JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **194**, 250–258 (1995)

Difference Equation for Modifications of Meixner Polynomials

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Submitted by George Gasper

Received February 8, 1994

We consider a modification of Meixner moment functional by adding a mass point at $x = 0$. We obtain the resulting orthogonal polynomials, identify them as hypergeometric ${}_3F_2$ functions, and derive the second order difference equation which these polynomials satisfy. In such a way we give the solution to a problem raised by R. Askey (1991) *in* "Orthogonal Polynomials and Their Applications," p. 418, Baltzer AG Scientific, (Basel).

1. INTRODUCTION

The study of orthogonal polynomials with respect to a modification of a linear functional in the linear space of polynomials with real coefficients via the addition of one or two delta Dirac measures has been performed by several authors. In particular, Chihara [3] considered some properties of such polynomials in terms of the location of the mass point with respect to the support of a positive measure. More recently Marcellán and Maroni [6] analyzed a more general situation for regular linear functionals, i.e., such that the principal submatrices of the corresponding infinite Hankel matrices for the moment sequences are nonsingular.

Special emphasis is given to modifications of classical linear functionals (Hermite, Laguerre, Jacobi, and Bessel). In [6], representation formulas for the new orthogonal polynomial sequences as well as the second order differential equation that such polynomials satisfy were deduced.

In the open problem section of the "Proceedings of the Third International Symposium on Orthogonal Polynomials and Their Applications" held in Erice (Italy), R. Askey raised the following question [1]: "Consider

0022-247X/95 $12.00

the Meixner Polynomials $M_n^{\gamma,\mu}(x)$, add or subtract a point mass at $x = 0$ and find the resulting polynomials. Identify them as hypergeometric functions and show that these polynomials satisfy a difference equation in x."

In the present paper we solve this problem. Very recently H. Bavinck and H. van Haeringen [2] found an infinite difference equation with polynomial coefficients that such generalized Meixner polynomials satisfy as well as a second order difference equation, which has a different form with respect to the equation that we find in this work.

We continue the algebraic approach presented by Godoy, Marcellán, Salto, and Zarzo [5] in a general theory based in the addition of a delta Dirac measure to a discrete semiclassical linear functional.

The structure of the paper is as follows. In Section 2, we deduce an expression of the generalized Meixner polynomials $M_n^{\gamma,\mu,A}(x)$ in terms of the nth Meixner polynomial and its first difference derivative. In Section 3, we obtain its representation as a hypergeometric function ${}_3F_2$. In Section 4, we find the second order difference equation which these generalized polynomials satisfy. Finally, an appendix with the required background concerning Meixner polynomials is included.

2. The Definition and Orthogonal Relation

Consider the linear functional U on the linear space of polynomials with real coefficients defined as

$$\langle U, P\rangle = \sum_{x\in\mathbf{N}} \frac{\mu^x\Gamma(\gamma + x)}{\Gamma(\gamma)\Gamma(1 + x)} P(x) + AP(0), \qquad x \in \mathbf{N}, A \geq 0 \qquad (1)$$

and

$$\langle M, P\rangle = \sum_{x\in\mathbf{N}} \frac{\mu^x\Gamma(\gamma + x)}{\Gamma(\gamma)\Gamma(1 + x)} P(x), \qquad x \in \mathbf{N}, \qquad (2)$$

where M is the Meixner moment functional, $0 < \mu < 1$, $\gamma > 0$, and $\mathbf{N} = \{0, 1, 2, ...\}$.

We will determine the monic polynomials $M_n^{\gamma,\mu,A}(x)$ which are orthogonal with respect to the functional U. These polynomials exist because U is a positive definite moment functional (see [4, p. 26, Definition 5.1]).

To obtain this we may write the generalized polynomials as a Fourier series

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) + \sum_{k=0}^{n-1} a_{n,k} M_k^{\gamma,\mu}(x), \tag{3}$$

where $M_n^{\gamma,\mu}(x)$ denotes the classical Meixner monic polynomial of degree n.

To find the unknown coefficients $a_{n,k}$ we will use the orthogonality of the polynomials $M_n^{\gamma,\mu,A}(x)$ with respect to U, i.e.,

$$\langle U, M_n^{\gamma,\mu,A}(x) M_k^{\gamma,\mu}(x)\rangle = 0 \qquad \forall k < n.$$

Now putting (3) in (1) we find

$$\begin{aligned}\langle U, M_n^{\gamma,\mu,A}(x) M_k^{\gamma,\mu}(x)\rangle = \langle M, M_n^{\gamma,\mu,A}(x) M_k^{\gamma,\mu}(x)\rangle \\ + A M_n^{\gamma,\mu,A}(0) M_k^{\gamma,\mu}(0).\end{aligned} \tag{4}$$

If we use the decomposition (3) and take into account the orthogonality of the classical Meixner polynomials with respect to the linear functional M, the coefficients $a_{n,k}$ are given by

$$a_{n,k} = -\frac{A M_n^{\gamma,\mu,A}(0) M_k^{\gamma,\mu}(0)}{d_k^2}, \tag{5}$$

where d_k^2 denotes the norm of the classical Meixner polynomials (24). Finally, Eq. (3) gives the expression

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) - A M_n^{\gamma,\mu,A}(0) \sum_{k=0}^{n-1} \frac{M_k^{\gamma,\mu}(0) M_k^{\gamma,\mu}(x)}{d_k^2}. \tag{6}$$

Now in order to obtain an explicit expression for these polynomials we need some properties of the classical Meixner polynomials. These formulas are enclosed in the Appendix.

Doing some algebraic calculations in (6) and taking into account formula (27), below, we obtain the following expression for the generalized Meixner polynomials:

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) - A M_n^{\gamma,\mu,A}(0) \frac{(-1)^n (\mu-1)^{n+\gamma-1}}{n!} \nabla M_n^{\gamma,\mu}(x). \tag{7}$$

In the above formula the polynomials $M_n^{\gamma,\mu,A}(x)$ evaluated in $x = 0$ appear. Then, to obtain the analytical expression of $M_n^{\gamma,\mu,A}(0)$ it is enough to evaluate (7) in $x = 0$. The solution of this equation is

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) + B_n \nabla M_n^{\gamma,\mu}(x) = (I + B_n \nabla) M_n^{\gamma,\mu}(x), \tag{8}$$

where

$$B_n = -A \frac{\mu^n (1-\mu)^{\gamma-1} (\gamma)_n}{n!\,(1 + A \sum_{k=0}^{n-1} ((M_k^{\gamma,\mu}(0))^2/d_k^2)}$$

From (6) we can conclude that the representation (8) exists for any value of the mass A. To obtain this is it enough to evaluate (6) in $x = 0$,

$$\left(1 + A \sum_{k=0}^{n-1} \frac{M_k^{\gamma,\mu}(0) M_k^{\gamma,\mu}(0)}{d_k^2}\right) M_n^{\gamma,\mu,A}(0) = M_n^{\gamma,\mu}(0) \neq 0,$$

and use the fact that $1 + A \sum_{n=0}^{n} (M_k^{\gamma,\mu}(0) M_k^{\gamma,\mu}(0)/d_k^2) > 0$ for $n \in \mathbf{N}$. The result then follows.

3. Representation as Hypergeometric Series

The classical Meixner polynomials are represented by the hypergeometric function

$$M_n^{\gamma,\mu}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n} {}_2F_1\left({-n,-x \atop \gamma}; 1 - \mu^{-1}\right). \tag{9}$$

where

$$ {}_pF_q\left({a_1, a_2, \ldots, a_p \atop b_1, b_2, \ldots, b_q}; x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

$$(a)_0 := 1,$$

$$(a)_k := a(a+1)(a+2)\cdots(a+k-1), \qquad k = 1, 2, 3, \ldots.$$

In this section we will prove the following:

PROPOSITION 1. *The orthogonal polynomial $M_n^{\gamma,\mu,A}(x)$ is, up to a constant factor, a generalized hypergeometric function. More precisely*

$$M_n^{\gamma,\mu,A}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n} {}_3F_2\left({-n, -x, 1 + xB_n^{-1} \atop \gamma, xB_n^{-1}}; 1 - \frac{1}{\mu}\right).$$

Proof. From (8) and the hypergeometric representation of the Meixner polynomials (9) we can write.

$$M_n^{\gamma,\mu,A}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n} \sum_{m=0}^{\infty} \frac{(-n)_m(-x)_m}{(\gamma)_m} \frac{z^m}{m!}$$
$$+ B_n(\gamma)_n \frac{\mu^n}{(\mu-1)^n} \sum_{m=0}^{\infty} \frac{(-n)_m(-x)_m}{(\gamma)_m} \frac{z^m}{m!}$$
$$- B_n(\gamma)_n \frac{\mu^n}{(\mu-1)^n} \sum_{m=0}^{\infty} \frac{(-n)_m(1-x)_m}{(\gamma)_m} \frac{z^m}{m!}$$

or, equivalently,

$$M_n^{\gamma,\mu,A}(x) = \frac{\mu^n}{(\mu-1)^n}(\gamma)_n \sum_{m=0}^{\infty} \frac{(-n)_m(-x)_m}{(\gamma)_m}\left[\frac{B_n m}{x}+1\right]\frac{z^m}{m!}, \tag{10}$$

where $z = 1 - 1/\mu$.

Taking into account the fact that $B_n \neq 0$ as well as the fact that the expression inside the quadratic brackets is a polynomial in m of degree 1, we can write (10) as

$$M_n^{\gamma,\mu,A}(x) = \frac{\mu^n}{(\mu-1)^n}(\gamma)_n \frac{B_n}{x} \sum_{m=0}^{\infty} \frac{(-n)_m(-x)_m}{(\gamma)_m}\left[m+\frac{x}{B_n}\right]\frac{z^m}{m!}. \tag{11}$$

Since

$$\left(m+\frac{x}{B_n}\right) = \frac{x(xB_n^{-1}+1)_m}{B_n(xB_n^{-1})_m}, \tag{12}$$

then (11) becomes

$$M_n^{\gamma,\mu,A}(x) = \frac{\mu^n}{(\mu-1)^n}(\gamma)_n \sum_{m=0}^{\infty} \frac{(-n)_m(-x)_m(xB_n^{-1}+1)_m}{(\gamma)_m(xB_n^{-1})_m}\frac{z^m}{m!}, \tag{13}$$

or in terms of the hypergeometric series

$$M_n^{\gamma,\mu,A}(x) = (\gamma)_n \frac{\mu^n}{(\mu-1)^n}\, {}_3F_2\left(\begin{matrix} -n, -x, 1+xB_n^{-1} \\ \gamma, xB_n^{-1} \end{matrix}; 1-\frac{1}{\mu}\right). \tag{14}$$

Here the coefficient xB_n^{-1} is, in general, a real number. In the case when xB_n^{-1} is a nonpositive integer we need to take the analytic continuation of the hypergeometric series (14).

It is straightforward to show that for $A = 0$ the hypergeometric function

(14) yields Eq. (9). So (14) can be considered as a generalization of the representation of the classical Meixner polynomials $M_n^{\gamma,\mu}(x)$ as hypergeometric series. ∎

4. A Second Order Difference Equation

We will prove the following:

Theorem 1. *The polynomial* $M_n^{\gamma,\mu,A}(x)$ *satisfies a second order linear difference equation*

$$
\begin{aligned}
&[x + B_n(B_n\lambda_n + \lambda_n - \tau(x))](x-1)\,\Delta\nabla P_n^A(x) + (x-1)\tau(x)\,\Delta P_n^A(x) \\
&\quad + B_n[(\tau(x) - B_n\lambda_n)(\lambda_n - 1 - \tau(x-1)) + \lambda_n(\lambda_n + B_n) \\
&\quad + (x + B_n\lambda_n)\,\Delta\tau(x)]\,\Delta P_n^A(x) + (x-1)\lambda_n P_n^A(x) \\
&\quad + B_n\lambda_n[\lambda_n - 1 - \tau(x-1) + B_n(\Delta\tau(x) + \lambda_n)]P_n^A(x) = 0,
\end{aligned}
$$

where

$$x = 0, 1, 2, \ldots, \qquad \tau(x) = \gamma\mu - x(1-\mu), \qquad \lambda_n = n(1-\mu)$$

and

$$\nabla f(x) = f(x) - f(x-1), \qquad \Delta f(x) = f(x+1) - f(x).$$

Proof. We will start from the representation (8) for the generalized polynomials

$$M_n^{\gamma,\mu,A}(x) = M_n^{\gamma,\mu}(x) + B_n\nabla M_n^{\gamma,\mu}(x).$$

Multiplying this expression by x and using the second order difference equation that the classical Meixner polynomials satisfy,

$$x\,\Delta\nabla M_n^{\gamma,\mu}(x) + \tau(x)\,\Delta M_n^{\gamma,\mu}(x) + \lambda_n M_n^{\gamma,\mu}(x) = 0, \tag{15}$$

we obtain

$$x M_n^{\gamma,\mu,A}(x) = (x + B_n\lambda_n)M_n^{\gamma,\mu}(x) + B_n(x + \tau(x))\,\Delta M_n^{\gamma,\mu}(x), \tag{16}$$

where the identity $\nabla M_n^{\gamma,\mu}(x) = \Delta M_n^{\gamma,\mu}(x) - \Delta\nabla M_n^{\gamma,\mu}(x)$ is used. Now if we apply the operator Δ to (16), from (15) the equation

$$x\,\Delta M_n^{\gamma,\mu,A}(x) = [x - B_n\tau(x)]\,\Delta M_n^{\gamma,\mu}(x) - B_n\lambda_n M_n^{\gamma,\mu,}(x) \tag{17}$$

is obtained. In the same way if we apply in (16) the operator ∇ and use (15) we find

$$\begin{aligned} x(x-1)\,\Delta\nabla P_n^A(x) = -[&(x-1)\tau(x) + B_n\tau(x)(\lambda_n - \tau(x-1) - 1) \\ &+ B_n x(\lambda_n + \Delta\tau(x))]\,\Delta P_n(x) \\ &-[x - 1 + B_n(\lambda_n - \tau(x-1) - 1)]\lambda_n P_n(x). \end{aligned} \tag{18}$$

Now from (16), (17), and (18) we can deduce that the determinant

$$\begin{vmatrix} xM_n^{\gamma,\mu,A}(x) & a(x) & b(x) \\ x\,\Delta M_n^{\gamma,\mu,A}(x) & c(x) & d(x) \\ x(x-1)\,\Delta\nabla M_n^{\gamma,\mu,A}(x) & e(x) & f(x) \end{vmatrix} = 0 \tag{19}$$

vanishes. Here, for $C = B_n$

$$\begin{aligned} a(x) &= (x + C\lambda_n), \\ b(x) &= C(x + \tau(x)) \\ c(x) &= -C\lambda_n, \\ d(x) &= x - C\tau(x) \\ e(x) &= -[x - 1 + C(\lambda_n - 1 - \tau(x-1))]\lambda_n \\ f(x) &= -[(x-1)\tau(x) + C[\tau(x)(\lambda_n - 1 - \tau(x-1)) + x(\lambda_n + \Delta\tau(x))]. \end{aligned}$$

If we expanding the determinant in (19) by the first column and divide by x the theorem follows. ∎

The difference equation of the previous theorem takes the form

$$\begin{aligned} \{x + B_n[&(1-\mu)(x + n + nB_n) - \gamma\mu]\}(x-1)\,\Delta\nabla M_n^{\gamma,\mu,A}(x) \\ &+ (x-1)[\gamma\mu - x(1-\mu)]\,\Delta M_n^{\gamma,\mu,A}(x) \\ &+ B_n\{(1-\mu)[\gamma\mu(n + nB_n + 2x - 1) \\ &+ (1-\mu)(x + n^2 - (x + nB_n)(x+n)) \\ &+ 2nB_n] - \gamma\mu(1 + \gamma\mu)\}\,\Delta M_n^{\gamma,\mu,A}(x) \\ &+ (x-1)n(1-\mu)M_n^{\gamma,\mu,A}(x) \\ &+ nB_n(1-\mu)[(1-\mu)(x + n + nB_n - B_n - 1) \\ &- 1 - \gamma\mu]M_n^{\gamma,\mu,A}(x) = 0. \end{aligned} \tag{20}$$

APPENDIX: THE CLASSICAL MEIXNER POLYNOMIALS

In this appendix we include some formulas for the classical Meixner polynomials which are useful for obtaining the generalized polynomial orthogonal with respect to the linear functional U. All the formulas and other properties for the classical Meixner polynomials can be found in the literature; see, for instance, the excellent monograph "Orthogonal Polynomials in Discrete Variables," by A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov [7].

In this work we will use monic polynomials, i.e., polynomials with leading coefficient equal to 1 ($P_n(x) = x^n +$ lower order terms).

The classical Meixner polynomials of a discrete variable are the polynomial solutions of the second order linear difference equation of hypergeometric type

$$x\,\Delta\nabla M_n^{\gamma,\mu}(x) + \tau(x)\,\Delta M_n^{\gamma,\mu}(x) + \lambda_n M_n^{\gamma,\mu}(x) = 0, \tag{21}$$

where

$$0 < \mu < 1, \qquad \gamma > 0, \qquad \tau(x) = \gamma\mu - x(1-\mu), \qquad \lambda_n = n(1-\mu)$$

and

$$\nabla f(x) = f(x) - f(x-1), \qquad \Delta f(x) = f(x+1) - f(x).$$

We will use the following two relations for the classical Meixner polynomials,

$$\frac{x}{n}\nabla M_n^{\gamma,\mu}(x) = \frac{\mu(\gamma+n-1)}{\mu-1} M_{n-1}^{\gamma,\mu}(x) - M_n^{\gamma,\mu}(x), \tag{22}$$

$$\Delta M_n^{\gamma,\mu}(x) = n M_{n-1}^{\gamma+1,\mu}(x). \tag{23}$$

These polynomials are orthogonal with respect to the linear functional M defined in (2). The orthogonality relation is

$$\sum_{x\in\mathbb{N}} M_n^{\gamma,\mu}(x) M_m^{\gamma,\mu}(x) \frac{\mu^x\Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(1+x)} = \delta_{nm}\, d_n^2, \tag{24}$$

where d_n^2 denotes the square of the norm of the classical Meixner polynomials

$$d_n^2 = \frac{n!(\gamma)_n \mu^n}{(1-\mu)^{\gamma+2n}}.$$

A consequence of the representation (9) is

$$M_n^{\gamma,\mu}(0) = \frac{\mu^n}{(\mu-1)^n}\frac{\Gamma(n+\gamma)}{\Gamma(\gamma)}. \tag{25}$$

From the Christoffel–Darboux formula (see [7, p. 14, formula [1.4.18]])

$$\sum_{m=0}^{n-1}\frac{M_m^{\gamma,\mu}(x)M_m^{\gamma,\mu}(y)}{d_m^2} = \frac{1}{x-y}\frac{M_n^{\gamma,\mu}(x)M_{n-1}^{\gamma,\mu}(y) - M_{n-1}^{\gamma,\mu}(x)M_n^{\gamma,\mu}(y)}{d_{n-1}^2}, \qquad n = 1,2,\ldots, \tag{26}$$

we will obtain a useful property for the kernels of the Meixner polynomials. We put $y = 0$ in (26) and use (25) and (22). Then

$$\sum_{m=0}^{n-1}\frac{M_m^{\gamma,\mu}(x)M_m^{\gamma,\mu}(0)}{d_m^2} = -\frac{(\mu-1)^{n-\gamma-1}}{n!}\nabla M_n^{\gamma,\mu}(x). \tag{27}$$

Acknowledgments

The research of the first author was supported by a grant of the Instituto de Cooperación Iberoamericana (ICI) of Spain. He is very grateful to the Departamento de Ingeniería of the Universidad Carlos III de Madrid for its kind hospitality. The research of the second author was supported by the Comisión Interministerial de Ciencia y Tecnología (CICYT) of Spain under Grant PB 93-0229-C02-01.

The authors thank the referees for their suggestions and remarks, as well as Professor George Gasper, who pointed out Ref. [2] to us during the revision of our paper.

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