v_2]$ is equal to the original function Z.

Result

The Mellin transform $M_S(\beta)$ of the dilatocycled form Z^D of a signal Z is equal to a regular sampled form of the Mellin transform of Z. Explicitly, we have

$$Z^D(v) = \left(\Delta_Q^r \vee Z\right)(v) \tag{11.267}$$

where

$$\Delta_Q^r(v) \equiv \sum_{n=-\infty}^{+\infty} Q^{nr}\,\delta\left(v - Q^{-n}\right) \tag{11.268}$$

and the result is

$$M_S(\beta) \equiv \frac{1}{\ln Q}\sum_{n=-\infty}^{+\infty} \mathcal{M}[Z](\beta)\,\delta\left(\beta - \frac{n}{\ln Q}\right) \tag{11.269}$$

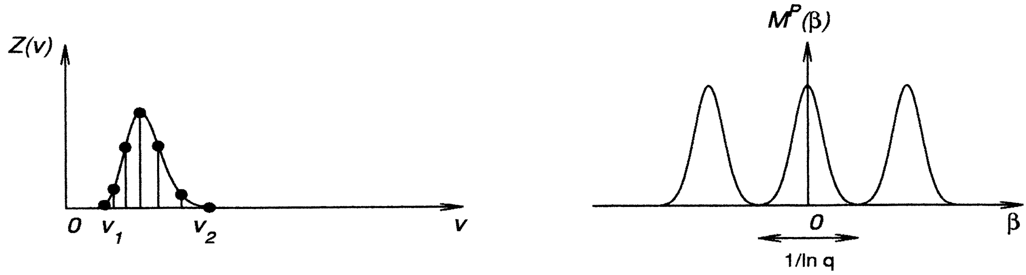

FIGURE 11.5 Correspondence between the geometrically sampled function and its Mellin transform.

Geometric Sampling in the Original Space

Given a function $Z(\nu)$, its geometrically sampled version is defined as the function Z_S equal to the invariant product (11.245) of Z with the geometric Dirac comb Δ_q^r, i.e., as

$$Z_S \equiv Z \circ \Delta_q^r \tag{11.270}$$

or, using the expression (11.238):

$$Z_S(\nu) = Z(\nu) \sum_{n=-\infty}^{+\infty} q^{-nr}\, \delta(\nu - q^n)\, \nu^{r+1} \tag{11.271}$$

$$= \sum_{n=-\infty}^{+\infty} q^n\, Z(q^n)\, \delta(\nu - q^n) \tag{11.272}$$

The result is a function made of impulses located at points forming a geometric progression in ν-space (Figure 11.5).

Let us compute the Mellin transform $M^P(\beta)$ of $Z_S(\nu)$. Using definition (11.270) and property (11.249), we can write:

$$M^P(\beta) \equiv \mathcal{M}[Z_S](\beta) \tag{11.273}$$

$$= \mathcal{M}[Z \circ \Delta_q^r](\beta) \tag{11.274}$$

$$= \left(\mathcal{M}[Z] * \mathcal{M}[\Delta_q^r]\right)(\beta) \tag{11.275}$$

Thus, function $M^P(\beta)$ is equal to the convolution between $\mathcal{M}[Z]$ and the transform $\mathcal{M}[\Delta_q^r]$ which has been shown in (11.244) to be a classical Dirac comb. As a consequence, it is equal to the classical periodized form of $\mathcal{M}[Z](\beta)$ which is given explicitly by:

$$M^P(\beta) \equiv \frac{1}{\ln q} \sum_{n=-\infty}^{+\infty} \mathcal{M}[Z]\left(\beta - \frac{n}{\ln q}\right) \tag{11.276}$$

If the function $M(\beta) \equiv \mathcal{M}[Z](\beta)$ is equal to zero outside the interval $[\beta_1,\beta_2]$, then to avoid aliasing, the period l/ln q must be chosen such that:

$$\frac{1}{\ln q} \geq \beta_2 - \beta_1 \qquad (11.277$$

In that case, the functions $M^P(\beta)$ and $(1/\ln q)M(\beta)$ coincide on the interval $[\beta_1,\beta_2]$.

Result

The geometrically sampled form of $Z(\nu)$ defined by:

$$Z_S(\nu) \equiv \sum_{n=-\infty}^{+\infty} q^n\, Z(q^n)\, \delta(\nu - q^n) \qquad (11.278)$$

is connected by Mellin's correspondence to the periodized form of $\mathcal{M}[Z](\beta)$ given by:

$$M^P(\beta) = \frac{1}{\ln q} \sum_{n=-\infty}^{\infty} \mathcal{M}[Z]\left(\beta - \frac{n}{\ln q}\right) \qquad (11.279)$$

11.3.2.2 The Discrete Mellin Transform

Let $Z(\nu)$ be a function with Mellin transform $M(\beta)$ and suppose that these functions can be approximated by their restriction to the intervals $[\nu_1, \nu_2]$ and $[[\beta_1,\beta_2]$, respectively, (see Figure 11.6a/b). For such functions, it is possible to write down a discretized form of the transform which is very similar to what is done for the Fourier transformation. One may obtain the explicit formulas by performing the following steps:

Dilatocycle function $Z(\nu)$ with ratio Q. This operation leads to the function Z^D defined by (11.267). To avoid aliasing, the real number Q must be chosen such that:

$$Q \geq \frac{\nu_2}{\nu_1} \qquad (11.280)$$

The Mellin transform of Z^D is the sampled function M_S defined by (11.269) in terms of $\mathcal{M}[Z](\beta) \equiv M(\beta)$ (Figure 11.6d).

Periodize $M_S(\beta)$ with a period $1/\ln q$. This is performed by rule (11.279) and yields a function $M_S^P(\beta)$ given by:

$$M_S^P(\beta) = \frac{1}{\ln q} \sum_{n=-\infty}^{\infty} M_S\left(\beta - \frac{n}{\ln q}\right) \qquad (11.281)$$

To avoid aliasing in β-space, the period must be chosen greater than the approximate support of $\mathcal{M}[Z](\beta)$ and this leads to the condition:

$$\frac{1}{\ln q} \geq \beta_2 - \beta_1 \qquad (11.282)$$

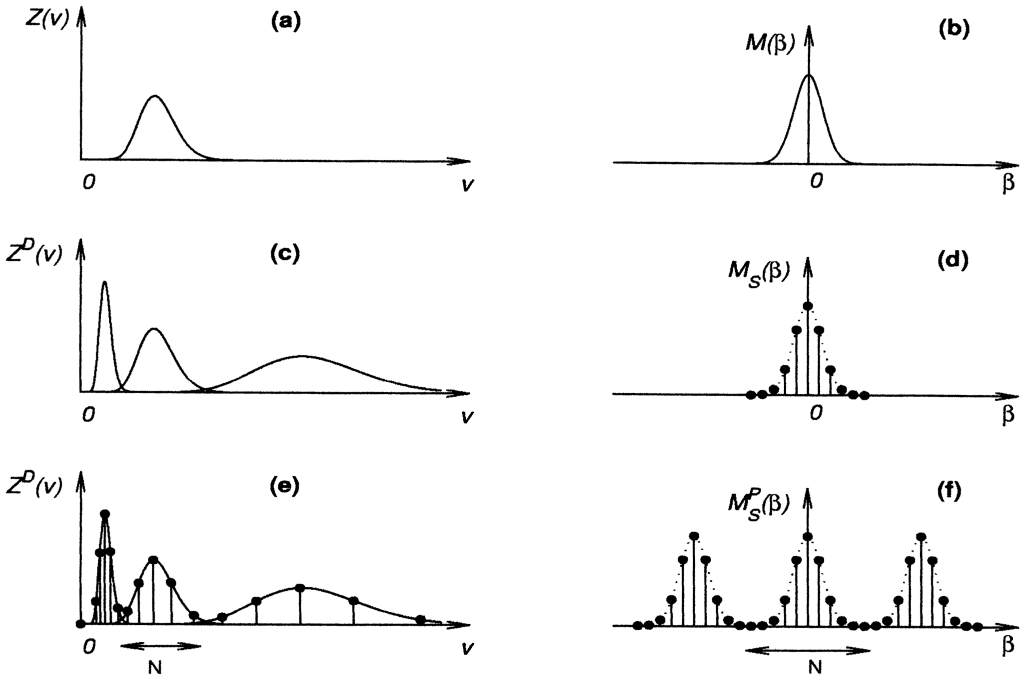

FIGURE 11.6 Steps leading to the Discrete Mellin Transform. Continuous form of the function $Z(v)$ (a) and its Mellin transform $M(\beta)$ (b). Dilatocycled function (c) and its Mellin transform (d). Correspondence between samples of a cycle (e) in v-space and samples of a period (f) in β-space.

The inverse Mellin transform of M_S^P is the geometrically sampled form of Z^D (Figure 11.6e) given, according to (11.278), by:

$$Z_S^D(v) = \sum_{n=-\infty}^{\infty} q^n \, Z^D(q^n) \, \delta(v - q^n) \tag{11.283}$$

The use of (11.269) allows to rewrite definition (11.281) as

$$M_S^P(\beta) = \frac{1}{\ln q \ln Q} \sum_{n,p=-\infty}^{\infty} M\left(\frac{p}{\ln Q}\right) \delta\left(\beta - \frac{n}{\ln q} - \frac{p}{\ln Q}\right) \tag{11.284}$$

We now impose that the real numbers q and Q be connected by the relation:

$$Q = q^N, \quad N \text{ positive integer} \tag{11.285}$$

This ensures that the function M_S^P defined by (11.281) is of periodic impulse type which can be written as:

$$M_S^P(\beta) = \frac{1}{\ln q \ln Q} \sum_{n,p=-\infty}^{\infty} M\left(\frac{p}{N \ln q}\right) \delta\left(\beta - \frac{nN + p}{N \ln q}\right) \tag{11.286}$$

or, changing the p-index to $k \equiv p + nN$:

$$M_S^P(\beta) = \frac{1}{\ln q \ln Q} \sum_{n,k=-\infty}^{\infty} M\left(\frac{k}{N \ln q} - \frac{n}{\ln q}\right) \delta\left(\beta - \frac{k}{N \ln q}\right) \tag{11.287}$$

Thus, recalling definition (11.279)

$$M_S^P(\beta) = \frac{1}{\ln Q} \sum_{k=-\infty}^{\infty} M^P\left(\frac{k}{\ln Q}\right) \delta\left(\beta - \frac{k}{\ln Q}\right) \tag{11.288}$$

Connect the v and β samples. This is done by writing explicitly that M_S^P as given by 11.288 is the Mellin transform (11.211) of Z_S^D and computing:

$$M_S^P(\beta) = \sum_{n=-\infty}^{\infty} q^{n(r+1)} Z^D(q^n) e^{2j\pi n\beta \ln q} \tag{11.289}$$

This formula shows that $q^{n(r+1)} Z^D(q^n)$ for different values of n are the Fourier series coefficients of the periodic function $M_S^P(\beta)$. They are computed as:

$$\begin{aligned} Z^D(q^n) &= q^{-n(r+1)} \ln q \int_0^{1/\ln q} \frac{1}{\ln Q} \sum_{k=-\infty}^{\infty} \delta\left(\beta - \frac{k}{\ln Q}\right) M^P\left(\frac{k}{\ln Q}\right) e^{-j2\pi n\beta \ln q}\, d\beta \\ &= \frac{q^{-n(r+1)}}{N} \sum_{k=K}^{K+N-1} M^P\left(\frac{k}{\ln Q}\right) e^{-2j\pi kn/N} \end{aligned} \tag{11.290}$$

where the summation is on these values of β lying inside the interval $[\beta_1, \beta_2]$. The integer K is thus given by the integer part of $\beta_1 \ln Q$.

Inversion of (11.290) is performed using the classical techniques of discrete Fourier transform. This leads to the discrete Mellin transform formula:

$$M^P\left(\frac{m}{\ln Q}\right) = \sum_{n=J}^{J+N-1} q^{n(r+1)} Z^D(q^n) e^{2j\pi nm/N} \tag{11.291}$$

where the integer J is given by the integer part of $\ln v_1/\ln q$. In fact, since the definition of the periodized M^P contains a factor $N/\ln Q = 1/\ln q$, the true samples of $M(\beta)$ are given by $(\ln Q/n) M^P(m/\ln Q)$.

It is clear on formulas (11.290) and (11.291) that their implementation can be performed with a Fast Fourier Transform (FFT) algorithm.

Choose the number of samples to handle. The number of samples N is related to q and Q according to (11.285) by:

$$N = \frac{\ln Q}{\ln q} \tag{11.292}$$

The conditions for nonaliasing given by (11.280) and (11.282) lead to the sampling condition:

$$N \geq \left(\beta_2 - \beta_1\right) \ln\left(\frac{\nu_2}{\nu_1}\right) \tag{11.293}$$

which gives the minimum number of samples to consider in terms of the spreads of $Z(\nu)$ and $\mathcal{M}[Z](\beta)$. In practice, the spread of the Mellin transform of a function is seldom known. However, as we will see in the applications, there are methods to estimate it.

11.3.2.3 Interpolation Formula in ν-Space

In the same way as the Fourier transformation is used to reconstruct a band-limited function from its regularly spaced samples, Mellin's transformation allows to recover a function $Z(\nu)$ with limited spread in the Mellin space from its samples spaced according to a geometric progression. If the Mellin transform $\mathcal{M}[Z]$ has a bounded support $[-\beta_0/2, \beta_0/2]$, it will be equal on this interval to its periodized form with period $1/\ln q = \beta_0$. Thus,

$$\mathcal{M}\left[Z\right]\left(\beta\right) = \sum_{n=-\infty}^{+\infty} \mathcal{M}\left[Z\right]\left(\beta - \frac{n}{\ln q}\right) g\left(\frac{\beta}{\beta_0}\right) \tag{11.294}$$

where the window function g is the characteristic function of the $[-1/2, 1/2]$-interval.

The inverse Mellin transform of this product is the multiplicative convolution of the two functions Z_1 and Z_2 defined as:

$$Z_1\left(\nu\right) = \ln q \sum_{n=-\infty}^{+\infty} q^n \; Z\left(q^n\right) \delta\left(\nu - q^n\right) \tag{11.295}$$

and

$$Z_2\left(\nu\right) = \int_{-\infty}^{\infty} g\left(\frac{\beta}{\beta_0}\right) \nu^{-2\,j\pi\beta - r - 1} \; d\beta \tag{11.296}$$

$$= \nu^{-r-1} \frac{\sin\left(\pi\beta_0 \ln \nu\right)}{\pi \ln \nu} \tag{11.297}$$

The multiplicative convolution between Z_1 and Z_2 takes the following form:

$$Z\left(\nu\right) = \int_0^{+\infty} \ln q \sum_{n=-\infty}^{+\infty} q^n \; Z\left(q^n\right) \delta\left(\nu' - q^n\right) \left(\frac{\nu}{\nu'}\right)^{-r-1} \frac{\sin\left(\pi\beta_0 \ln\left(\frac{\nu}{\nu'}\right)\right)}{\pi \ln\left(\frac{\nu}{\nu'}\right)} \frac{d\nu'}{\nu'} \tag{11.298}$$

which reduces to:

$$Z(v) = v^{-r-1} \sum_{n=-\infty}^{+\infty} q^{n(r+1)} \, Z(q^n) \frac{\sin \pi\left(\left(\frac{\ln v}{\ln q}\right) - n\right)}{\pi\left(\left(\frac{\ln v}{\ln q}\right) - n\right)} \tag{11.299}$$

where the relation $\beta_0 = 1/\ln q$ has been used.

This is the interpolation formula of a function $Z(v)$ from its geometrically spaced samples $Z(q^n)$.

11.3.3 Practical Use in Signal Analysis

11.3.3.1 Preliminaries

As seen above, Mellin's transformation is essential in problems involving dilations. Thus, it is not surprising that it has come to play a dominant role in the development of analytical studies of wide-band signals. In fact, expressions involving dilations arise in signal theory any time the approximation of small relative bandwidth is not appropriate. Recent examples of the use of the Mellin transform in this context can be found in time-frequency analysis where it has contributed to the introduction of several classes of distributions.[29-36] This fast growing field cannot be explored here but an illustration of the essential role played by Mellin's transformation in the analysis of wide-band signals will be given in Section 11.3.3.2 where Cramer-Rao bound for velocity estimation is derived.[37]

Numerical computation of Mellin's transform has been undertaken in various domains such as signal analysis,[38, 39] optical image processing,[40] or pattern recognition.[41, 43] In the past, however, all these applications have been restricted by the difficulty of assessing the validity of the results, due to the lack of definite sampling rules. Such a limitation does not exist any more as we will show in Section 11.3.3.3 by deriving a sampling theorem and a practical way to use it. The technique will be applied in Sections 11.3.3.4 and 11.3.3.5 to the computation of a wavelet coefficient and of an affine time-frequency distribution.[47-49]

11.3.3.2 Computation of Cramer-Rao Bounds for Velocity Estimation in Radar Theory[37]

In a classical radar or sonar experiment, a real signal is emitted and its echo is processed in order to find the position and velocity of the target. In simple situations, the received signal will differ from the original one only by a time shift and a Doppler compression. In fact, the signal will also undergo an attenuation and a phase shift; moreover, the received signal will be embedded in noise.

The usual procedure, which is adapted to narrow-band signals, is to represent the Doppler effect by a frequency shift.[44] This approximation will not be made here so that the results will be valid whatever the extent of the frequency band. Describing the relevant signals by their positive frequency parts (so-called analytic signals), we can write the expression of the received signal $x(t)$ in terms of the emitted signal $z(t)$ and noise $n(t)$ as:

$$x_{\mathbf{a}'}(t) = a_1'^{-1/2} \, A_0 \, z\left(a_1'^{-1} t - a_2'\right) e^{j\phi} + n(t) \tag{11.300}$$

where A_0 and ϕ characterize the unknown changes in amplitude and phase and the vector $\mathbf{a}' \equiv (a_1', a_2')$ represents the unknown parameters to be estimated. The parameter a_2' is the delay and a_1' is the Doppler compression given in terms of the target velocity v by

$$a_1' = \frac{c+v}{c-v}, \quad c \text{ velocity of light} \tag{11.301}$$

The noise $n(t)$ is supposed to be a zero mean Gaussian white noise with variance equal to σ^2. Relation (11.300) can be written in terms of the Fourier transforms Z, X, N of z, x, n (defined by (11.19)):

$$X_{\mathbf{a}'}(f) = a_1'^{1/2}\, A_0\, e^{-j\pi f a_1' a_2'}\, Z(a_1' f)\, e^{j\phi} + N(f) \tag{11.302}$$

The signal $Z(f)$ is supposed normalized so that:

$$\left\|Z(f)\right\|^2 \equiv \int_0^{\infty} \left|Z(f)\right|^2 df = 1 \tag{11.303}$$

Hence, the delayed and compressed signal will also be of norm equal to one. Remark that here we work in the space $L^2(\mathrm{IR}^+, f^{2r+1}\, df)$ with $r = -1/2$ (cf Section 11.3.1.1).

We will consider the maximum-likelihood estimates $\hat{a}_i$ of the parameters a_i'. They are obtained by maximizing the likelihood function $\Lambda(\mathbf{a}', \mathbf{a})$ which is given in the present context by:

$$\Lambda(\mathbf{a}', \mathbf{a}) \equiv \frac{1}{2\sigma^2} \left|A(\mathbf{a}', \mathbf{a})\right|^2 \tag{11.304}$$

where

$$A(\mathbf{a}', \mathbf{a}) \equiv \int_0^{+\infty} X_{\mathbf{a}'}(f)\, Z^*(a_1 f)\, e^{2j\pi a_1 a_2 f}\, df \tag{11.305}$$

is the broad-band ambiguity function.[45]

The efficiency of an estimator $\hat{a}_i$ is measured by its variance σ_{ij}^2 defined by:

$$\sigma_{ij}^2 \equiv E\left[\left(\hat{a}_i - a_i\right)\left(\hat{a}_j - a_j\right)\right] \tag{11.306}$$

where the mean value operation E includes an average on noise.

For an unbiased estimator ($E(\hat{a}_i) = a_i$), this variance satisfies the Cramer–Rao inequality[46] given by:

$$\sigma_{ij}^2 \geq \left(J^{-1}\right)_{ij} \tag{11.307}$$

where the matrix J, the so-called Fisher information matrix, is defined by:

$$J_{ij} = \left(-E\left[\frac{\partial^2 \Lambda}{\partial a_i \partial a_j}\right]\right)_{ij} \tag{11.308}$$

with the partial derivatives evaluated at the true values of the parameters. The minimum value of the variance given by:

$$\left(\sigma_{ij}^0\right)^2 = \left(J^{-1}\right)_{ij} \tag{11.309}$$

is called the Cramer-Rao bound and is attained in the case of an efficient estimator such as the maximum-likelihood one.

The determination of the matrix (11.308) by classical methods is intricate and does not lead to an easily interpretable result. On the contrary, we shall see how the use of Mellin's transformation allows a direct computation and leads to a physical interpretation of the matrix coefficients.

The computation of J is done in the vicinity of the value $\mathbf{a} = \mathbf{a}'$ which maximizes the likelihood function Λ and, without loss of generality, all partial derivatives will be evaluated at the point $a_1 = 1$, $a_2 = 0$. Using Parseval's formula (11.213), we can write the ambiguity function $A(\mathbf{a}', \mathbf{a})$ as:

$$A(\mathbf{a}', \mathbf{a}) = \int_{-\infty}^{+\infty} \mathcal{M}[X](\beta)\, \mathcal{M}^*[Z_{a_2}](\beta)\, a_1^{2j\pi\beta}\, d\beta \tag{11.310}$$

with

$$Z_{a_2} \equiv Z(f)\, e^{-2j\pi a_2 f} \tag{11.311}$$

On this form, the partial derivatives with respect to $\mathbf{a}$ are easily computed and the result is

$$\left\{\frac{\partial A}{\partial a_1}\right\} = 2j\pi \int_{-\infty}^{+\infty} \beta\, \mathcal{M}[X](\beta)\, \mathcal{M}^*[Z](\beta)\, d\beta \tag{11.312}$$

$$\left\{\frac{\partial A}{\partial a_2}\right\} = 2j\pi \int_{-\infty}^{+\infty} \mathcal{M}[X](\beta)\, \mathcal{M}^*[fZ(f)](\beta)\, d\beta \tag{11.313}$$

$$= 2j\pi \int_{0}^{+\infty} f\, X(f)\, Z^*(f)\, df \tag{11.314}$$

$$\left\{\frac{\partial^2 A}{\partial a_1\, \partial a_2}\right\} = -4\pi^2 \int_{-\infty}^{+\infty} \beta\, \mathcal{M}[X](\beta)\, \mathcal{M}^*[fZ(f)](\beta)\, d\beta \tag{11.315}$$

$$\left\{\frac{\partial^2 A}{\partial a_1^2}\right\} = 2j\pi \int_{-\infty}^{+\infty} \beta(2j\pi\beta - 1)\mathcal{M}[X](\beta)\, \mathcal{M}^*[Z](\beta)\, d\beta \tag{11.316}$$

$$\left\{\frac{\partial^2 A}{\partial a_2^2}\right\} = -4\pi^2 \int_{0}^{+\infty} f^2\, X(f)\, Z^*(f)\, df \tag{11.317}$$

where the curly brackets mean that the functions are evaluated for the values $a_1 = a_1' = 1$, $a_2 = a_2' = 0$.

The corresponding Fisher information matrix can now be computed. To obtain J_{11}, we substitute the expression (11.304) in definition (11.308) and use (11.312) and (11.316):

$$J_{11} = -\frac{1}{\sigma^2} E\left[Re\left(A^* \left\{\frac{\partial^2 A}{\partial a_1^2}\right\} + \left\{\left|\frac{\partial A}{\partial a_1}\right|^2\right\}\right)\right] \tag{11.318}$$

$$= -\frac{1}{\sigma^2} Re \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E\left[\mathcal{M}[X](\beta_1)\mathcal{M}^*[X](\beta_2)\right] \mathcal{M}^*[Z](\beta_1)\mathcal{M}[Z](\beta_2)$$
$$\times \left[2j\pi\beta_1\left(2j\pi\beta_1 - 1\right) + 4\pi^2 \beta_1 \beta_2\right] d\beta_1\, d\beta_2 \quad (11.319)$$

The properties of the zero mean white Gaussian noise $n(t)$ lead to the following expression for the covariance of the Mellin transform of X:

$$E\left[\mathcal{M}[X](\beta_1)\mathcal{M}^*[X](\beta_2)\right] = A_0^2\, \mathcal{M}[Z](\beta_1)\mathcal{M}^*[Z](\beta_2) + \sigma^2\, \delta(\beta_1 - \beta_2) \quad (11.320)$$

Substituting this relation in (11.319), we obtain the expression of the J_{11} coefficient:

$$J_{11} = \frac{4\pi^2 A_0^2}{\sigma^2}\, \sigma_\beta^2 \quad (11.321)$$

where the variance σ_β^2 of parameter β defined in (11.217) is given explicitly by:

$$\sigma_\beta^2 = \int_{-\infty}^{+\infty} \left(\beta - \bar{\beta}\right)^2 \left|\mathcal{M}[Z](\beta)\right|^2 d\beta, \quad \bar{\beta} = \int_{-\infty}^{+\infty} \beta \left|\mathcal{M}[Z](\beta)\right|^2 d\beta \quad (11.322)$$

The computation of the J_{22} coefficient is performed in the same way and leads to:

$$J_{22} = \frac{4\pi^2 A_0^2}{\sigma^2}\, \sigma_f^2 \quad (11.323)$$

where

$$\sigma_f^2 = \int_{-\infty}^{+\infty} \left(f - \bar{f}\right)^2 \left|Z(f)\right|^2 df, \quad \bar{f} = \int_{-\infty}^{+\infty} f \left|Z(f)\right|^2 df \quad (11.324)$$

The computation of the symmetrical coefficient $J_{12} = J_{21}$ is a little more involved. Writing the definition in the form:

$$J_{12} = -\frac{1}{\sigma^2}\, E\left[Re\left(A^*\left\{\frac{\partial^2 A}{\partial a_1\, \partial a_2}\right\} + \left\{\frac{\partial A^*}{\partial a_1}\, \frac{\partial A}{\partial a_2}\right\}\right)\right] \quad (11.325)$$

and using relations (11.312)-(11.315), (11.320), we get:

$$J_{12} = \frac{4\pi^2 A_0^2}{\sigma^2}\, Re\left[\int_{-\infty}^{+\infty} \beta_1\, \mathcal{M}^*\left[f\, Z(f)\right](\beta_1)\, \mathcal{M}[Z](\beta_1)\, d\beta_1\right.$$
$$\left. - \int_{-\infty}^{+\infty} \mathcal{M}^*\left[f\, Z(f)\right](\beta_1)\, \mathcal{M}[Z](\beta_1)\, d\beta_1 \int_{-\infty}^{+\infty} \beta_2 \left|\mathcal{M}[Z](\beta_2)\right|^2 d\beta_2\right] \quad (11.326)$$

This expression is then transformed to the frequency domain using the Parseval formula (11.213) and the property (11.208) of the operator $\mathcal{B}$ defined by equation (11.187) (with $r = -1/2$). The result is

$$\begin{aligned} J_{12} &= \frac{4\pi^2 A_0^2}{\sigma^2}\left[Re\int_0^{+\infty} \mathcal{B}Z(f)\, f\, Z^*(f)\, df - \overline{\beta}\overline{f}\right] \\ &= \frac{4\pi^2 A_0^2}{\sigma^2}\left[M - \overline{\beta}\overline{f}\right] \end{aligned} \tag{11.327}$$

where M is the broad-band modulation index defined by:

$$M \equiv \frac{1}{2\pi}\, Im\int_0^{+\infty} f^2 \frac{dZ^*}{df}\, Z(f)\, df \tag{11.328}$$

The inversion of the matrix J just obtained leads according to (11.307) to the explicit expression of the Cramer-Rao bound for the case of delay and velocity estimation with broad-band signals:

$$\left(\sigma_{ij}^0\right)^2 = \frac{\sigma^2}{4\pi^2 A_0^2\left(\sigma_f^2\sigma_\beta^2 - \left(M - \overline{\beta}\overline{f}\right)^2\right)}\begin{pmatrix} \sigma_f^2 & \overline{\beta}\overline{f} - M \\ \overline{\beta}\overline{f} - M & \sigma_\beta^2 \end{pmatrix} \tag{11.329}$$

Relation (11.301) allows to deduce from this result the minimum variance of the velocity estimator:

$$E\left[\left(v - \hat{v}\right)^2\right] = \frac{c^2}{4}\, E\left[\left(a_1 - \hat{a}_1\right)^2\right] \tag{11.330}$$

$$= \frac{c^2}{4}\left(\sigma_{11}^0\right)^2 \tag{11.331}$$

Comparing these results to the narrow-band case, we see that the delay resolution measured by σ_{22}^0 is still related to the spread of the signal in frequency:

$$\left(\sigma_{22}^0\right)^2 \geq \frac{\sigma^2}{4\pi^2 A_0^2}\,\frac{1}{\sigma_f^2} \tag{11.332}$$

while the velocity resolution now depends in an essential way on the spread in Mellin's space:

$$E\left[\left(v - \hat{v}\right)^2\right] \geq \frac{c^2\sigma^2}{16\pi^2 A_0^2\sigma_\beta^2} \tag{11.333}$$

Thus, for wide-band signals, it is not the duration of the signal that determines the velocity resolution, but the spread in the dual Mellin variable measured by the variance σ_β^2.

As an illustrative example, consider the hyperbolic signal defined by:

$$Z(f) = f^{-2j\pi\beta_0 - 1/2} \tag{11.334}$$

Its Mellin transform which is equal to $\delta(\beta - \beta_0)$ can be considered to have zero spread in β. Hence, such a signal cannot be of any help if seeking a finite velocity resolution.

These remarks can be developed and applied to the construction of radar codes with given characteristics in the variables f and β.[37]

The above results can be seen as a generalization to arbitrary signals of a classical procedure since, in the limit of narrow band, the variance of the velocity estimator can be shown to tend toward its usual expression:

$$E\left[\left(v-\hat{v}\right)^2\right] = \frac{c^2\sigma^2}{16\pi^2 A_0^2 f_0^2} \frac{\sigma_f^2}{\sigma_t^2\sigma_f^2 - \left(m - f_0 t_0\right)^2} \tag{11.335}$$

where the modulation index m is given by:

$$m = \frac{1}{2\pi} Im \int_{-\infty}^{+\infty} t\, z^*\left(t\right) \frac{dz}{dt}\, dt = \frac{1}{2\pi} Im \int_{-\infty}^{+\infty} t\, Z\left(f\right) \frac{dZ^*}{df}\, df \tag{11.336}$$

and the variance σ_t^2 by:

$$\sigma_t^2 = \int_{-\infty}^{\infty} \left(t - \bar{t}\right)^2 \left|z\left(t\right)\right|^2 dt, \qquad \bar{t} = \int_{-\infty}^{\infty} t \left|z\left(t\right)\right|^2 dt \tag{11.337}$$

11.3.3.3 Interpretation of the Dual Mellin Variable in Relation to Time and Frequency

Consider a signal defined by a function of time $z(t)$ such that its Fourier transform $Z(f)$ has only positive frequencies (so-called analytic signal). In that case a Mellin transformation can be applied to $Z(f)$ and yields a function $\mathcal{M}[Z](\beta)$. But while variables t and f have a well defined physical meaning as time and frequency, the interpretation of variable β and its relation to physical parameters of the signal has still to be worked out. This will be done in the present paragraph, thus allowing a formulation of the sampling condition (11.293) for the Mellin transform in terms of the time and frequency spreads of the signal.

As seen in Section 11.3.1.1, the Mellin transform $\mathcal{M}[Z](\beta)$ gives the coefficients of the decomposition of Z on the basis $\{E_\beta(f)\}$:

$$Z\left(f\right) = \int_{-\infty}^{\infty} \mathcal{M}\left[Z\right]\left(\beta\right) E_\beta\left(f\right) d\beta \tag{11.338}$$

The elementary parts:

$$E_\beta\left(f\right) = f^{-2\pi j\beta - r - 1} \equiv f^{-r-1} e^{j\phi\left(f\right)} \tag{11.339}$$

can be considered as filters with group delay given by:

$$\begin{aligned} T\left(f\right) &\equiv -\frac{1}{2\pi} \frac{d\phi\left(f\right)}{df} \\ &= \frac{\beta}{f} \end{aligned} \tag{11.340}$$

As seen on this expression, the variable β has no dimension and labels hyperbolas in a time-frequency half-plane $f > 0$. Hyperbolas displaced in time, corresponding to a group delay law $t = \xi + \beta/f$ are obtained by time shifting the filters E_β to $E_\beta^\xi(f)$ defined by:

$$E_\beta^\xi(f) = e^{-2\pi j\xi f} f^{-2\pi j\beta - r - 1} \tag{11.341}$$

A more precise characterization of signals (11.339) and, hence, of variable β is obtained from a study of a particular affine time-frequency distribution which is to dilations what Wigner-Ville's is to frequency translations. We give only the practical results of the study, referring the interested reader to the literature.[29-31] The explicit form of the distribution is

$$P_0(t, f) = f^{2r+2} \int_{-\infty}^{+\infty} \left(\lambda(u)\lambda(-u)\right)^{r+1} Z\left(f\lambda(u)\right) Z^*\left(f\lambda(-u)\right) e^{2j\pi ftu}\, du \tag{11.342}$$

where function λ is given by:

$$\lambda(u) = \frac{ue^{u/2}}{2\sinh u/2} \tag{11.343}$$

This distribution realizes an exact localization of hyperbolic signals defined by (11.341) on hyperbolas of the time-frequency half-plane as follows:

$$Z(f) = e^{-2\pi j\xi f} f^{-r-1} f^{-2j\pi\beta} \rightarrow P_0(t, f) = f^{-1}\delta\left(t - \xi - \beta/f\right) \tag{11.344}$$

It can be shown that the affine time-frequency distribution (11.342) has the so-called tomographic property[29-31] which reads:

$$\int_{-\infty}^{+\infty} dt \int_0^{+\infty} P_0(t, f)\, \delta\left(t - \xi - \beta/f\right) f^{-1}\, df = \left|\mathcal{M}[Z](\beta)\right|^2 \tag{11.345}$$

Formulas (11.344) and (11.345) are basic for the interpretation of the β variable. It can be shown that for a signal $z(t) \leftrightarrow Z(f)$ having a duration $T = t_2 - t_1$ and a bandwidth $B = f_2 - f_1$, distribution P_0 has a support approximately localized in a bounded region of the half-plane $f > 0$ (see Figure 11.7) around the time $\xi = (t_1 + t_2)/2$ and the mean frequency $f_0 = (f_1 + f_2)/2$. Writing that the hyperbolas at the limits of this region have the equation:

$$t = \xi \pm \beta_0/f \tag{11.346}$$

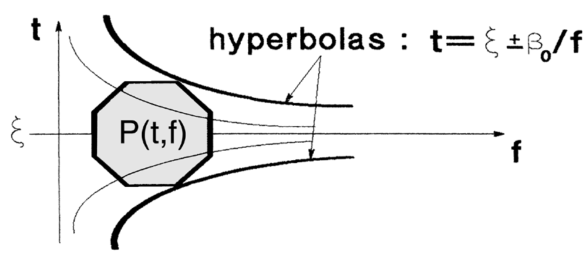

FIGURE 11.7 Time-frequency localization of a signal between hyperbolas with equations $t = \xi + \beta_0/f$ and $t = \xi - \beta_0/f$.

and pass through the points of coordinates $\xi \pm T/2$, $f_0 + B/2$, we find:

$$\beta_0 = \left(f_0 + B/2\right)\left(T/2\right) \tag{11.347}$$

The support $[\beta_1, \beta_2]$ of the Mellin transform $\mathcal{M}[Z](\beta)$ thus can be written in terms of B and T as:

$$\beta_2 - \beta_1 = 2\beta_0 \tag{11.348}$$

The condition (11.293) to avoid aliasing when performing a discrete Mellin transform can now be written in terms of the time-bandwidth product BT and the relative bandwidth R defined by:

$$R \equiv \frac{B}{f_0} \tag{11.349}$$

The result giving the minimum number of samples to treat is

$$N \geq BT\left(\frac{1}{2} + \frac{1}{R}\right)\ln\frac{1 + R/2}{1 - R/2} \tag{11.350}$$

11.3.3.4 The Mellin Transform and the Wavelet Transform[47,48]

The Mellin transform is well suited to the computation of expressions containing dilated functions and, in particular, of scalar products such as:

$$\left(Z_1, \mathcal{D}_a Z_2\right) = a^{r+1}\int_0^{+\infty} Z_1\left(f\right) Z_2^*\left(af\right) f^{2r+1}\, df \tag{11.351}$$

Because of the dilation parameter, a numerical computation of these functions of a by standard techniques (such as DFT) requires the use of oversampling and interpolation. By contrast, the Mellin transform allows a direct and more efficient treatment. The method will be explained on the example of the wavelet transform for one-dimensional signals. But it can also be used in more general situations such as those encountered in radar imaging.[47,48]

Let $s(t)$ be a real signal with Fourier transform $S(f)$ defined by:

$$S\left(f\right) = \int_{-\infty}^{\infty} s\left(t\right) e^{-2j\pi tf}\, dt \tag{11.352}$$

The reality of s implies that:

$$S\left(-f\right) = S^*\left(f\right) \tag{11.353}$$

Given a real function $\phi(t)$ (the so-called mother wavelet), one defines the continuous wavelet transform of signal $s(t)$ as a function $C(a, b)$ of two variables $a > 0$, b real given by:

$$C\left(a, b\right) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} z\left(t\right) \phi^*\left(\frac{t-b}{a}\right) dt \tag{11.354}$$

Transposed to the frequency domain by a Fourier transformation and the use of property (11.353), the definition becomes:

$$C(a, b) = 2\,Re\left\{\sqrt{a}\int_{0}^{+\infty} Z(f)\,\Phi^{*}(af)\,e^{2j\pi fb}\,df\right\} \tag{11.355}$$

where Φ denotes the Fourier transform of ϕ.

If we define the function $Z_b(f)$ by:

$$Z_b(f) \equiv Z(f)\,e^{2j\pi bf} \tag{11.356}$$

the scale invariance property (11.215) of the Mellin transform with $r = -1/2$ and the unitarity property (11.213) allow to write (11.355) in Mellin's space as:

$$C(a, b) = 2Re\left\{\int_{-\infty}^{+\infty} \mathcal{M}\left[Z_b\right](\beta)\,\mathcal{M}^{*}\left[\Phi\right](\beta)\,a^{2j\pi\beta}\,d\beta\right\} \tag{11.357}$$

In this form, there are no more dilations and the computation of the wavelet coefficient reduces to Fourier and Mellin transforms which can all be performed using an FFT algorithm. First the Mellin transform of the wavelet is computed once and for all. Then, for each value of b, one computes the Mellin transform of Z_b and the inverse Fourier transform with respect to β of the product $\mathcal{M}[Z_b](\beta)\ \mathcal{M}^{*}[\Phi](\beta)$. The complexity of this algorithm is given by $(2M + 1)$ FFT with $2N$ points if the wavelet coefficients are discretized in (N, M) points on the (a,b) variables. The signal and the mother wavelet are supposed geometrically sampled with the same geometric ratio q.

The same procedure can be applied to the computation of the broad band ambiguity function.[45] This function is used in problems of radar theory involving target detection and estimation of its characteristics (range, velocity, angle, …). It is defined for an analytic signal $z(t)$ with Fourier transform $Z(f)$ by:

$$X(a, b) = \frac{1}{\sqrt{a}}\int_{-\infty}^{+\infty} z(t)\,z^{*}\left(\frac{t}{a} - b\right)dt \tag{11.358}$$

$$= \sqrt{a}\int_{-\infty}^{+\infty} Z(f)\,Z^{*}(af)\,e^{2j\pi abf}\,df \tag{11.359}$$

The parameters a and b are respectively called the Doppler compression factor and the time shift.

11.3.3.5 Numerical Computation of Affine Time-Frequency Distributions[49]

In this section, the Mellin transformation is applied to the fast computation of the affine time-frequency distribution[29-31] given by:

$$P_0(t, f) = f^{2r+2-q}\int_{-\infty}^{+\infty}\left(\lambda(u)\,\lambda(-u)\right)^{r+1} Z\left(f\lambda(u)\right)\,Z^{*}\left(f\lambda(-u)\right)\,e^{2j\pi ftu}\,du \tag{11.360}$$

where the function λ is defined by:

$$\lambda(u) = \frac{ue^{\frac{u}{2}}}{2\sinh\left(\frac{u}{2}\right)} \tag{11.361}$$

and where r and q are real numbers.

Setting

$$\gamma = ft \qquad \text{and} \qquad \tilde{P}_0(\gamma, f) = P_0(t, f) \tag{11.362}$$

one can write (11.360) as:

$$f^{-r-1+q}\, \tilde{P}_0(\gamma, f) = \int_{-\infty}^{+\infty} \big(\lambda(u)\lambda(-u)\big)^{r+1} \Big[f^{r+1}\, Z\big(f\lambda(u)\big)\, Z^*\big(f\lambda(-u)\big)\Big]\, e^{2j\pi\gamma u}\, du \tag{11.363}$$

To perform the Mellin transformation of this expression with respect to f, we notice that the term in brackets represents the invariant product of the two functions of f defined by $Z(f\lambda(u))$ and $Z^*(f\lambda(-u))$. By relation (11.249), we know that the Mellin transform of this product is equal to the convolution of the functions $\mathcal{M}[Z(f\lambda(u))]$ and $\mathcal{M}[Z^*(f\lambda(-u))]$. Besides, the scaling property (11.215) allows to write:

$$\mathcal{M}\Big[Z\big(f\lambda(u)\big)\Big](\beta) = \lambda(u)^{-2j\pi\beta-r-1}\, \mathcal{M}\big[Z\big](\beta) \tag{11.364}$$

and

$$\mathcal{M}\Big[Z^*\big(f\lambda(-u)\big)\Big](\beta) = \lambda(-u)^{-2j\pi\beta-r-1}\, \mathcal{M}^*\big[Z\big](-\beta) \tag{11.365}$$

where

$$\mathcal{M}^*\big[Z\big](-\beta) \equiv \Big[\mathcal{M}\big[Z\big](-\beta)\Big]^* \tag{11.366}$$

Introducing the notation:

$$X(\beta, u) \equiv \lambda(u)^{-2j\pi\beta}\, \mathcal{M}\big[Z\big](\beta) \tag{11.367}$$

we can write the convolution between (11.364) and (11.365) as:

$$\big(\lambda(u)\lambda(-u)\big)^{-r-1} \int_{-\infty}^{+\infty} X(\beta_1, u)\, X^*(\beta_1 - \beta, -u)\, d\beta_1 \tag{11.368}$$

The Mellin transform of expression (11.363) is now written as:

$$\mathcal{M}\Big[f^{-r-1+q}\, \tilde{P}_0(\gamma, f)\Big](\beta) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} X(\beta_1, u)\, X^*(\beta_1 - \beta, -u)\, d\beta_1\right] e^{2j\pi\gamma u}\, du \tag{11.369}$$

The cross-correlation inside brackets is computed in terms of the Fourier transform of $X(\beta, u)$ defined by:

$$F(\theta, u) = \int_{-\infty}^{+\infty} X(\beta, u)\, e^{-2j\pi\theta\beta}\, d\beta \tag{11.370}$$

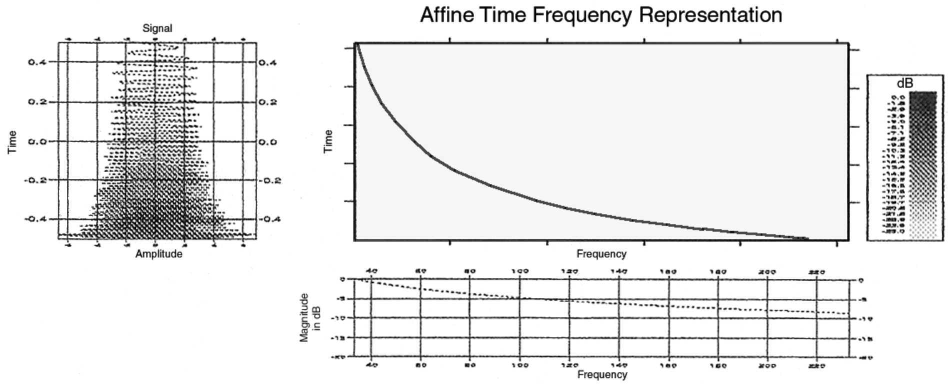

FIGURE 11.8 Affine time-frequency representation of a hyperbolic signal.

and (11.369) becomes:

$$\mathcal{M}\left[f^{-r-1+q}\,\tilde{P}_0\left(\gamma,\, f\right)\right]\left(\beta\right) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F\left(\theta,\, u\right)\, F^{*}\left(\theta,\, -u\right)\, e^{2j\pi\theta\beta}\, e^{2j\pi\gamma u}\, d\theta du \qquad (11.371)$$

Finally, inverting the Mellin transform by (11.212), recalling (11.362) and taking into account the property of the integrand in the change $u \to -u$, one obtains the following form of the affine Wigner function P_0:

$$P_0\left(t,\, f\right) = 2\,Re\left\{f^{-q}\int_0^{+\infty} F\left(\ln f,\, u\right)\, F^{*}\left(\ln f,\, -u\right)\, e^{2j\pi tfu}\, du\right\} \qquad (11.372)$$

where Re denotes the real part operation. In this form, the numerical computation of P_0 has been reduced to a Fourier transform. The operations leading from Z to F are a Fourier and a Mellin transform, both of which are performed using the Fast Fourier Transform algorithm. The approximate complexity of the whole algorithm for computing P_0 can be expressed in terms of the number of FFT performed. If the time-frequency distribution $P_0(t,f)$ is characterized by (M,N) points, respectively, in time and frequency, we have to deal with $2M$ + FFT of $2N$ points and N FFT of M points. The Figure 11.8 gives an example of affine distribution computed by this method.

Appendix A. Some Special Functions Frequently Occurring as Mellin Transforms

The Gamma Function (see also Chapter 1)

Definition. The gamma function $\Gamma(s)$ is defined on the complex half-plane $Re(s) > 0$ by the integral:

$$\Gamma\left(s\right) = \int_0^{\infty} e^{-t}\, t^{s-1}\, dt \qquad (11.373)$$

Analytic continuation. The analytically continued gamma function is holomorphic in the whole plane except at the points $s = -n$, $n = 0,1,2,\ldots$ where it has a simple pole.

Residues at the poles

$$\text{Re}\, s_{s=-n}\left(\Gamma(s)\right) = \frac{(-1)^n}{n!} \tag{11.374}$$

Relation to the factorial

$$\Gamma(n+1) = n! \tag{11.375}$$

Functional relations

$$\Gamma(s+1) = s\Gamma(s) \tag{11.376}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \tag{11.377}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \tag{11.378}$$

$$\Gamma(2s) = \pi^{-1/2}\, 2^{2s-1}\, \Gamma(s)\Gamma(s+1/2) \tag{11.379}$$

(Legendre's duplication formula)

$$\Gamma(ms) = m^{ms-1/2}\,(2\pi)^{(1-m)/2} \prod_{k=0}^{m-1} \Gamma(s+k/m), \quad m = 2, 3, \ldots \tag{11.380}$$

(Gauss – Legendre multiplication formula[5])

Stirling asymptotic formula

$$\Gamma(s) \sim \sqrt{2\pi}\, s^{s-1/2} \exp\left[-s\left(1+\frac{1}{12s}+O\left(s^{-2}\right)\right)\right] \quad s \to \infty, \quad \left|\arg(s)\right| < \pi \tag{11.381}$$

The Beta Function

Definition

$$B(x, y) \equiv \int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt \tag{11.382}$$

Relation to the gamma function

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{11.383}$$

The Psi Function (Logarithmic Derivative of the Gamma Function)

Definition

$$\psi(s) \equiv \frac{d}{ds} \ln \Gamma(s) \tag{11.384}$$

$$= -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{s+n} \right) \tag{11.385}$$

Euler's constant γ, also called C, is defined by:

$$\gamma \equiv -\Gamma'(1) / \Gamma(1) \tag{11.386}$$

and has value $\gamma \cong 0.577 \ldots$.

Riemann's Zeta Function[12, 13]

Definition

$$\zeta(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad \text{Re}(z) > 1 \tag{11.387}$$

Other forms of the definition, which are valid for all complex values of z, have been written down. They coincide with (11.387) for $Re(z) > 1$ and allow the continuation of $\zeta(z)$ as a meromorphic function in the whole complex z-plane. The resulting function has only one pole, situated in $z = 1$; it is simple, with residue equal to +1. In addition, the ζ-function has simple zeros at $z = -2n$, $n \neq 0$. All other zeros are in the strip $0 \leq Re(z) \leq 1$.

Functional equation

$$\pi^{-z/2} \Gamma(z/2) \zeta(z) = \pi^{1/2(z-1)} \Gamma\left(\frac{1-z}{2} \right) \zeta(1-z) \tag{11.388}$$

Other forms of this equation can be obtained by using the properties of the Gamma function.

Asymptotic estimates. Let $\sigma \equiv Re(z)$ and $t \equiv Im(z)$. The behavior of $\zeta(z)$ when $|t| \to \infty$ is such that:

$$|\zeta(z)| < C(\epsilon) |t|^{\mu(\sigma)+\epsilon}, \quad \epsilon > 0 \tag{11.389}$$

where $C(\epsilon)$ is a constant and $\mu(\sigma)$ is a function defined as follows:

$$\begin{aligned} \mu(\sigma) &= 0 & \sigma > 1 \\ \mu(\sigma) &\leq \frac{1-\sigma}{2} & 0 < \sigma < 1 \\ \mu(\sigma) &= \frac{1}{2} - \sigma & \sigma < 0 \end{aligned} \tag{11.390}$$

For $\sigma = 1/2$, more precise estimates have been proven. In particular:[12]

$$\zeta\left(1/2 + it\right) = O\left(\left|t\right|^{9/56+\epsilon}\right), \ \epsilon > 0 \tag{11.391}$$

Appendix B. Summary of Properties of the Mellin Transformation

Definition. The Mellin transformation of a function $f(t)$, $0 < t < \infty$ is defined by:

$$\mathcal{M}\left[f;\, s\right] \equiv \int_0^\infty f\left(t\right) t^{s-1}\, dt$$

and the result is a function holomorphic in the strip S_f of the complex plane s.

When the real part $Re(s) \equiv r + 1$ of s is held fixed, the Mellin transform is defined by:

$$\mathcal{M}\left[f\right]\left(\beta\right) \equiv \mathcal{M}\left[f;\, r + 1 + 2\pi j\beta\right]$$

In that case, it is an isomorphism between the space $L^2(\mathrm{IR}^+, t^{2r+1}\, dt)$ of functions $f(t)$ on $(0, \infty)$ equipped with the scalar product:

$$\left(f,\, g\right) \equiv \int_0^\infty f\left(t\right) g^*\left(t\right) t^{2r+1}\, dt$$

and the space $L^2(\mathrm{IR})$ of functions $\mathcal{M}[f](\beta)$.

Moreover, the scaled function defined by:

$$\mathcal{D}_a f\left(t\right) \equiv a^{r+1}\, f\left(at\right)$$

is transformed according to:

$$\mathcal{M}\left[\mathcal{D}_a f\right]\left(\beta\right) = a^{-j2\pi\beta}\, \mathcal{M}\left[f\right]\left(\beta\right)$$

Inversion formulas

$$f\left(t\right) = \left(1/2\pi j\right) \int_{a-j\infty}^{a+j\infty} \mathcal{M}\left[f;\, s\right] t^{-s}\, ds$$

$$f\left(t\right) = \int_{-\infty}^{+\infty} \mathcal{M}\left[f\right]\left(\beta\right) t^{-2j\pi\beta - r - 1}\, d\beta$$

Parseval formulas

$$\int_0^\infty f\left(t\right) g\left(t\right) dt = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathcal{M}\left[f;\, s\right] \mathcal{M}\left[g;\, 1 - s\right] ds$$

$$\int_0^\infty f(t)\, g^*(t)\, t^{2r+1}\, dt = \int_\infty^\infty \mathcal{M}[f](\beta)\, \mathcal{M}^*[g](\beta)\, d\beta$$

Other basic formulas involving the Mellin transforms $\mathcal{M}[f;s]$ and $\mathcal{M}[f](\beta)$ are recalled in Tables 11.1 and 11.2.

Multiplicative convolution. It is defined by:

$$(f \vee g)(t) \equiv \int_0^\infty f(\tau)\, f(t/\tau)\, (d\tau/\tau)$$

$$f \vee \delta(t-1) = f$$

TABLE 11.1 Properties of the Mellin Transform in s Variable (Definition (11.2.1.1))

Original Function $f(t),\ t>0$	Mellin Transform $\mathcal{M}[f;s] \equiv \int_0^\infty f(t)\, t^{s-1}\, dt$	Strip of Holomorphy
$f(t)$	$F(s)$	S_f
$f(at), \quad a>0$	$a^{-s}\, F(s)$	S_f
$f(t^a)$, a real $\neq 0$	$\lvert a\rvert^{-1} F(a^{-1} s)$	$a^{-1} s \in S_f$
$(\ln t)^k\, f(t)$	$\dfrac{d^k}{ds^k} F(s)$	$s \in S_f$
$(t)^z\, f(t)$, z complex	$F(s+z)$	$s+z \in S_f$
$\dfrac{d^k}{dt^k} f(t)$	$(-1)^k (s-k)_k\, F(s-k)$ $(s-k)_k \equiv (s-k)(s-k+1)\ldots(s-1)$	$s-k \in S_f$
$\left(t\dfrac{d}{dt}\right)^k f(t)$	$(-1)^k\, s^k\, F(s)$	$s \in S_f$
$\dfrac{d^k}{dt^k} t^k\, f(t)$	$(-1)^k (s-k)_k\, F(s)$	$s \in S_f$
$t^k \dfrac{d^k}{dt^k} f(t)$	$(-1)^k (s)_k\, F(s)$ $(s)_k \equiv s(s+1)\ldots(s+k-1)$	$s \in S_f$
$\int_t^\infty f(x)\, dx$	$s^{-1}\, F(s+1)$	
$\int_0^t f(x)\, dx$	$-s^{-1}\, F(s+1)$	
$\int_0^\infty f_1(\tau)\, f_2(t/\tau)\, (d\tau/\tau)$	$F_1(s)\, F_2(s)$	$s \in S_{f_1} \cap S_{f_2}$

Note: Here k is a positive integer.

TABLE 11.2 Some Properties of the Mellin Transform in β Variable (Definition (11.200))

Original Function	Mellin Transform
$f(t),\ t>0$	$\mathcal{M}[f](\beta) \equiv \int_0^\infty f(t)\, t^{2\pi j\beta + r}\, dt$
$f(t)$	$M(\beta)$
$\mathcal{D}_a f(t) \equiv a^{r+1} f(at),\quad a>0$	$a^{-2\pi j\beta} M(\beta)$
$t^{2\pi jc} f(t),\ c$ real	$M(\beta + c)$
$\dfrac{-1}{2j\pi}\left(t\dfrac{d}{dt} + r + 1\right) f(t)$	$\beta M(\beta)$

$$\left(t\frac{d}{dt}\right)^k (f \vee g) = \left[\left(t\frac{d}{dt}\right)^k f\right] \vee g$$

$$= f \vee \left[\left(t\frac{d}{dt}\right)^k g\right]$$

$$(\ln t)(f \vee g) = [(\ln t) f] \vee g + f \vee [(\ln t) g]$$

$$\delta(t-a) \vee f = a^{-1} f(a^{-1}t)$$

$$\delta(t-p) \vee \delta(t-p') = \delta(t-pp'),\quad p, p' > 0$$

$$\delta^{(k)}(t-1) \vee f = (d/dt)^k (t^k f)$$

Invariant product. It is defined by:

$$(f \circ g)(t) \equiv t^{r+1} f(t)\, g(t)$$

$$\mathcal{D}_a[f] \circ \mathcal{D}_a[g] = \mathcal{D}_a[f \circ g]$$

$$\mathcal{M}[f \circ g](\beta) = (\mathcal{M}[f] * \mathcal{M}[g])(\beta)$$

Useful formulas for discretization. In the following, the variable v goes from 0 to ∞ and $Z(v)$ is a (possibly generalized) function.

Geometric Dirac comb:

$$\Delta_Q^r(\nu) \equiv \sum_{n=-\infty}^{+\infty} Q^{-nr}\,\delta(\nu - Q^n), \quad , Q>0$$

Dilatocycled form of function *Z*:

$$Z^D(\nu) \equiv \sum_{n=-\infty}^{+\infty} Q^{n(r+1)}\,Z(Q^n \nu),\ \ Q>0$$

$$Z^D(\nu) = \left[\Delta_Q^r \vee Z\right](\nu)$$

$$\mathcal{M}\left[Z^D\right](\beta) = \frac{1}{\ln Q}\sum_{n=-\infty}^{+\infty}\mathcal{M}[Z](\beta)\,\delta\left(\beta - \frac{n}{\ln Q}\right)$$

Geometrically sampled form of function *Z*:

$$Z_S(\nu) \equiv \sum_{n=-\infty}^{+\infty} q^n\,Z(q^n)\,\delta(\nu - q^n)$$

$$Z_S(\nu) = \left(Z \circ \Delta_q^r\right)(\nu)$$

$$\mathcal{M}\left[Z_S\right](\beta) = \frac{1}{\ln q}\sum_{n=-\infty}^{+\infty}\mathcal{M}[Z]\left(\beta - \frac{n}{\ln q}\right)$$

Discrete Mellin transform pair.

$$M^P\left(\frac{m}{N \ln q}\right) = \sum_{n=M}^{M+N-1} q^{n(r+1)}\,Z^D(q^n)\,e^{2j\pi nm/N}$$

$$Z^D(q^n) = \frac{q^{-n(r+1)}}{N}\sum_{k=K}^{K+N-1} M^P\left(\frac{k}{N \ln q}\right)e^{2j\pi kn/N}$$

where N is the number of samples. In practice, the choice of the ratio q is facilitated by the time-frequency interpretation of the signal $Z(\nu)$ (see Section 11.3.3.3).

TABLE 11.3 Some Standard Mellin Transform Pairs

Original Function	Mellin Transform	
$f(t),\ t>0$	$\mathcal{M}[f;s]\equiv\int_0^\infty f(t)\,t^{s-1}\,dt$	Strip of Holomorphy
$e^{-pt},\ p>0$	$p^{-s}\,\Gamma(s)$	$Re(s)>0$
$H(t-a)t^b,\ a>0$	$-\frac{a^{b+s}}{b+s}$	$Re(s)<-Re(b)$
$(H(t-a)-H(t))t^b$	$-\frac{a^{b+s}}{b+s}$	$Re(s)>-Re(b)$
$(1+t)^{-1}$	$\frac{\pi}{\sin(\pi s)}$	$0<Re(s)<1$
$(1+t)^{-a}$	$\frac{\Gamma(s)\Gamma(a-s)}{\Gamma(a)}$	$0<Re(s)<Re(a)$
$(1-t)^{-1}$	$\pi\cot(\pi s)$	$0<Re(s)<1$
$H(1-t)(1-t)^{b-1},\ Re(b)>0$	$\frac{\Gamma(s)\Gamma(b)}{\Gamma(s+b)}$	$Re(s)>0$
$H(t-1)(t-1)^{-b}$	$\frac{\Gamma(b-s)\Gamma(1-b)}{\Gamma(1-s)}$	$Re(s)<Re(b)<1$
$H(t-1)\sin(a\ln t)$	$\frac{a}{s^2+a^2}$	$Re(s)<-\lvert Im(a)\rvert$
$H(1-t)\sin(-a\ln t)$	$\frac{a}{s^2+a^2}$	$Re(s)>\lvert Im(a)\rvert$
$(H(t)-H(t-p))\ln(p/t),\ \ p>0$	$\frac{p^s}{s^2}$	$Re(s)>0$
$\ln(1+t)$	$\frac{\pi}{s\sin(\pi s)}$	$-1<Re(s)<0$
$H(p-t)\ln(p-t)$	$-p^s\,s^{-1}\left[\psi(s+1)+p^{-1}\ln\gamma\right]$	$Re(s)>0$
$t^{-1}\ln(1+t)$	$\frac{\pi}{(1-s)\sin(\pi s)}$	$0<Re(s)<1$
$\ln\left\lvert\frac{1+t}{1-t}\right\rvert$	$(\pi/s)\ tan(\pi s$	$-1<Re(s)<1$
$(e^t-1)^{-1}$	$\Gamma(s)\zeta(s)$	$Re(s)>1$
$t^{-1}\,e^{-t^{-1}}$	$\Gamma(1-s)$	$-\infty<Re(s)<1$
e^{-x^2}	$(1/2)\Gamma(s/2)$	$0<Re(s)<+\infty$
e^{iat}	$a^{-s}\,\Gamma(s)e^{i\pi(s/2)}$	$0<Re(s)<1$

TABLE 11.3 (continued) Some Standard Mellin Transform Pairs

Original Function	Mellin Transform	
$f(t),\ t>0$	$\mathcal{M}[f;s]\equiv\int_0^\infty f(t)\,t^{s-1}\,dt$	Strip of Holomorphy
$\tan^{-1}(t)$	$\dfrac{-\pi}{2s\cos(\pi s/2)}$	$-1<Re(s)<0$
$\text{cotan}^{-1}(t)$	$\dfrac{\pi}{2s\cos(\pi s/2)}$	$0<Re(s)<1$
$\delta(t-p),\ p>0$	p^{s-1}	Whole plane
$\sum_{n=1}^{\infty}\delta(t-pn),\ p>0$	$p^{s-1}\zeta(1-s)$	$Re(s)<0$
$J_\nu(t)$	$\dfrac{2^{s-1}\Gamma[(s+\nu)/2]}{\Gamma[(1/2)(\nu-s)+1]}$	$-\nu<Re(s)<3/2$
$\sum_{n=-\infty}^{+\infty}p^{-nr}\delta(t-p^n)$ $p>0,\ r$ real	$\dfrac{1}{\ln p}\sum_{n=-\infty}^{+\infty}\delta\left(\beta-\dfrac{n}{\ln p}\right)$ $\beta=Im(s)$	$s=r+j\beta$
t^b	$\delta\ (b+s)$	None (analytic functional)

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