Difference Equations for Generalized Meixner Polynomials

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DEDICATED TO RICHARD ASKEY ON HIS 60TH BIRTHDAY

In this paper is introduced a system of polynomials orthogonal with respect to the classical discrete weight function for Meixner polynomials with an extra point mass added at $x = 0$. A difference operator of infinite order is constructed for which these new polynomials are eigenfunctions and a second-order difference equation is given with polynomial coefficients, n-dependent and of at most degree 2, which these polynomials satisfy. © 1994 Academic Press, Inc.

1. INTRODUCTION

For the polynomials $\{L_n^{\alpha, N}(x)\}_{n=0}^{\infty}$ that are orthogonal on $[0, \infty)$ with respect to the weight function

$$\frac{1}{\Gamma(\alpha + 1)} x^{\alpha} e^{-x} + N \delta(x), \qquad \alpha > -1, \, N \geqslant 0$$

(see Koornwinder [7]), Koekoek and Koekoek [6] found a differential equation of the form

$$N \sum_{i=0}^{\infty} a_i(x) \, y^{(i)}(x) + xy''(x) + (\alpha + 1 - x) \, y'(x) + ny(x) = 0,$$

where the coefficients $a_i(x)$, $i \in \{1, 2, 3, ...\}$, are independent of n and $a_0(x) = a_0(n, \alpha)$ depends on n but is independent of x. For a more constructive approach to this differential equation see [2].

At a conference held in Erice (May 1990), Askey [1] posed the problem of finding difference equations of a similar form for generalizations of the discrete orthogonal polynomials that are orthogonal with respect to a

453

classical weight function at which a point mass at the point $x = 0$ is added. In [3] a solution to this problem for Charlier polynomials is given and in the present paper we deal with the more complicated case of Meixner polynomials. Moreover, we construct a second-order difference equation with polynomial coefficients, n-dependent and of degree at most 2, which the generalized Meixner polynomials satisfy.

2. MEIXNER POLYNOMIALS

Taking a normalization slightly different from the one used in [4], we define the classical Meixner polynomials $M_n(x; \beta, c)$ by the generating function

$$\sum_{n=0}^{\infty} M_n(x; \beta, c) t^n = \left(1 - \frac{t}{c}\right)^x (1 - t)^{-x - \beta}, \tag{2.1}$$

from which it easily follows that

$$M_n(x; \beta, c) = (-1)^n \sum_{k=0}^{n} \binom{x}{k}\binom{-x-\beta}{n-k} c^{-k} \tag{2.2}$$

$$= \frac{(\beta)_n}{n!} \, {}_2F_1\left(\begin{matrix} -n, & -x \\ \beta \end{matrix} \middle| 1 - \frac{1}{c}\right), \qquad n = 0, 1, 2, \dots . \tag{2.3}$$

Note that (2.2) can be used for all values of β and c except $c = 0$. Formula (2.3) is not defined for $\beta = 0, -1, -2, \dots$, and for $c = 0$. Obviously

$$M_n(0; \beta, c) = \frac{(\beta)_n}{n!}, \qquad n = 0, 1, 2, \dots . \tag{2.4}$$

Meixner polynomials are closely related to Jacobi polynomials. In fact, we have

$$M_n(x; \beta, c) = (-c)^{-n} P_n^{(x-n, \beta-1)}(1 - 2c), \qquad n = 0, 1, 2, \dots . \tag{2.5}$$

For $\beta > 0$ and $0 < c < 1$ the Meixner polynomials satisfy the orthogonality relation

$$(1 - c)^{\beta} \sum_{x=0}^{\infty} M_n(x; \beta, c) M_p(x; \beta, c) \frac{c^x (\beta)_x}{x!}$$

$$= \frac{c^{-n}}{n!} (\beta)_n \, \delta_{np}, \qquad n, p = 0, 1, 2, \dots \tag{2.6}$$

and the second-order difference equation

$$x \, \Delta\nabla y(x) + [\beta c - x(1-c)] \, \Delta y(x) + n(1-c) \, y(x) = 0, \qquad (2.7)$$

where $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$.

Direct consequences of (2.1) are the following formulae which are valid for all real x, β, c (except $c = 0$), and v, and for all $n \in \{0, 1, 2, ...\}$:

$$M_n(x+v; \beta - v, c) = \sum_{k=0}^{n} \binom{v}{k} \left(-\frac{1}{c} \right)^k M_{n-k}(x; \beta, c), \qquad (2.8)$$

$$M_n(x; \beta - v, c) = \sum_{k=0}^{n} \binom{v}{k} (-1)^k M_{n-k}(x; \beta, c), \qquad (2.9)$$

$$\Delta M_n(x; \beta, c) = \left(\frac{c-1}{c} \right) M_{n-1}(x; \beta + 1, c). \qquad (2.10)$$

In the sequel we always take $\beta > 0$, $0 < c < 1$, and since c is kept fixed during the whole paper we simplify the notation putting $M_n(x; \beta)$ instead of $M_n(x; \beta, c)$.

3. GENERALIZED MEIXNER POLYNOMIALS

Let P denote the space of all polynomials with real coefficients. We consider the inner product

$$\langle f(x), g(x) \rangle = (1-c)^\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} f(x) \, g(x)$$

$$+ Nf(0) \, g(0), \qquad N \geqslant 0, f, g \in P. \qquad (3.1)$$

We show that coefficients A_n and B_n can be chosen in such a way that the polynomials $M_n^N(x; \beta) = M_n^N(x; \beta, c)$ that are orthogonal with respect to the inner product (3.1) can be written as

$$M_n^N(x; \beta) = A_n M_n(x; \beta) + B_n M_{n-1}(x-1; \beta+1).$$

Suppose that $n \geqslant 2$ and $p(x) = xq(x)$ with degree $[q(x)] \leqslant n-2$. Then we obtain using (2.6)

$$\langle p(x), M_n^N(x; \beta) \rangle$$

$$= B_n(1-c)^\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} xq(x) M_{n-1}(x-1; \beta+1)$$

$$= B_n(1-c)^\beta c\beta \sum_{x=1}^{\infty} \frac{c^{x-1}(\beta+1)_{x-1}}{(x-1)!} q(x) M_{n-1}(x-1; \beta+1) = 0.$$

Hence for $n \geqslant 1$ the coefficients A_n and B_n have to fulfill only the following condition:

$$0 = \langle 1, M_n^N(x; \beta) \rangle$$

$$= B_n (1-c)^\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} M_{n-1}(x-1; \beta+1)$$

$$+ N A_n M_n(0; \beta) + N B_n M_{n-1}(-1; \beta+1).$$

By using (2.8) with $v = -1$ and (2.6) it follows that

$$(1-c)^\beta \sum_{x=0}^{\infty} \frac{c^x(\beta)_x}{x!} M_{n-1}(x-1; \beta+1) = c^{-n+1}.$$

So a possible choice for A_n and B_n is $A_n = N c^{n-1} M_{n-1}(-1; \beta+1) + 1$ and $B_n = -N c^{n-1} M_n(0; \beta)$, and we put

$$M_n^N(x; \beta) = [N c^{n-1} M_{n-1}(-1; \beta+1) + 1] M_n(x; \beta)$$
$$- N c^{n-1} M_n(0; \beta) M_{n-1}(x-1; \beta+1), \qquad n = 0, 1, 2, \dots.$$

$$(3.2)$$

Here and in the sequel we use $M_{-k}(x; \beta, c) = 0$, $k = 1, 2, \dots$. Note that $M_n^0(x; \beta) = M_n(x; \beta)$ and that $M_n^N(0; \beta) = M_n(0; \beta)$.

4. THE DIFFERENCE EQUATION

We are looking for a difference equation of the form

$$N \sum_{i=0}^{\infty} a_i(x) \Delta^i y(x) + x \Delta\nabla y(x) + [\beta c - x(1-c)]$$

$$\times \Delta y(x) + n(1-c) y(x) = 0 \qquad (4.1)$$

for the polynomials $\{M_n^N(x; \beta)\}_{n=0}^{\infty}$ given by (3.2), where the coefficients $\{a_i(x)\}_{i=1}^{\infty} = \{a_i(x, \beta, c)\}_{i=1}^{\infty}$ are functions of x, β, and c, but are independent of the degree n. Moreover, since we want the polynomials $\{M_n^N(x; \beta)\}_{n=0}^{\infty}$ to be eigenfunctions of a difference operator, we assume $a_0(x) = a_0(n, \beta, c)$ to be independent of x but dependent on n. So we insert

$$y(x) = M_n^N(x; \beta)$$
$$= [N c^{n-1} M_{n-1}(-1; \beta+1) + 1] M_n(x; \beta)$$
$$- N c^{n-1} M_n(0; \beta) M_{n-1}(x-1; \beta+1)$$

into (4.1). Using the difference equation (2.7) for the classical Meixner polynomials, (2.10), and (2.8) in the case $v = 1$, we find

$$N[Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\sum_{i=0}^{\infty}a_i(x)\Delta^i M_n(x;\beta)$$

$$-N^2 c^{n-1}M_n(0;\beta)\sum_{i=0}^{\infty}a_i(x)\Delta^i M_{n-1}(x-1;\beta+1)$$

$$-Nc^{n-1}M_n(0;\beta)(1-c)M_{n-1}(x-2;\beta+2)=0.$$

This relation has to hold for all values of n, β, c, and $N > 0$. Its left-hand side is a polynomial in N, so each coefficient must be zero. Thus we find

$$M_{n-1}(-1;\beta+1)\sum_{i=0}^{\infty}a_i(x)\Delta^i M_n(x;\beta)$$

$$-M_n(0;\beta)\sum_{i=0}^{\infty}a_i(x)\Delta^i M_{n-1}(x-1;\beta+1)=0 \qquad (4.2)$$

and

$$\sum_{i=0}^{\infty}a_i(x)\Delta^i M_n(x;\beta)-c^{n-1}(1-c)M_n(0;\beta)M_{n-1}(x-2;\beta+2)=0.$$

This leads to the following systems of equations:

$$\sum_{i=0}^{\infty}a_i(x)\Delta^i M_n(x;\beta)=c^{n-1}(1-c)M_n(0;\beta)M_{n-1}(x-2;\beta+2),$$
$$(4.3)$$

$$\sum_{i=0}^{\infty}a_i(x)\Delta^i M_{n-1}(x-1;\beta+1)=c^{n-1}(1-c)$$

$$\times M_{n-1}(-1;\beta+1)M_{n-1}(x-2;\beta+2).$$
$$(4.4)$$

Formula (4.2) can be rewritten as

$$\sum_{i=1}^{\infty}a_i(x)[M_{n-1}(-1;\beta+1)\Delta^i M_n(x;\beta)$$

$$-M_n(0;\beta)\Delta^i M_{n-1}(x-1;\beta+1)]$$

$$=a_0(n,\beta,c)[M_{n-1}(x-1;\beta+1)M_n(0;\beta)$$

$$-M_n(x;\beta)M_{n-1}(-1;\beta+1)].$$

The right-hand side is 0 for $x = 0$ and since this holds for all values of n, β, and c, we conclude step by step that $a_i(0) = a_i(0, \beta, c) = 0$ for all $i \in \{1, 2, ...\}$. Therefore, setting $x = 0$ in formula (4.3), we obtain

$$a_0(x) = a_0(0) = a_0(n, \beta, c) = c^{n-1}(1-c) M_{n-1}(-2; \beta+2), \qquad (4.5)$$

and Eqs. (4.3) and (4.4) can be rewritten as

$$\sum_{i=1}^{\infty} a_i(x) \Delta^i M_n(x; \beta) = c^{n-1}(1-c) [M_n(0; \beta) M_{n-1}(x-2; \beta+2)$$

$$- M_n(x; \beta) M_{n-1}(-2; \beta+2)], \qquad (4.6)$$

and

$$\sum_{i=1}^{\infty} a_i(x) \Delta^i M_{n-1}(x-1; \beta+1)$$

$$= c^{n-1}(1-c) [M_{n-1}(-1; \beta+1) M_{n-1}(x-2; \beta+2)$$

$$- M_{n-1}(x-1; \beta+1) M_{n-1}(-2; \beta+2)]. \qquad (4.7)$$

We now show that any solution of the system (4.7) also satisfies (4.6). Since by (2.8) with $v = 1$ we have

$$\sum_{i=1}^{\infty} a_i(x) \Delta^i M_n(x; \beta) = \sum_{i=1}^{\infty} a_i(x) \Delta^i M_n(x-1; \beta+1)$$

$$- \frac{1}{c} \sum_{i=1}^{\infty} a_i(x) \Delta^i M_{n-1}(x-1; \beta+1),$$

it remains to be proved that for $n \geq 1$,

$$c^{n-1}(1-c) [M_n(0; \beta) M_{n-1}(x-2; \beta+2) - M_n(x; \beta) M_{n-1}(-2; \beta+2)]$$

$$= c^n(1-c) [M_n(-1; \beta+1) M_n(x-2; \beta+2) - M_n(x-1; \beta+1)$$

$$\times M_n(-2; \beta+2)] - c^{n-2}(1-c) [M_{n-1}(-1; \beta+1)$$

$$\times M_{n-1}(x-2; \beta+2) - M_{n-1}(x-1; \beta+1) M_{n-1}(-2; \beta+2)].$$

This can be shown by combining terms and applying the formula (2.8) with $v = 1$ for different values of x and β.

5. A FORMULA FOR THE COEFFICIENTS $a_i(x)$

We now solve the system (4.7). Writing n instead of $n-1$ and using (2.10), we get

$$\sum_{i=1}^{\infty} a_i(x) \left(\frac{c-1}{c}\right)^i M_{n-i}(x-1; \beta+i+1)$$

$$= c^n(1-c) [M_n(-1; \beta+1) M_n(x-2; \beta+2)$$

$$- M_n(x-1; \beta+1) M_n(-2; \beta+2)], \quad n=1, 2, \dots. \quad (5.1)$$

If we consider $a_i(x)(1-1/c)^i$ as unknown, the matrix T of the system (5.1) is triangular with entries t_{ij} for which we have

$$t_{ij} = M_{i-j}(x-1; \beta+j+1), \quad \text{for} \quad i, j = 1, 2, \dots.$$

We show that the entries u_{ij} of the inverse matrix are

$$u_{ij} = M_{i-j}(-x+1; -\beta-i), \quad \text{for} \quad i, j = 1, 2, \dots. \quad (5.2)$$

In order to prove (5.2) we use the generating function (2.1) to find that

$$\sum_{n=0}^{\infty} M_n(-x+1; -\beta-i) t^n \sum_{n=0}^{\infty} M_n(x-1; \beta+j+1) t^n = (1-t)^{i-j-1}.$$

Equating the coefficients of t^{i-j} $(i \geqslant j)$ on both sides we obtain

$$\sum_{k=j}^{i} M_{i-k}(-x+1; -\beta-i) M_{k-j}(x-1; \beta+j+1) = \delta_{ij}.$$

We conclude that the unique solution of the system (5.1) reads

$$a_i(x) = \left(\frac{c-1}{c}\right)^{-i} \sum_{k=1}^{i} M_{i-k}(-x+1; -\beta-i)$$

$$\times c^k(1-c) [M_k(-1; \beta+1) M_k(x-2; \beta+2)$$

$$- M_k(x-1; \beta+1) M_k(-2; \beta+2)], \quad i=1, 2, \dots. \quad (5.3)$$

The difference equation (4.1), with $a_0(x)$ given by (4.5) and $a_i(x)$ by (5.3) for $i = 1, 2, \dots$, is of infinite order for all values of β and c $(0 < c < 1)$ and $N > 0$. We prove this by evaluating the coefficient $k_i = k_i(\beta, c)$ of x^i in $a_i(x)$, $i \geqslant 1$. From (2.3) we derive that

$$M_n(x; \beta, c) = \frac{1}{n!} \left(\frac{c-1}{c}\right)^n x^n + \text{terms with lower powers of } x. \quad (5.4)$$

Furthermore, in the case $v = n$, formula (2.8) can be rewritten as

$$(-c)^n M_n(x+n; \beta-n, c) = \sum_{k=0}^{n} \binom{n}{k} (-c)^k M_k(x; \beta, c). \quad (5.5)$$

Hence we can write, using (5.3), (5.4), (5.5), and (2.8) with $v = 1$,

$$
\begin{aligned}
k_i &= \left(\frac{c}{c-1}\right)^i \sum_{k=1}^{i} \frac{1}{(i-k)!} \left(-\frac{c-1}{c}\right)^{i-k} c^k (1-c) \\
&\quad \times \frac{1}{k!} \left(\frac{c-1}{c}\right)^k [M_k(-1; \beta+1) - M_k(-2; \beta+2)] \\
&= \frac{(-1)^i}{i!} (1-c) \sum_{k=0}^{i} \binom{i}{k} (-c)^k [M_k(-1; \beta+1) - M_k(-2; \beta+2)] \\
&= \frac{c^i(1-c)}{i!} [M_i(i-1; \beta-i+1) - M_i(i-2; \beta-i+2)] \\
&= \frac{c^{i-1}(c-1)}{i!} M_{i-1}(i-2; \beta-i+2).
\end{aligned} \tag{5.6}
$$

Hence using (2.5) we obtain

$$
\begin{aligned}
k_i &= (-1)^i \frac{1-c}{i!} P_{i-1}^{(-1, \beta - i+1)}(1-2c) \\
&= \frac{c-1}{i!} P_{i-1}^{(\beta - i+1, -1)}(2c-1).
\end{aligned}
$$

In particular, $k_1 = c - 1$. Expressing the Jacobi polynomials in terms of $_2F_1$'s and using Euler's transformation,

$$
{}_2F_1\left(\begin{matrix} a, & b \\ & c \end{matrix} \Big| z\right) = (1-z)^{c-a-b} \, {}_2F_1\left(\begin{matrix} c-a, & c-b \\ & c \end{matrix} \Big| z\right),
$$

we obtain the relation

$$
2n P_n^{(\beta, -1)}(z) = (n+\beta)(z+1) P_{n-1}^{(\beta, 1)}(z).
$$

Hence

$$
k_i(\beta, c) = \frac{\beta c(c-1)}{i!\,(i-1)} P_{i-2}^{(\beta - i+1, 1)}(2c-1), \qquad \text{for } i \geq 2. \tag{5.7}
$$

In order to show that the difference equation (4.1) is of infinite order we prove that no pair of consecutive coefficients (k_i, k_{i+1}) can vanish simultaneously. This even holds for arbitrary complex values of β and c, provided that $\beta c(c-1) \neq 0$.

We start with two known relations for Jacobi polynomials (see [5, p. 173, formula (33), with n replaced by $n-1$, and formula (35)]):

$$nP_n^{(\alpha,\beta)}(z)$$
$$= -(\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z) + \tfrac{1}{2}(\alpha+\beta+2n)(1+z)\,P_{n-1}^{(\alpha,\beta+1)}(z),$$

$$(\alpha+\beta+2n)\,P_n^{(\alpha-1,\beta)}(z)$$
$$= (\alpha+\beta+n)\,P_n^{(\alpha,\beta)}(z) - (\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z).$$

We eliminate $P_n^{(\alpha,\beta)}(z)$ from these two relations to find

$$(\beta+n)\,P_{n-1}^{(\alpha,\beta)}(z) + nP_n^{(\alpha-1,\beta)}(z) = \frac{1}{2}(\alpha+\beta+n)(1+z)\,P_{n-1}^{(\alpha,\beta+1)}(z)$$

$$= (1+z)\frac{d}{dz}\,P_n^{(\alpha-1,\beta)}(z),$$

and we take the special case $\beta := 1$, $n := i-1$, $\alpha := \beta - i + 1$:

$$iP_{i-2}^{(\beta-i+1,1)}(z) + (i-1)\,P_{i-1}^{(\beta-i,1)}(z) = (1+z)\frac{d}{dz}\,P_{i-1}^{(\beta-i,1)}(z). \qquad (5.8)$$

Now suppose that $k_i(\beta,c) = k_{i+1}(\beta,c) = 0$ for some $i \geqslant 2$. Putting for convenience $2c - 1 = z$, we then get from (5.7)

$$P_{i-2}^{(\beta-i+1,1)}(z) = P_{i-1}^{(\beta-i,1)}(z) = 0 \qquad \text{for some} \quad i \geqslant 2.$$

Hence from (5.8) we conclude that both $P_{i-1}^{(\beta-i,1)}(z)$ and its derivative vanish. Since the Jacobi polynomial satisfies a second-order differential equation, this is impossible if z is a regular point of the differential equation, for then *all* its derivatives would vanish. The only singular points are $z = -1$ and $z = 1$ corresponding to $c = 0$ and $c = 1$.

It is well known that the Charlier polynomials $C_n^{(a)}(x)$ can be regarded as limits of Meixner polynomials:

$$C_n^{(a)}(x) = \lim_{\beta\to\infty} \left(-\frac{a}{\beta}\right)^n M_n\!\left(x;\beta,\frac{a}{\beta}\right).$$

By using this relation in (4.5), (5.3), and (5.6), we retrieve the results of [3] for generalized Charlier polynomials.

6. A Second-Order Difference Equation

In this section we show that the polynomials $M_n^N(x;\beta,c)$ satisfy a second-order difference equation with polynomial coefficients, n-dependent,

and of at most second degree. We construct this difference equation in a
way similar to the method derived in [7, Prop. 6.1], for obtaining differen-
tial equations for systems of orthogonal polynomials.

First, using (2.10) twice, we write (3.2) in the form

$$M_n^N(x;\beta) = [Nc^{n-1}M_{n-1}(-1;\beta+1)+1]\,M_n(x-1;\beta)$$

$$+\left[1-N\frac{c^n}{c-1}M_n(-1;\beta)\right]\Delta M_n(x-1;\beta), \qquad n=0,1,2,...,$$

$$(6.1)$$

and from (2.7) it is not difficult to derive that

$$c(x+\beta)\,\Delta^2 M_n(x-1;\beta) + [(c-1)(x-n)+c\beta]\,\Delta M_n(x-1;\beta)$$

$$+ (1-c)\,nM_n(x-1;\beta) = 0, \qquad n=0,1,2,...\,. \qquad (6.2)$$

If we put

$$u := M_n^N(x;\beta), \qquad\qquad p := 1+Nc^{n-1}M_{n-1}(-1;\beta+1),$$

$$q := 1-N\frac{c^n}{c-1}M_n(-1;\beta), \qquad y := M_n(x-1;\beta),$$

$$(6.3)$$

(6.1) and (6.2) can be rewritten as

$$u = py + q\,\Delta y, \qquad (6.4)$$

$$(cx+c\beta)\,\Delta^2 y + [(c-1)x+c(\beta-n)+n]\,\Delta y + (1-c)\,ny = 0. \qquad (6.5)$$

We assume $q \neq 0$. We eliminate Δy and $\Delta^2 y$ from (6.4), (6.5), and the equa-
tion obtained by taking the difference of (6.4). This leads to the relation

$$q(cx+c\beta)\,\Delta u + (ax+b)u + (dx+f)\,y = 0, \qquad (6.6)$$

with

$$a := q(c-1)-pc, \qquad b := \beta c(q-p)+qn(1-c), \qquad d := p^2 c - pq(c-1),$$

$$f := p^2 c\beta - pq(c\beta-cn+n)+q^2(1-c)n. \qquad (6.7)$$

Next we eliminate y and Δy from (6.4), (6.6), and the first difference of
(6.6). We finally obtain

$$q^2 c(dx+f)(x+\beta+1)\,\Delta^2 u$$

$$+ [q(ax+a+b+qc)(dx+f)+qc(x+\beta)(dpx+dp+fp-dq)]\,\Delta u$$

$$+ [(dx+aq+d+f)(dx+f)+(ax+b)(dpx+dp+fp-dq)]\,u = 0,$$

where u, p, and q are given in (6.3) and a, b, d, and f in (6.7).

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