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A NOTE ON GENERATING FUNCTIONS AND SUMMATION FORMULAE FOR MEIXNER POLYNOMIALS OF SEVERAL VARIABLES

Abstract. The present paper deals with certain generating functions and various elegant summation formulae for Meixner polynomials of several variables.

1. Introduction

Generalized functions occupy the place of pride in literature on special functions. Their importance which is mounting everyday stems from the fact that they generalize well-known one variable special functions namely Hermite polynomials, Laguerre polynomials, Legendre polynomials, Gegenbauer polynomials, Jacobi polynomials, Rice polynomials, generalized Sylvester polynomials, Meixner polynomials etc. All these polynomials are closely associated with problems of applied nature. For example, Gegenbauer polynomials are deeply connected with axially symmetric potentials in n dimensions and contain the Legendre and Chebyshev polynomials as special cases. The hypergeometric functions of which the Jacobi polynomials are a special case, are important in many cases of mathematical analysis and its applications. Further, Bessel functions are closely associated with problems possessing circular or cylindrical symmetry. For example, they arise in the study of free vibrations of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic theory and numerous other areas of physics and engineering.

The following results are required in this paper

DEFINITIONS:
Following the work of Riordan [5] (p. 90 et seq.), one denotes by $S(n, k)$ the Stirling numbers of the second kind, defined by

$$(1.1) \qquad S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

so that

$$(1.2) \qquad S(n,0) = \begin{cases} 1 \ (n=0), \\ 0 \ (n \in N), \end{cases}$$

and

$$S(n,1) = S(n,n) = 1 \quad \text{and} \quad S(n,n-1) = \binom{n}{2}.$$

Recently, several authors (see, for example, Gabutti and Lyness [2], Mathis and Sismondi [8], and Srivastava [3]) considered various families of generating functions associated with the Stirling numbers $S(n,k)$. We choose to recall here the following general results on these families of generating functions, which were given by Srivastava [3].

THEOREM 1. ([3], p. 754) *Let the sequence* $\{S_n(x)\}_{n=0}^{\infty}$ *be generated by*

$$\sum_{n=0}^{\infty} \binom{n+k}{k} S_{n+k}(x)t^k = f(x,t)\{g(x,t)\}^{-n} S_n(h(x,t)) (n \in N_0 := N \cup \{0\}),$$

where f, g *and* h *are suitable functions of* x *and* t.

    Then, in terms of the Stirling numbers $S(n,k)$ *defined by (1.1), the following family of generating functions holds true:*

$$(1.3) \quad \sum_{k=0}^{\infty} k^n S_k(h(x,-z)) \left(\frac{z}{g(x,-z)} \right)^k$$

$$= \{f(x,-z)\}^{-1} \sum_{k=0}^{n} k! S(n,k) S_k(x) z^k \quad (n \in N_0),$$

provided that each member of (1.3) exists.

THEOREM 2. ([4], p. 403) *Let the function* $F(z)$ *be holomorphic on a domain* D *of the complex* z-*plane, and define*

$$(1.4) \qquad f_n(z) = \frac{1}{n!} D_z^n \{(az+b)^n F(z)\}, \qquad D_z = \frac{d}{dz}, \quad n = 0,1,2,\ldots,$$

where a *and* b *are complex constants such that* $|a| + |b| > 0$. *Also let* $\{\lambda_n\}$ *be any sequence of complex numbers for which*

$$(1.5) \qquad R^{-1} = \lim_{n \to \infty} \sup |\lambda_n|^{1/n}$$

is finite or for which

$$(1.6) \qquad R = \lim_{n \to \infty} \left| \frac{\lambda_n}{\lambda_{n+1}} \right|$$

exists and is positive. Suppose further that

$$(1.7) \qquad A_n = \sum_{k=0}^{[n/N]} \binom{n}{Nk} \lambda_k w^k$$

for some positive integer N and some complex number w. Then

$$(1.8) \qquad \sum_{n=0}^{\infty} A_n f_n(z) t^n = \frac{1}{1-at} \sum_{k=0}^{\infty} \lambda_k f_{Nk} \left(\frac{z+bt}{1-at} \right) \left[\frac{wt^N}{(1-at)^N} \right]^k$$

for some domain in the complex t-plane that includes the origin.

LAGRANGE'S EXPANSION FORMULA. ([4], p. 355) *If $\phi(z)$ is holomorphic at $z = z_0$ and $\phi(z_0) \neq 0$, and if*

$$(1.9) \qquad z = z_0 + w\phi(z),$$

then an analytic function $f(z)$, which is holomorphic at $z = z_0$, can be expanded as a power series in w by the Lagrange formula [Whittaker and Watson (1927), p. 133]

$$(1.10) \qquad f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{w^n}{n!} D_z^{n-1} \{f'(z)[\phi(z)]^n\}) \mid_{z=z_0},$$

where $D_z = d/dz$.

If we differentiate both sides of (1.10) with respect to w, using the relationship (1.9), and replace $f'(z)\phi(z)$ in the resulting equation by $f(z)$, we can write (1.10) in the form [cf. Polya and Szego (1972), p. 146, problem 207]:

$$(1.11) \qquad \frac{f(z)}{1-w\phi'(z)} = \sum_{n=0}^{\infty} \frac{w^n}{n!} D_z^n \{f(z)[\phi(z)]^n\} \mid_{z=z_0},$$

which is usually more suitable to apply than (1.10).

For $\phi(z) \equiv 1$, both (1.10) and (1.11) evidently yield Taylor,s expansion

$$(1.12) \qquad f(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!} f^{(n)}(z_0),$$

where, as usual,

$$f^{(n)}(z_0) = D_z^n \{f(z)\} \mid_{z=z_0}.$$

THEOREM 3. (Carlitz's [6], p. 521) *Let $A(z)$, $B(z)$ and $z^{-1}C(z)$ be arbitrary functions which are analytic in the neighborhood of the origin, and assume that*

$$(1.13) \qquad A(0) = B(0) = C(0) = 1.$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ by means of

$$(1.14) \qquad A(z)[B(z)]^{\alpha} exp\ (xC(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x) \frac{z^n}{n!},$$

where α and x are arbitrary complex numbers independent of z. Then, for arbitrary parameters λ and y independent of z,

$$(1.15) \qquad \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\frac{t^n}{n!} = \frac{A(\zeta)[B(\zeta)]^{\alpha} exp\ (xC(\zeta))}{1 - \zeta\{\lambda[B'(\zeta)/B(\zeta)] + yC'(\zeta)\}}$$

where ζ is a function of t defined by

$$(1.16) \qquad \zeta = t[B(\zeta)]^{\lambda} exp\ (yC(\zeta)).$$

2. Meixner polynomials

The Meixner polynomials are denoted by $m_n(x; \beta, c)$ and are defined as (see [10])

$$m_n(x; \beta, c) = (\beta)_n\ {}_2F_1 \begin{bmatrix} -n, & x; \\ & & 1-c^{-1} \\ \beta; & \end{bmatrix}$$

$$(2.1) \qquad = \sum_{k=0}^{n} \frac{(\beta)_n(-x)_k\ n!}{k!\ (n-k)!(\beta)_k}\left(\frac{1}{c} - 1\right)^k$$

where $\beta > 0$, $0 < c < 1$ and $x = 0, 1, 2, \ldots$

Agarwal and Manocha [1] defined the polynomials $m_n(x; \beta, c)$ by the generating relations

$$(2.2) \qquad \sum_{n=0}^{\infty} m_n(x; \beta, c)\frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}, |t| < min(1, |c|),$$

$$(2.3) \quad \sum_{n=0}^{\infty} m_{n+k}(x; \beta, c)\frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta-k} m_k\left(x; \beta, \frac{c-t}{1-t}\right)$$

$$(n \in N_0;\ |t| < \min\{1, |c|\}).$$

The following generating function is well-known [4], p. 443, problem 5(ii)

$$(2.4) \quad \sum_{n=0}^{\infty} A_n m_n(x; \beta, c)\frac{t^n}{n!}$$

$$= \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta} \sum_{k=0}^{\infty} \frac{\lambda_k}{(Nk)!} m_{Nk}\left(x; \beta, \frac{c-t}{1-t}\right)\left[\frac{wt^N}{(1-t)^N}\right]^k.$$

By applying Theorem 1, one immediately obtain the following (presumably new) generating function for the Meixner polynomials defined by (2.1):

$$(2.5) \quad \sum_{k=0}^{\infty} \frac{k^n}{k!} m_k\left(x; \beta, \frac{c+z}{1+z}\right) \left(\frac{z}{1+z}\right)^k$$

$$= (1+z)^{x+\beta} \left(1 + \frac{z}{c}\right)^{-x} \sum_{k=0}^{n} S(n,k) m_k(x; \beta, c) z^k \ (n \in N_0; \ |z| < \min\{1, |c|\})$$

where $S(n,k)$ and A_n are given by (1.1) and (1.7) of this paper.

3. Applications of Carlitz's theorem

Let γ and δ be arbitrary constants. Then the polynomials $m_n(x; \beta, c)$ defined by (2.2) above satisfy the following generating relations:

$$(3.1) \quad \sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} = \frac{\left(1 - \frac{u}{c}\right)^x (1-u)^{-x-\beta}}{1 - u\gamma(1-u)^{-1}},$$

where u is a function of t defined by

$$u = t(1-u)^{-\gamma}, \quad u(0) = 0.$$

Proof of (3.1). We know generating function

$$(i) \quad \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}.$$

Expanding the function on the R.H.S. of (i) by Taylor's theorem.

$$(ii) \quad \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n} \left\{ \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta} \right\} \right]_{t=0} \frac{t^n}{n!}.$$

Replacing β by $\beta + \gamma n$ in (ii) we get

$$(iii) \quad \sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n} \left\{ \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}[(1-t)^{-\gamma}]^n \right\} \right]_{t=0} \frac{t^n}{n!}.$$

We know that the Lagrange's expansion formula:

$$(iv) \quad \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n} \{f(t)[\phi(t)]^n\} \right]_{t=0} \frac{u^n}{n!} = \frac{f(t)}{1 - u\phi'(t)}; \quad u = \frac{t}{\phi(t)}$$

$$(v) \quad \sum_{n=0}^{\infty} m_n(x; \beta + \gamma n, c) \frac{t^n}{n!} = \frac{f(u)}{1 - t\phi'(u)}; \quad t = \frac{u}{\phi(u)}.$$

Here

$$f(u) = \left(1 - \frac{u}{c}\right)^x (1-u)^{-x-\beta}, \qquad \phi(u) = (1-u)^{-\gamma}, \qquad \phi'(u) = \gamma(1-u)^{-\gamma-1}.$$

Therefore we get (3.1)

(3.2) $$\sum_{n=0}^{\infty} m_n(x+\gamma n; \beta, c)\frac{t^n}{n!} = \frac{\left(1-\frac{v}{c}\right)^x (1-v)^{-x-\beta}}{1 - v\gamma[\frac{-1}{c}(1-\frac{v}{c})^{-1} + (1-v)^{-1}]},$$

where v is a function of t defined by

$$v = t\left(1 - \frac{v}{c}\right)^\gamma (1-v)^{-\gamma}, \quad v(0) = 0. \quad \blacksquare$$

Proof of (3.2). We know generating function

(i) $$\sum_{n=0}^{\infty} m_n(x; \beta, c)\frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}.$$

Expanding the function on the R.H.S. of (i) by Taylor's theorem

(ii) $$\sum_{n=0}^{\infty} m_n(x; \beta, c)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left\{\left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}\right\}\right]_{t=0} \frac{t^n}{n!}.$$

Replacing x by $x + \gamma n$ in (ii) we get

(iii) $$\sum_{n=0}^{\infty} m_n(x+\gamma n; \beta, c)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left\{\left(1 - \frac{t}{c}\right)^x (1-t)^{-x-\beta}\left[\left(1 - \frac{t}{c}\right)^\gamma (1-t)^{-\gamma}\right]^n\right\}\right]_{t=0} \frac{t^n}{n!}.$$

We know that the Lagrange's expansion formula:

(iv) $$\sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\{f(t)[\phi(t)]^n\}\right]_{t=0} \frac{v^n}{n!} = \frac{f(t)}{1 - v\phi'(t)}; \qquad v = \frac{t}{\phi(t)},$$

(v) $$\sum_{n=0}^{\infty} m_n(x+\gamma n; \beta, c)\frac{t^n}{n!} = \frac{f(v)}{1 - t\phi'(v)}; \qquad t = \frac{v}{\phi(v)}.$$

Here

$$f(v) = \left(1 - \frac{v}{c}\right)^x (1-v)^{-x-\beta},$$

$$\phi(v) = \left(1 - \frac{v}{c}\right)^\gamma (1-v)^{-\gamma},$$

$$\phi'(v) = \phi(v)\gamma\left[\frac{-1}{c}\left(1 - \frac{v}{c}\right)^{-1} + (1-v)^{-1}\right].$$

Therefore we get (3.2) and

(3.3) $$\sum_{n=0}^{\infty} m_n(x + \gamma n; \beta + \delta n, c)\frac{t^n}{n!}$$

$$= \frac{\left(1 - \frac{w}{c}\right)^x (1 - w)^{-x-\beta}}{1 - w[\frac{-\gamma}{c}(1 - \frac{w}{c})^{-1} + (\gamma + \delta)(1 - w)^{-1}]},$$

where w is a function of t defined by

$$w = t\left(1 - \frac{w}{c}\right)^\gamma (1 - w)^{-\gamma-\delta}, \quad w(0) = 0. \blacksquare$$

Proof of (3.3). We know generating function

(i) $$\sum_{n=0}^{\infty} m_n(x; \beta, c)\frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1 - t)^{-x-\beta}.$$

Expanding the function on the R.H.S. of (i) by Taylor's theorem

(ii) $$\sum_{n=0}^{\infty} m_n(x; \beta, c)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left\{\left(1 - \frac{t}{c}\right)^x (1 - t)^{-x-\beta}\right\}\right]_{t=0} \frac{t^n}{n!}.$$

Replacing x by $x + \gamma n$ and β by $\beta + \delta n$ in (ii) we get

(iii) $$\sum_{n=0}^{\infty} m_n(x + \gamma n; \beta + \delta n, c)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\left\{\left(1 - \frac{t}{c}\right)^x (1 - t)^{-x-\beta} \left[\left(1 - \frac{t}{c}\right)^\gamma (1 - t)^{-(\gamma+\delta)}\right]^n\right\}\right]_{t=0} \frac{t^n}{n!}.$$

We know that the Lagrange's expansion formula:

(iv) $$\sum_{n=0}^{\infty} \left[\frac{d^n}{dt^n}\{f(t)[\phi(t)]^n\}\right]_{t=0} \frac{w^n}{n!} = \frac{f(t)}{1 - w\phi'(t)}; \quad w = \frac{t}{\phi(t)}$$

(v) $$\sum_{n=0}^{\infty} m_n(x + \gamma n; \beta + \delta n, c)\frac{t^n}{n!} = \frac{f(w)}{1 - t\phi'(w)}; \quad t = \frac{w}{\phi(w)}.$$

Here

$$f(w) = \left(1 - \frac{w}{c}\right)^x (1 - w)^{-x-\beta},$$

$$\phi(w) = \left(1 - \frac{w}{c}\right)^\gamma (1 - w)^{-(\gamma+\delta)},$$

$$\phi'(w) = \phi(w)\left[\frac{-\gamma}{c}\left(1 - \frac{w}{c}\right)^{-1} + (\gamma + \delta)(1 - w)^{-1}\right].$$

Therefore we get (3.3). $\blacksquare$

Formulas (3.1), (3.2), and (3.3) are applications of Carlitz's Theorem (3) for Meixner Polynomials.

Particular cases:

(i) Taking $\delta = \gamma$ in (3.3) we get

$$
(3.4) \quad \sum_{n=0}^{\infty} m_n(x + \gamma n; \beta + \gamma n, c) \frac{t^n}{n!}
$$

$$
= \frac{\left(1 - \frac{w}{c}\right)^x (1 - w)^{-x - \beta}}{1 - w\left[\frac{-\gamma}{c}(1 - \frac{w}{c})^{-1} + 2\gamma(1 - w)^{-1}\right]}, \quad w = t\left(1 - \frac{w}{c}\right)^\gamma (1 - w)^{-2\gamma}.
$$

(ii) Taking $\delta = 0$ and w is replace by v in (3.3) we get

$$
(3.5) \quad \sum_{n=0}^{\infty} m_n(x + \gamma n; \beta, c) \frac{t^n}{n!}
$$

$$
= \frac{\left(1 - \frac{v}{c}\right)^x (1 - v)^{-x - \beta}}{1 - v\gamma\left[\frac{-1}{c}(1 - \frac{v}{c})^{-1} + (1 - v)^{-1}\right]}, \quad v = t\left(1 - \frac{v}{c}\right)^\gamma (1 - v)^{-\gamma}.
$$

(iii) Taking $\gamma = 0$, and w is replace by u in (3.3) we get

$$
(3.6) \quad \sum_{n=0}^{\infty} m_n(x; \beta + \delta n, c) \frac{t^n}{n!}
$$

$$
= \frac{\left(1 - \frac{u}{c}\right)^x (1 - u)^{-x - \beta}}{1 - u\delta(1 - u)^{-1}}, \quad u = t(1 - u)^{-\delta}.
$$

(iv) Taking $\gamma = 0$, $\delta = 0$ and $w = t$ in (3.3) we get

$$
(3.7) \quad \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!} = \left(1 - \frac{t}{c}\right)^x (1 - t)^{-x - \beta}.
$$

4. Summation formulae for Meixner polynomials

The following summation formulae are easily derivable from known results in view of the relationship (2.2):

$$
(4.1) \quad m_n(x; \beta, c) = \sum_{r=0}^{n} \binom{n}{r} (\beta - \gamma)_r m_{n-r}(x; \gamma, c),
$$

$$
(4.2) \quad m_n(x_1 + x_2; \beta_1 + \beta_2, c) = \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1; \beta_1, c) \, m_r(x_2; \beta_2, c),
$$

$$
(4.3) \quad m_n(x; \beta_1 + \beta_2, c) = \sum_{r=0}^{n} \binom{n}{r} (\beta_2)_r m_{n-r}(x; \beta_1, c),
$$

(4.4) $m_n(x_1 + x_2; 2(\beta_1 + \beta_2), c)$

$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1; \beta_1 + \beta_2, c) \ m_r(x_2; \beta_1 + \beta_2, c),$$

(4.5) $m_{n+1}(x; \beta, c) = \sum_{r=0}^{n} \binom{n}{r} n! \ m_{n-r}(x; \beta, c) \left\{ (x + \beta) - \dfrac{x}{c^{r+1}} \right\},$

(4.6) $m_n(x; \beta\gamma, c) = \sum_{r=0}^{n} \binom{n}{r} (\beta(\gamma - 1))_{n-r} m_r(x; \beta, c).$

Proof of (4.1).

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(\beta - \gamma)_r m_{n-r}(x; \gamma, c)}{r! \ (n-r)!} t^n$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(\beta - \gamma)_r m_n(x; \gamma, c)}{r! \ n!} t^{n+r} = \sum_{r=0}^{\infty} (\beta - \gamma)_r \frac{t^r}{r!} \sum_{n=0}^{\infty} m_n(x; \gamma, c) \frac{t^n}{n!}$$

$$= (1 - t)^{-\beta + \gamma} (1 - t)^{-x - \gamma} \left(1 - \frac{t}{c} \right)^x = (1 - t)^{-x - \beta} \left(1 - \frac{t}{c} \right)^x$$

$$= \sum_{n=0}^{\infty} m_n(x; \beta, c) \frac{t^n}{n!}.$$

Equating the coefficient of t^n we get (4.1)

$$m_n(x; \beta, c) = \sum_{r=0}^{n} \binom{n}{r} (\beta - \gamma)_r m_{n-r}(x; \gamma, c).$$

Similarly we can prove (4.2), (4.3), (4.4), (4.5) and (4.6). ∎

5. Meixner polynomials of two variables

The Meixner polynomials of two variables is denoted by $m_n(x, y; \beta, c, d)$ and are defined in terms of Kampe de Feriet double hypergeometric functions as (see [4]) follows:

(5.1) $m_n(x, y; \beta, c, d) = (\beta)_n \ F \begin{bmatrix} -n : & x; -y; \\ & & 1 - c^{-1}, 1 - d^{-1} \\ \beta : & -; -; \end{bmatrix}$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(\beta)_n (-n)_{r+s} (-x)_r (-y)_s (1 - c^{-1})^r (1 - d^{-1})^s}{(\beta)_{r+s} r! \ s!}$$

where $\beta > 0$, $0 < c < 1$, $0 < d < 1$, $x = 0, 1, 2, \dots$ and $y = 0, 1, 2, \dots$.

The following generating functions holds true for the Meixner polynomials defined by (5.1)

(5.2) $\displaystyle\sum_{n=0}^{\infty} m_n(x, y; \beta, c, d)\frac{t^n}{n!}$

$$= (1 - t)^{-\beta-x-y} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y, \ |t| < \min(1, |c|, |d|),$$

(5.3) $\displaystyle\sum_{n=0}^{\infty} m_{n+k}(x, y; \beta, c, d)\frac{t^n}{n!}$

$$= (1 - t)^{-\beta-k-x-y} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y m_k\left(x, y; \beta, \frac{c-t}{1-t}, \frac{d-t}{1-t}\right),$$

(5.4) $\displaystyle\sum_{n=0}^{\infty} A_n m_n(x, y; \beta, c, d)\frac{t^n}{n!}$

$$= \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y (1 - t)^{-x-y-\beta}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\lambda_k}{(Nk)!} m_{Nk}\left(x, y; \beta, \frac{c-t}{1-t}, \frac{d-t}{1-t}\right) \left[\frac{wt^N}{(1-t)^N}\right]^k,$$

(5.5) $\displaystyle\sum_{k=0}^{\infty} \frac{k^n}{k!} m_k\left(x, y; \beta, \frac{c+z}{1+z}, \frac{d+z}{1+z}\right) \left(\frac{z}{1+z}\right)^k$

$$= (1 + z)^{x+y+\beta} \left(1 + \frac{z}{c}\right)^{-x} \left(1 + \frac{z}{d}\right)^{-y} \sum_{k=0}^{n} S(n, k) m_k(x, y; \beta, c, d) z^k$$

$$(n \in N_0; \ |z| < \min\{1, |c|, |d|\}).$$

6. Summation formulae for Meixner polynomials of two variables

The following summation formulae holds for (5.2):

(6.1) $\displaystyle m_n(x, y; \beta, c, d) = \sum_{r=0}^{n} \binom{n}{r} (\beta - \gamma)_r m_{n-r}(x, y; \gamma, c, d,$

(6.2) $m_n(x_1 + x_2, y_1 + y_2; \beta_1 + \beta_2, c, d)$

$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, y_1; \beta_1, c, d) \, m_r(x_2, y_2; \beta_2, c, d),$$

$$(6.3) \qquad m_n(x, y; \beta_1 + \beta_2, c, d) = \sum_{r=0}^{n} \binom{n}{r} (\beta_2)_r m_{n-r}(x, y; \beta_1, c, d),$$

(6.4) $\quad m_n(x_1 + x_2, y_1 + y_2; 2(\beta_1 + \beta_2), c, d)$

$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, y_1; \beta_1 + \beta_2, c, d) \ m_r(x_2, y_2; \beta_1 + \beta_2, c, d),$$

(6.5) $\quad m_{n+1}(x, y; \beta, c, d)$

$$= \sum_{r=0}^{n} \binom{n}{r} n! \ m_{n-r}(x, y; \beta, c, d) \left\{ (x + y + \beta) - \frac{x}{c^{r+1}} - \frac{y}{d^{r+1}} \right\},$$

$$(6.6) \qquad m_n(x, y; \beta\gamma, c, d) = \sum_{r=0}^{n} \binom{n}{r} (\beta(\gamma - 1))_{n-r} m_r(x, y; \beta, c, d).$$

Proof of (6.2).

$$\sum_{n=0}^{\infty} m_n(x_1 + x_2, y_1 + y_2; \beta_1 + \beta_2, c, d) \frac{t^n}{n!}$$

$$= \left(1 - \frac{t}{c}\right)^{(x_1+x_2)} \left(1 - \frac{t}{d}\right)^{(y_1+y_2)} (1 - t)^{-\{(x_1+x_2)+(y_1+y_2)\}-(\beta_1+\beta_2)}$$

$$= \left(1 - \frac{t}{c}\right)^{x_1} \left(1 - \frac{t}{d}\right)^{y_1} (1 - t)^{-x_1-y_1-\beta_1} \left(1 - \frac{t}{c}\right)^{x_2}$$

$$\cdot \left(1 - \frac{t}{c}\right)^{y_2} (1 - t)^{-x_2-y_2-\beta_2}$$

$$= \sum_{n=0}^{\infty} m_n(x_1, y_1; \beta_1, c, d) \frac{t^n}{n!} \sum_{r=0}^{\infty} m_r(x_2, y_2; \beta_2, c, d) \frac{t^r}{r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} m_n(x_1, y_1; \beta_1, c, d) \ m_r(x_2, y_2; \beta_2, c, d) \frac{t^{n+r}}{n! \ r!}$$

$$= \sum_{n=0}^{\infty} \sum_{r=0}^{n} m_{n-r}(x_1, y_1; \beta_1, c, d) \ m_r(x_2, y_2; \beta_2, c, d) \frac{t^n}{(n-r)! \ r!}.$$

Equating the coefficient of t^n we get (6.2).

Similarly we can prove (6.1), (6.3), (6.4), (6.5) and (6.6). ∎

7. Meixner polynomials of three variables

The Meixner polynomials of three variables $m_n(x, y, z; \beta, c, d, e,)$ are defined in terms of Kampe de Feriet triple hypergeometric functions as (see [4]) follows:

(7.1) $m_n(x, y, z; \beta, c, d, e,)$

$$= (\beta)_n F \begin{bmatrix} -n :: & -;-;- : & -x; -y; -z; \\ & & 1 - c^{-1}, 1 - d^{-1}, 1 - e^{-1} \\ \beta :: & -;-;- : & -;-;-; \end{bmatrix}$$

$$= \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} \frac{(\beta)_n (-n)_{r+s+k}(-x)_r(-y)_s(-z)_k(1-c^{-1})^r(1-d^{-1})^s(1-e^{-1})^k}{(\beta)_{r+s+k} r! s! k!},$$

where $\beta > 0$, $0 < c < 1$, $0 < d < 1$, $0 < e < 1$, $x = 0, 1, 2, \ldots$, $y = 0, 1, 2, \ldots$ and $z = 0, 1, 2, \ldots$.

The following generating functions holds for (7.1)

(7.2) $\displaystyle\sum_{n=0}^{\infty} m_n(x, y, z; \beta, c, d, e) \frac{t^n}{n!}$

$$= (1-t)^{-\beta-x-y-z} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y \left(1 - \frac{t}{e}\right)^z, \ |t| < \min(1, |c|, |d|, |e|),$$

(7.3) $\displaystyle\sum_{n=0}^{\infty} m_{n+k}(x, y, z; \beta, c, d, e) \frac{t^n}{n!}$

$$= (1-t)^{-\beta-k-x-y-z} \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y \left(1 - \frac{t}{e}\right)^z$$

$$\cdot m_k \left(x, y, z; \beta, \frac{c-t}{1-t}, \frac{d-t}{1-t}, \frac{e-t}{1-t}\right),$$

(7.4) $\displaystyle\sum_{n=0}^{\infty} A_n m_n(x, y, z; \beta, c, d, e) \frac{t^n}{n!}$

$$= \left(1 - \frac{t}{c}\right)^x \left(1 - \frac{t}{d}\right)^y \left(1 - \frac{t}{e}\right)^z (1-t)^{-x-y-z-\beta}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\lambda_k}{(Nk)!} m_{Nk} \left(x, y, z; \beta, \frac{c-t}{1-t}, \frac{d-t}{1-t}, \frac{e-t}{1-t}\right) \left[\frac{wt^N}{(1-t)^N}\right]^k,$$

(7.5) $\displaystyle\sum_{k=0}^{\infty} \frac{k^n}{k!} m_k \left(x_1, x_2, x_3; \beta, \frac{c+z}{1+z}, \frac{d+z}{1+z}, \frac{e+z}{1+z}\right) \left(\frac{z}{1+z}\right)^k$

$$= (1+z)^{x_1+x_2+x_3+\beta} \left(1 + \frac{z}{c}\right)^{-x_1} \left(1 + \frac{z}{d}\right)^{-x_2} \left(1 + \frac{z}{e}\right)^{-x_3}$$

$$\cdot \sum_{k=0}^{n} S(n, k) m_k(x_1, x_2, x_3; \beta, c, d, e) z^k \quad (n \in N_0; \ |z| < \min\{1, |c|, |d|, |e|\}).$$

8. Summation formulae for Meixner polynomials of three variables

The following summation formulae holds for the generating functions (7.2):

$$(8.1) \qquad m_n(x, y, z; \beta, c, d, e) = \sum_{r=0}^{n} \binom{n}{r} (\beta - \gamma)_r m_{n-r}(x, y, z; \gamma, c, d, e),$$

$$(8.2) \quad m_n(x_1 + x_2, y_1 + y_2, z_1 + z_2; \beta_1 + \beta_2, c, d, e)$$
$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, y_1, z_1; \beta_1, c, d, e) \, m_r(x_2, y_2, z_2; \beta_2, c, d, e),$$

$$(8.3) \quad m_n(x, y, z; \beta_1 + \beta_2, c, d, e) = \sum_{r=0}^{n} \binom{n}{r} (\beta_2)_r m_{n-r}(x, y, z; \beta_1, c, d, e),$$

$$(8.4) \quad m_n(x_1 + x_2, y_1 + y_2, z_1 + z_2; 2(\beta_1 + \beta_2), c, d, e)$$
$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, y_1, z_1; \beta_1 + \beta_2, c, d, e) \, m_r(x_2, y_2, z_2; \beta_1 + \beta_2, c, d, e),$$

$$(8.5) \quad m_{n+1}(x, y, z; \beta, c, d, e) = \sum_{r=0}^{n} \binom{n}{r} n! \, m_{n-r}(x, y, z; \beta, c, d, e)$$
$$\left\{ (x + y + z + \beta) - \frac{x}{c^{r+1}} - \frac{y}{d^{r+1}} - \frac{z}{e^{r+1}} \right\},$$

$$(8.6) \quad m_n(x, y, z; \beta\gamma, c, d, e) = \sum_{r=0}^{n} \binom{n}{r} (\beta(\gamma - 1))_{n-r} m_r(x, y, z; \beta, c, d, e).$$

9. Meixner polynomials of s-variables

The Meixner polynomials of s-variables are defined as follows (see [7]):

$$(9.1) \quad m_n(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)$$
$$= \sum_{r_1=0}^{n} \sum_{r_2=0}^{n-r_1} \cdots \sum_{r_s=0}^{n-r_1-r_2-\ldots-r_s} \frac{(\beta)_n (-n)_{r_1+r_2+\ldots+r_s} \prod_{j=1}^{s} (-x_j)_{r_j} \prod_{j=1}^{s} (1 - c_j^{-1})^{r_j}}{(\beta)_{r_1+r_2+\ldots+r_s} \prod_{j=1}^{s} (r_j)!},$$

where $\beta > 0$, $0 < c_i < 1$, $i = 1, 2, \ldots, s$, $x_j = 0, 1, 2, \ldots$, and $j = 1, 2, \ldots, s$.

The following generating functions holds true for the Meixner polynomials defined by (9.1)

(9.2) $\displaystyle\sum_{n=0}^{\infty} m_n(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)\frac{t^n}{n!}$

$$= (1-t)^{-\beta-x_1-x_1-\cdots-x_s} \prod_{j=1}^{s}\left(1-\frac{t}{c_j}\right)^{x_j}, \ |t| < min(1, |c_1|, |c_1|, \ldots, |c_s|),$$

(9.3) $\displaystyle\sum_{n=0}^{\infty} m_{n+k}(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)\frac{t^n}{n!}$

$$= (1-t)^{-\beta-k-x_1-x_1-\cdots-x_s} \prod_{j=1}^{s}\left(1-\frac{t}{c_j}\right)^{x_j}$$

$$\cdot m_k\left(x_1, x_2, \ldots, x_s; \beta, \frac{c_1-t}{1-t}, \frac{c_2-t}{1-t}, \ldots, \frac{c_s-t}{1-t}\right),$$

(9.4) $\displaystyle\sum_{n=0}^{\infty} A_n m_n(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)\frac{t^n}{n!}$

$$= (1-t)^{-\beta-x_1-x_1-\cdots-x_s} \prod_{j=1}^{s}\left(1-\frac{t}{c_j}\right)^{x_j}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\lambda_k}{(Nk)!} m_{Nk}\left(x_1, x_2, \ldots, x_s; \beta, \frac{c_1-t}{1-t}, \frac{c_2-t}{1-t}, \ldots, \frac{c_s-t}{1-t}\right)\left[\frac{wt^N}{(1-t)^N}\right]^k,$$

(9.5) $\displaystyle\sum_{k=0}^{\infty} \frac{k^n}{k!} m_k\left(x_1, x_2, \ldots, x_s; \beta, \frac{c_1+z}{1+z}, \frac{c_2+z}{1+z}, \ldots, \frac{c_s+z}{1+z}\right)\left(\frac{z}{1+z}\right)^k$

$$= (1+z)^{x_1+x_2+..+x_s+\beta} \prod_{j=1}^{s}\left(1+\frac{z}{c_j}\right)^{-x_j}$$

$$\cdot \sum_{k=0}^{n} S(n,k)m_k(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)z^k$$

$$(n \in N_0; \ |z| < \min\{1, |c_1|, |c_1|, \ldots, |c_s|\}).$$

10. Summation formulae for Meixner polynomials of s-variables

The following summation formulae holds for the generating functions (9.2):

(10.1) $m_n(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r}(\beta - \gamma)_r m_{n-r}(x_1, x_2, \ldots, x_s; \gamma, c_1, c_2, \ldots, c_s),$$

(10.2) $m_n(x_1 + x_2, x_1 + x_2, ..x_1 + x_2; \beta_1 + \beta_2, c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, x_1, ..x_1; \beta_1, c_1, c_2, \ldots, c_s) m_r(x_2, x_2, \ldots, x_2; \beta_2, c_1, c_2, \ldots, c_s),$$

(10.3) $m_n(x_1, x_2, \ldots, x_s; \beta_1 + \beta_2, c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r} (\beta_2)_r m_{n-r}(x_1, x_2, \ldots, x_s; \beta_1, c_1, c_2, \ldots, c_s),$$

(10.4) $m_n(x_1 + x_2, x_1 + x_2, \ldots, x_1 + x_2; 2(\beta_1 + \beta_2), c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r} m_{n-r}(x_1, x_1, \ldots, x_1; \beta_1 + \beta_2, c_1, c_2, \ldots, c_s) \, m_r(x_2, x_2, ..x_2; \beta_1 + \beta_2, c_1, c_2, \ldots, c_s),$$

(10.5) $m_{n+1}(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r} n! \, m_{n-r}(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s)$$

$$\cdot \left\{ (x_1 + x_2 + .. + x_s + \beta) - \frac{x_1}{c_1^{r+1}} - \frac{x_2}{c_2^{r+1}} - \cdots - \frac{x_s}{c_s^{r+1}} \right\},$$

(10.6) $m_n(x_1, x_2, \ldots, x_s; \beta\gamma, c_1, c_2, \ldots, c_s)$

$$= \sum_{r=0}^{n} \binom{n}{r} (\beta(\gamma - 1))_{n-r} m_r(x_1, x_2, \ldots, x_s; \beta, c_1, c_2, \ldots, c_s).$$

For more results on Meixner's polynomials of several variables one is also referred to [9] and [11].

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