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The Symmetric Meixner-Pollaczek Polynomials

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Abstract

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The Symmetric Meixner-Pollaczek polynomials are considered. We denote these polynomials in this thesis by $p_n^{(\lambda)}(x)$ instead of the standard notation $p_n^{(\lambda)}(x/2, \pi/2)$, where $\lambda > 0$. The limiting case of these sequences of polynomials $p_n^{(0)}(x) = \lim_{\lambda \to 0} p_n^{(\lambda)}(x)$, is obtained, and is shown to be an orthogonal sequence in the strip, $S = \{z \in \mathbb{C} : -1 \leq \Im(z) \leq 1\}$.

From the point of view of Umbral Calculus, this sequence has a special property that makes it unique in the Symmetric Meixner-Pollaczek class of polynomials: it is of convolution type. A convolution type sequence of polynomials has a unique associated operator called a delta operator. Such an operator is found for $p_n^{(0)}(x)$, and its integral representation is developed. A convolution type sequence of polynomials may have associated Sheffer sequences of polynomials. The set of associated Sheffer sequences of the sequence $p_n^{(0)}(x)$ is obtained, and is found to be $\mathbb{P} = \{\{p_n^{(\lambda)}(x)\}_{n=0}^{\infty} : \lambda \in \mathbb{R}\}$. The major properties of these sequences of polynomials are studied.

The polynomials $\{p_n^{(\lambda)}(x)\}_{n=0}^{\infty}$, $\lambda < 0$, are not orthogonal polynomials on the real line with respect to any positive real measure for failing to satisfy Favard's three term recurrence relation condition. For every $\lambda \leq 0$, an associated non-standard inner product is defined with respect to which $p_n^{(\lambda)}(x)$ is orthogonal.

Finally, the connection and linearization problems for the Symmetric Meixner-Pollaczek polynomials are solved. In solving the connection problem the convolution property of the polynomials is exploited, which in turn helps to solve the general linearization problem.

Key words and phrases. Meixner-Pollaczek polynomial, Orthogonal polynomial, Polynomial operator, Inner product, Umbral Calculus, Sheffer polynomial, Convolution type polynomial, Connection and Linearization problem.

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Dedicated to my brother Tedros
and my friend Mekonnen

This thesis consists of a summary and the following four papers:

I. The Meixner-Pollaczek Polynomials and a System of Orthogonal Polynomials in a Strip (submitted).

II. Umbral Calculus and the Symmetric Meixner-Pollaczek Polynomials.

III. The Symmetric Meixner-Pollaczek Polynomials with real parameter.

IV. Linearization and Connection problems for the Symmetric Meixner-Pollaczek Polynomials.

Contents

1. Introduction

This thesis is mainly concerned about the Meixner-Pollaczek polynomials. These are the polynomials first discovered by Meixner [23] and are known in the literature as the Meixner polynomials of the second kind (see Chihara [9]). These polynomials were later studied by Pollaczek [25]. The polynomials are denoted by $p_n^{(\lambda)}(x,\phi)$, and have a hypergeometric representation:

$$p_n^{(\lambda)}(x,\phi) = \frac{(2\lambda)_n}{n!} e^{in\phi}\, {}_2F_1\left(\begin{matrix} -n,\ \lambda+ix \\ 2\lambda \end{matrix}\,\middle|\, 1-e^{-i2\phi}\right), \quad \lambda > 0,\ 0 < \phi < \pi,$$

where

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, x\right) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!}, \quad \text{and}$$

$$(a)_k := a(a+1)\dots(a+k-1).$$

The polynomials are completely described by the recurrence formula:

$$p_{-1}^{(\lambda)}(x,\phi) = 0, \quad p_0^{(\lambda)}(x,\phi) = 1,$$

$$(n+1)p_{n+1}^{(\lambda)}(x,\phi) - 2[x\sin\phi + (n+\lambda)\cos\phi]p_n^{(\lambda)}(x,\phi) + (n+2\lambda-1)p_{n-1}^{(\lambda)}(x,\phi) = 0$$

for $n \geq 1$, and have a generating function

$$G_\lambda(x,t) = (1-e^{i\phi}t)^{-\lambda+ix}(1-e^{-i\phi}t)^{-\lambda-ix} = \sum_{n=0}^{\infty} p_n^{(\lambda)}(x,\phi)t^n.$$

Erdélyi [13] and Szegö [29] briefly mentioned these polynomials. Their major properties are discussed by Chihara [9] and Koekoek and Swarttouw [18]. Asymptotic properties of these polynomials and their zeros are studied by Li and Wong [21]. The applications of these polynomials are also studied by many of them. For example: The connection between the Heisenberg algebra and Meixner-Pollaczek polynomials are studied by Bender, Mead and Pinsky [8] and Koornwinder [20]. The combinatorial interpretation of the linearization coefficients of these polynomials is discussed by Zeng [30]. The interpretation of the Meixner-Pollaczek polynomials as overlap coefficients in the positive discrete series representation of the Lie algebra $\boldsymbol{SU}(1,1)$ are discussed by Koelink and Van der Jeugt [19].

An area of interest in connection with orthogonal polynomials is limit relations. The report by Koekoek and Swarttouw [18] is a good source of information in this direction, Askey and Wilson [5] is another. Both papers illustrate the Askey-Scheme of hypergeometric orthogonal polynomials from the highest levels Wilson and Racah with degree of freedom 4 to the lowest level Hermite with degree of freedom 0. They consider (from and to) limit relations for the intermediate levels. The Meixner-Pollaczek polynomials have their place in this scheme in the third row with two free parameters. Those below Meixner-Pollczek are Laguerre and Hermite polynomials, with defining formulas [18]:
The Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}\,\middle|\, x\right). \tag{1.1}$$

The Hermite polynomials

$$H_n(x) = (2x)^n\, {}_2F_0\left(\begin{matrix} -n/2, -(n-1)/2 \\ - \end{matrix}\,\middle|\, -1/x^2\right). \tag{1.2}$$

In particular, one is interested to know the limiting cases of $p_n^{(\lambda)}(x,\phi)$, say as $\phi \to 0$, $\phi \to \pi$, $\lambda \to \infty$ or $\lambda \to 0$. Using an appropriate scaling of the variable, we have

Meixner-Pollaczek $\to$ Laguerre:
Making the substitution $\lambda = (\alpha+1)/2$, $x \mapsto -x/(2\phi)$, and letting $\phi \to 0$,

$$\lim_{\phi\to 0} p_n^{(\frac{\alpha+1}{2})}(-\frac{x}{2\phi},\phi) = L_n^{(\alpha)}(x).$$

Meixner-Pollaczek $\to$ Hermite:
Making the substitution $x \mapsto \frac{x\sqrt{\lambda}-\lambda\cos\phi}{\sin\phi}$ and letting $\lambda \to \infty$,

$$\lim_{\lambda\to\infty} \lambda^{-n/2} p_n^{(\lambda)}(\frac{x\sqrt{\lambda}-\lambda\cos\phi}{\sin\phi},\phi) = \frac{H_n(x)}{n!}.$$

The case $\phi \to \pi$, produces the trivial polynomial system. This leaves us with the last limit situation, i.e., $\lambda \to 0$.

In Section 2 we tackle this problem for a fixed value of ϕ. In fact from now on we fix the value of the parameter ϕ to be $\pi/2$, and further we make the scaling of the variable so that we have $p_n^{(\lambda)}(x/2,\pi/2)$. The resulting polynomials are called the the Symmetric Meixner-Pollaczek polynomials, and this thesis is mainly concerned with the set of these kinds of polynomials and their extensions. In the sequel we denote these polynomials by $p_n^{(\lambda)}(x)$. Section 2 considers the limiting case of these polynomials, i.e., $p_n^{(0)}(x) := \lim_{\lambda\to 0} p_n^{(\lambda)}(x)$, and shows that these polynomials are orthogonal polynomials in a strip, which is one of the main results of Paper 2. Besides, the polynomials $p_n^{(0)}(x)$ are found to be important polynomials.

Section 3 starts with the Symmetric Meixner-Pollaczek polynomials, $\mathbb{P}^+ = \{\{p_n^{(\lambda)}(x)\}_{n=0}^\infty : \lambda > 0\}$ plus the new system mentioned in the preceding paragraph, $\{p_n^{(0)})(x)\}_{n=0}^\infty$. It extends this class to include sequences of polynomials $\{\{p_n^{(\lambda)}(x)\}_{n=0}^\infty : \lambda < 0\}$, so that the extended class becomes $\mathbb{P} = \{\{p_n^{(\lambda)}(x)\}_{n=0}^\infty : \lambda \in \mathbb{R}\}$. It employs Umbral Calculus [27, 11, 10] to identify the special properties of the polynomials $\{p_n^{(0)}(x)\}_{n=0}^\infty$, and to study the connection between $\{p_n^{(0)}(x)\}_{n=0}^\infty$ and the other members of $\mathbb{P}$. Furthermore, it examines the major properties of the Symmetric Meixner-Pollaczek polynomials which are shared by $\{\{p_n^{(\lambda)}(x)\}_{n=0}^\infty : \lambda < 0\}$.

Unfortunately, the polynomials $\{p_n^{(\lambda)}(x)\}_{n=0}^\infty$, $\lambda < 0$, mentioned in the preceding paragraph are not orthogonal polynomials on the real line with respect to any positive real measure for failing to satisfy Favard's [14] positivity condition. However, for each $\lambda \le 0$, defining an inner product with respect to which $p_n^{(\lambda)}(x)$ is an orthogonal system is of interest though its real application is not known.

Motivated by the Sobolev type orthogonal polynomials [24, 22] corresponding to the Sobolev type inner product (4.7), in Section 4 we consider $\mathbb{P} = \{\{p_n^{(\lambda)}(x)\}_{n=0}^\infty : \lambda \in \mathbb{R}\}$. For every $\lambda \in \mathbb{R}$, we define in an analogous way a corresponding inner product with respect to which the system $\{p_n^{(\lambda)}(x)\}_{n=0}^\infty$ becomes orthogonal. For $\lambda > 0$ these inner products coincide with the standard inner products for the Meixner-Pollaczek polynomials.

Another area of interest in connection with orthogonal polynomials is the Fourier expansion of functions with respect to an orthogonal polynomial system, i.e.,

$$f(x) = \sum_{k=0}^{\infty} C_k p_k(x), \quad \text{where } f \text{ satisfies certain conditions.}$$

Particular cases of this expansion are when f is a polynomial in a different class, or is a product of two or more polynomials. These are what are called connection and linearization problems [3, 4, 7, 15, 16], respectively. In Section 5 we solve these problems for the Symmetric Meixner-Pollaczek polynomials.

2. A limiting case of the Symmetric Meixner-Pollaczek polynomials

Let $w(x) = 1/(2\cosh(\pi x/2))$. Then the function $w(x)$ is the density function of a probability measure. Furthermore, it has interesting properties that make it useful as a weight function for orthogonal polynomials. The most useful property of the weight function $w(x)$ is that it can be interpreted as a Poisson kernel [28], namely we have the following;

Proposition 1. *Let the function f be continuous and harmonic in the strip $S = \{\ z : -1 \leq Im(z) \leq 1\}$, and suppose further that $|f(z)| < Ce^{a|z|}$, for some a, $0 \leq a < \pi/2$. Then*

$$f(0) \quad = \quad \int_{-\infty}^{\infty} \frac{f(x+i)+f(x-i)}{2} \frac{dx}{2\cosh\frac{\pi}{2}x}. \tag{2.1}$$

Since the weight w is so closely related to the strip S, we describe an orthogonal basis for the space $H^2(S, \mathcal{P})$ where $\mathcal{P}$ is the Poisson measure for 0. This is summarized in the following theorem (Paper 1 and [17]):

Theorem 1. *Let the system $\{\sigma_k\}_{k=0}^{\infty}$ be given by the following recursion relation:*

$$\sigma_{-1} \;=\; 0,\ \sigma_0 \;=\; 1 \ \text{ and } \ \sigma_{k+1}(z) \;=\; z\sigma_k(z) - k(k-1)\sigma_{k-1}(z). \tag{2.2}$$

Then:

1. *the function $\sigma_k(z)$ is a monic polynomial of degree k.*
2. *the sequence of polynomials $\{\ (k!)^{-1}\sigma_k(z)\ \}_0^{\infty}$ is an orthogonal basis in the Hilbert space $H^2(S, \mathcal{P})$.*
3. *the norm of $(k!)^{-1}\sigma_k$ is 1 for $k = 0$ and $\sqrt{2}$ for $k \geq 1$.*
4. *the polynomials $\sigma_k(z)$ have an exponential generating function*

$$\sum_{k=0}^{\infty} \frac{\sigma_k(z)}{k!} s^k = e^{z \arctan s}.$$

Another important result of Paper 1 is that $\widetilde{\sigma}_k := (k!)^{-1}\sigma_k$ is the limiting case of the *Symmetric Meixner-Pollaczek* polynomial systems, $p_k^{(\lambda)}(x)$, as the parameter $\lambda \to 0$, and it has a hypergeometric representation. This is the content of the next proposition.

Proposition 2.

$$\widetilde{\sigma}_k(x) \quad = \quad \lim_{\lambda \to 0^+} p_k^{(\lambda)}(x) \; = p_k^{(0)}(x),$$

$$p_0^{(0)}(x) = 1, \ \text{ and } \ p_k^{(0)}(x) = \quad i^{k-1} x\, {}_2F_1\left(\left. \begin{matrix} 1-k,\ 1+\frac{ix}{2} \\ 2 \end{matrix} \right| 2 \right), \quad k \geq 1.$$

3. Extending the parameter λ to the whole real line

The polynomial sequence $\{p_n^{(0)}(x)\}_{n=0}^{\infty}$ introduced in Section 2 has a special property in the Symmetric Meixner-Pollaczek polynomial class. This is seen from the generating function of these polynomials, and leads to:

$$p_n^{(0)}(x+y) = \sum_{k=0}^{n} p_k^{(0)}(y) p_{n-k}^{(0)}(x). \tag{3.1}$$

This is the only polynomial sequence in the Meixner-Pollaczek class with this property. A polynomial sequence with such a property is called a convolution type polynomial. Convolution type polynomials have a unique associated polynomial operator which is called a delta operator. We denote the delta operator for $p_n^{(0)}(x)$ by Q. This is the operator which maps $p_1^{(0)}(x)$ to 1 and $Qp_n^{(0)}(x) = p_{n-1}^{(0)}(x)$. An integral representation of this operator is found, which is one of the results in Paper 2. It is described by

$$Qf = -\frac{1}{\sinh \frac{\pi x}{2}} * f(x) = \int_{-\infty}^{\infty} \frac{f(x+y)}{2\sinh \frac{\pi y}{2}} dy.$$

In Umbral language, a convolution type sequence of polynomials may have associated sequences of Sheffer polynomials. In the case of $\{p_n^{(0)}(x)\}_{n=0}^{\infty}$, these are the sequences of polynomials $\{q_n(x)\}_{n=0}^{\infty}$ satisfying:

$$q_n(x+y) = \sum_{k=0}^{n} p_k^{(0)}(x) q_{n-k}(y).$$

However, every sequence of polynomials whose generating function is of the form $e^{x \arctan t}/(1+t^2)^{\lambda}$, where $\lambda \in \mathbb{R}$ satisfies the above mentioned property. These are the polynomials completely described by the recurrence relation

$$\begin{gathered} p_{-1}^{(\lambda)}(x) = 0, \;\; p_0^{(\lambda)}(x) = 1 \quad \text{and} \\ (n+1)p_{n+1}^{(\lambda)}(x) - xp_n^{(\lambda)}(x) + (n-1+2\lambda)p_{n-1}^{(\lambda)}(x) = 0, \;\; \text{n=1, 2, } \ldots . \end{gathered} \tag{3.2}$$

Now, if we take the whole class $\mathbb{P} = \{\{p_n^{(\lambda)}(x)\}_{n=0}^{\infty} : \lambda \in \mathbb{R}\}$, then the convolution property of $\{p_n^{(0)}(x)\}_{n\in\mathbb{N}}$ justifies that for each $\lambda \in \mathbb{R}$, there is an associated linear shift-invariant polynomial operator denoted by P^{λ} and defined by $P^{\lambda} : \{p_n^{(0)}(x)\}_{n\in\mathbb{N}} \mapsto \{p_n^{(\lambda)}(x)\}_{n\in\mathbb{N}}$ such that $P^{\lambda}p_n^{(0)} := p_n^{(\lambda)}$. The set of all these operators make up an algebra of shift-invariant polynomial operators. The main results in Paper 2 include:

Proposition 3. *$p_n^{(0)}(x)$ is the basic sequence with respect to the delta operator Q and conversely.*

An immediate consequence of which is:

Proposition 4. *For each $\lambda \in \mathbb{R}$ the following statements are equivalent:*
1) $p_n^{(\lambda)}(x)$ is a Sheffer sequence with respect to $p_n^{(0)}(x)$.
2) $p_n^{(\lambda)}(x)$ is a Sheffer sequence with respect to Q.

Another important result is:

Proposition 5. *Each of the operators P^λ, $\lambda \in \mathbb{R}$ and Q has a power series representation in powers of the differential operator D, moreover each operator has a closed form representation given by*

$$P^\lambda = \cos^{2\lambda}(D), \qquad Q = \tan D.$$

4. Inner product for the extended symmetric Meixner-Pollaczek class

Proposition 1 makes it natural to consider the following two operators:

$$Rf(x) := \frac{1}{2}(f(x+i) + f(x-i)) \tag{4.1}$$

$$Jf(x) := \frac{1}{2i}(f(x+i) - f(x-i)) \tag{4.2}$$

These operators happen to connect the polynomials in $\mathbb{P}$, as stated in the following proposition.

Proposition 6. *Given any $\lambda \geq 0$, the following relations hold true:*

$$Rp_n^{(\lambda)}(x) = p_n^{(\lambda+1/2)}(x), \tag{4.3}$$

$$Jp_n^{(\lambda)}(x) = p_{n-1}^{(\lambda+1/2)}(x). \tag{4.4}$$

We also consider the operator R on the product of two functions, say f and g as follows:

$$R(fg) := \frac{f(x+i)g(\overline{x+i}) + f(x-i)g(\overline{x-i})}{2},$$

which may also be written as:

$$R(fg) = f(x-i)Rg(x) + iJf(x)g(x-i).$$

Furthermore, powers of R are considered where $R^{r+1}f := R[R^r f]$, for which a simple induction gives

$$R^r f = \frac{1}{2^r}\sum_{k=0}^{r}\binom{r}{k} f(x + i(r-2k)). \tag{4.5}$$

Applying this to the polynomials in $\mathbb{P}$, we have

Proposition 7. *Suppose that $p_n^{(\lambda)}, p_m^{(\lambda)}$ are the symmetric polynomials and r is a positive integer, then:*

$$R^r(p_n^{(\lambda)}p_m^{(\lambda)}) = \sum_{k=0}^{r}\binom{r}{k} p_{n-k}^{(\lambda+\frac{r}{2})}p_{m-k}^{(\lambda+\frac{r}{2})}. \tag{4.6}$$

For each real number $\lambda \leq 0$ we define, $\mathbb{N}_\lambda := \{n| \ \ n \in \mathbb{N} \text{ and } \ \lambda + n/2 > 0\}$, then $\mathbb{N}_\lambda$ has a least element. We denote the associated least element by m_λ, where $m_\lambda = \min_{n\in\mathbb{N}}\{n : \lambda + n/2 > 0\}$. In what follows we will be interested in the results of Proposition 7 where r is replaced by m_λ.

Inner-products other than the standard one are often used, particularly when a non-standard inner-product is more natural. Orthogonal polynomials with respect to such inner products can also be considered. For example, Sobolev type orthogonal polynomials appear in the works of Milovanović [24], Marcellán and

Álvarez-Nodarse [22] and the references therein. In general, the Sobolev type inner product is defined by:

$$\langle f, g\rangle = \sum_{k=0}^{m} \int_{\mathbb{R}} f^{(k)}(t) g^{(k)}(t) d\mu_k(t), \tag{4.7}$$

where $d\mu_k(t),\ k = 0, 1, \dots, m$ are given positive measures on $\mathbb{R}$.

Now, let $\lambda \leq 0$ be given and let m_λ be the associated least positive integer, then we define the associated inner product as follows:

$$\begin{aligned}
\langle f, g\rangle_\lambda = \int_\lambda fg d(\mathcal{P}_\lambda(x)) &:= \int_{-\infty}^{\infty} R^{m_\lambda}(fg)\omega_{\lambda+\frac{m_\lambda}{2}}(x)dx, \\
&= \frac{1}{2^{m_\lambda}} \sum_{k=0}^{m_\lambda} \binom{m_\lambda}{k} \times \\
&\quad \int_{-\infty}^{\infty} f(x + i(m_\lambda - 2k)) g(x - i(m_\lambda - 2k)) \omega_{\lambda+\frac{m_\lambda}{2}}(x)dx,
\end{aligned}$$

$$\text{where} \quad \omega_{\lambda+\frac{m_\lambda}{2}}(x) := \frac{|\Gamma(\lambda + \frac{m_\lambda}{2} + \frac{ix}{2}|^2}{2\pi}, \tag{4.8}$$

is the weight function associated with the polynomials $p_n^{(\lambda+\frac{m_\lambda}{2})}(x)$, in the Symmetric Meixner-Pollaczek class, and $R^{m_\lambda}f$ is as defined in formula (4.5).

The preceding inner product in (4.8) is analogous to the Sobolev type inner product in (4.7) where the differential operator is replaced by the operator R, and the positive measures $d\mu_k(t)$ for $k = 0, 1, \dots, m_\lambda$ are replaced by $\omega_{\lambda+\frac{m_\lambda}{2}}(t)dt$. One of the major results of Paper 3 is summarized in the following theorem:

Theorem 2. *For each $\lambda \leq 0$, the corresponding polynomial system $p_n^{(\lambda)}(x)$, is an orthogonal polynomial system with respect to the inner product (4.8).*

Proposition 8. *For each $\lambda \in \mathbb{R}$, the corresponding orthogonal polynomial system with respect to the associated inner product satisfies the following relation:*

$$\langle p_n^{(\lambda)}(x), p_n^{(\lambda)}(x)\rangle_\lambda = \begin{cases} \sum_{k=0}^{n} \binom{m_\lambda}{k} \frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & \text{if } n < m_\lambda, \\ \\ \sum_{k=0}^{m_\lambda} \binom{m_\lambda}{k} \frac{2^{1-2\mu}\Gamma(n-k+2\mu)}{(n-k)!}, & \text{if } n \geq m_\lambda, \end{cases}$$

where $\mu = \lambda + \frac{m_\lambda}{2}$.

5. Linearization and connection problems for the Symmetric Meixner-Pollaczek polynomials

Paper 4 is concerned about the linearization and connection problems for the Symmetric Meixner-Pollaczek polynomials. The main results in this paper include:

Proposition 9. *Let $\lambda > 0$ be given, then for any $p, q \in \mathbb{N}$*

$$p_p^{(\lambda)}(x) p_q^{(\lambda)}(x) = \sum_{\nu=|p-q|}^{p+q} C_{pq\nu}\, p_\nu^{(\lambda)}(x), \quad \text{where} \tag{5.1}$$

$$C_{pq\nu} = \frac{\nu!}{\Gamma(2\lambda + \nu)} \frac{\Gamma(2\lambda + \frac{p+q+\nu}{2})}{(\frac{\nu+p-q}{2})!(\frac{p+q-\nu}{2})!(\frac{\nu+q-p}{2})!}. \tag{5.2}$$

The sequence of linearization coefficients $C_{pq\nu}$ in (5.2) satisfy the recurrence relation:

$$C_{p,q,\nu+2} = \frac{(\nu+1)_2}{(2\lambda+\nu)_2} \frac{(p+q-\nu)(4\lambda+p+q+\nu)}{(2+p+\nu-q)(2+q+\nu-p)}\, C_{p,q,\nu}. \tag{5.3}$$

In Paper 4 a variant solution is obtained for the linearization problem using the Rodrigues' formula of the polynomials, which is summarized in the following proposition.

Proposition 10. *Let $\lambda > 0$ be given, then for any $n, m \in \mathbb{N}$*

$$p_n^{(\lambda)}(x) p_m^{(\lambda)}(x) = \sum_{r=0}^{n+m} C_{nmr} p_r^{(\lambda)}(x), \ \text{where} \tag{5.4}$$

$$\begin{aligned} C_{nmr} = \ & \frac{2^{r+2\lambda-1}}{\Gamma(2\lambda+r)} \sum_{k=0}^{r} \sum_{l=0}^{r-k} \sum_{q=0}^{k} \binom{r}{k} \binom{r-k}{l} \binom{k}{q} \times \\ & i^l (-i)^q \int_{-\infty}^{\infty} p_{n-k-l}^{(\lambda+\frac{r}{2})}(x) p_{m+k-r-q}^{(\lambda+\frac{r}{2})}(x) w_{\lambda+\frac{r}{2}}(x) dx. \end{aligned} \tag{5.5}$$

The connection problem has also been considered. Its solution has brought about the convolution property of the polynomials into the play. This in turn leads us to remark that it is easy to solve the general linearization problem for these polynomials using this property, i.e., if $\lambda, \ \mu \in \mathbb{R}$, and $\nu > 0$, then

$$p_n^{(\lambda)} p_n^{(\mu)} = \sum_{k=|n-m|}^{n+m} C_{nmk} p_k^{(\nu)},$$

and the coefficients can be solved using the convolution property and (5.1) (or (5.4)).

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