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A FURTHER NOTE ON LUCASIAN NUMBERS

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1. INTRODUCTION

This paper will extend and unify the results in [4] by completely determining all Lucasian numbers which are terms in certain Lucas sequences. Our specification of all Lucasian numbers will be based on results obtained in [1] in which all terms in particular Lucas sequences which do not have any primitive prime divisors are found.

Before proceeding, we will recall some definitions and known results. Our notation will be the same as that in [4]. Let $u(r,s)$ and $v(r,s)$ be Lucas sequences which satisfy the same second-order recursion relation

$$w_{n+2} = rw_{n+1} + sw_n \tag{1}$$

and have initial terms $u_0 = 0,\ u_1 = 1,\ v_0 = 2,\ v_1 = r$ respectively, where r and s are integers. Associated with the sequences $u(r,s)$ and $v(r,s)$ is the characteristic polynomial

$$f(x) = x^2 - rx - s \tag{2}$$

with characteristic roots α and β. Let $D = r^2 + 4s = (\alpha - \beta)^2$ be the discriminant of both $u(r,s)$ and $v(r,s)$. By the Binet formulas

$$u_n = (\alpha^n - \beta^n)/(\alpha - \beta) \tag{3}$$

and

$$v_n = \alpha^n + \beta^n. \tag{4}$$

The recurrences $u(r,s)$ and $v(r,s)$ are said to be degenerate if $\alpha\beta = -s = 0$ or α/β is a root of unity. It follows from (3) and (4) that u_n or v_n can be equal to zero for $n \geq 1$ only if both $u(r,s)$ and $v(r,s)$ are degenerate.

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The integer m is a divisor of the recurrence $w(r,s)$ satisfying the relation (1) if $m \mid w_n$ for some $n \geq 1$. The prime p is a primitive prime divisor of w_n, $n \geq 1$, if $p \mid w_n$ but $p \nmid w_i$ for $1 \leq i < n$. Given the Lucas sequence $v(r,s)$, the integer m is called Lucasian if m is a divisor of $v(r,s)$. In our main theorem, Theorem 2.6, we will show that if $u(r,s)$ and $v(r,s)$ are nondegenerate and $gcd(r,s) = 1$, then u_n is not Lucasian if $n \geq 27$. Theorem 2.6 will also find all terms u_a and v_b such that $a \geq 1$, $b \geq 1$, and $u_a \mid v_b$.

A related question is to determine all a and b such that $a \geq 1$, $b \geq 1$, and $v_a \mid u_b$. We will answer this question completely in Theorem 2.8 when both $v(r,s)$ and $u(r,s)$ are nondegenerate and $gcd(r,s) = 1$.

2. THE MAIN RESULTS

For reference, we will give the main results of [4] in Theorems 2.1-2.4. Theorems 2.6 and 2.8 will then generalize Theorems 2.1, 2.2 and 2.4.

Theorem 2.1: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) = 1$ and $D > 0$. Let a and b be positive integers. Then $u_a \mid v_b$ if and only if one of the following conditions also holds. For convenience, the value of u_a is given in these conditions. The expressions $r = *$ given below means that r can be any integer.
(i) $a = 1,\ r = *,\ s = *,\ b = *,\ u = 1;$
(ii) $a = 2, |\, r \,| = 1$ or $2,\ s = *,\ b = *,\ u_2 = r = v_1;$
(iii) $a = 2,\ |\, r \,| \geq 3,\ s = *,\ b \equiv 1 \pmod 2), u_2 = r = v_1;$
(iv) $a = 3, |\, r \,| = 1,\ s = 1,\ b \equiv 0 \pmod 3),\ u_3 = 2;$
(v) $a = 4,\ |\, r \,| = 1, s = *,\ b \equiv 2 \pmod 4),\ u_4 = v_2 = r^2 + 2s.$
In particular, u_n is not Lucasian if $n \geq 5$.

Theorem 2.2: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) = 1$ and $D < 0$. Then u_n is not Lucasian for $n > e^{452}2^{68}$.

Theorem 2.3: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) > 1$. Then there exists a constant $N(r,s)$ dependent on r and s such that u_n is not Lucasian for $n \geq N(r,s)$.

Theorem 2.4: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) = 1$ and $D > 0$. Let a and b be positive integers. If $|v_a| \geq 3$, then $v_a | u_b$ if and only if $2a|b$. If $|v_a| \leq 2$, then $v_a|u_b$ if and only if one of the following two conditions also holds (the value of v_a is given for convenience):
(i) $a = 1,\ |r| = 1,\ s = *,\ b = *,\ v_1 = r;$
(ii) $a = 1,\ |r| = 2,\ s \equiv 1 \pmod 2),\ b \equiv 0 \pmod 2),\ v_1 = r = u_2.$

Remark 2.5: The proofs of Theorem 2.1 and 2.4 given in [4] depend partly on the fact that if $D > 0$, then $u(r,s)$ is strictly increasing for $n \geq 2$ and $v(r,s)$ is strictly increasing for $n \geq 1$.

Theorem 2.6: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) = 1$. Let a and b be positive integers. Then u_a is Lucasian and $u_a|v_b$ if and only if one of the following conditions also holds. For convenience, the value of u_a is given in these conditions.
(i) $a = 1,\ r = *,\ s = *,\ b = *,\ u_1 = 1;$
(ii) $a = 2,\ |r| = 1$ or $2,\ s = *,\ b = *,\ u_2 = v_1 = r;$
(iii) $a = 2,\ |r| \geq 3,\ s = *,\ b \equiv 1 \pmod 2),\ u_2 = v_1 = r;$
(iv) $a = 3,\ r = *,\ s = \pm 1 - r^2, b = *, u_3 = \pm 1;$
(v) $a = 3,\ r \equiv 1 \pmod 2),\ s = \pm 2 - r^2, b \equiv 0 \pmod 3),\ u_3 = \pm 2;$

(vi) $a = 4,\ |r| = 1,\ s = *,\ b \equiv 2 (\text{mod } 4),\ u_4 = \pm v_2$;
(vii) $a = 4,\ r \equiv 1 (\text{mod } 2),\ s = (\pm 1 - r^2)/2,\ b \equiv 1 (\text{mod } 2),\ u_4 = \pm u_2 = \pm v_1 = \pm r$;
(viii) $a = 5,\ |r| = 1,\ s = -2,\ b = *,\ u_5 = -1$;
(ix) $a = 5,\ |r| = 1,\ s = -3,\ b = *,\ u_5 = 1$;
(x) $a = 5,\ |r| = 12,\ s = -55,\ b = *,\ u_5 = 1$;
(xi) $a = 5,\ |r| = 12,\ s = -377,\ b = *,\ u_5 = 1$;
(xii) $a = 6,\ r = *,\ s = \pm 1 - r^2,\ b \equiv 3 (\text{mod } 6),\ u_6 = \pm v_3$;
(xiii) $a = 7,\ |r| = 1,\ s = -5,\ b = *,\ u_7 = 1$;
(xiv) $a = 8,\ |r| = 1,\ s = -2,\ b \equiv 2 (\text{mod } 4),\ u_8 = u_4 = \pm v_2 = \pm 3$;
(xv) $a = 10,\ |r| = 1,\ s = -2,\ b \equiv 5 (\text{mod } 10),\ u_{10} = -v_5 = \pm 11$;
(xvi) $a = 10,\ |r| = 1,\ s = -3,\ b \equiv 5 (\text{mod } 10),\ u_{10} = v_5 = \pm 31$;
(xvii) $a = 10,\ |r| = 12,\ s = -55,\ b \equiv 5 (\text{mod } 10),\ u_{10} = v_5 = \pm 44868$;
(xviii) $a = 10,\ |r| = 12,\ s = -377,\ b \equiv 5 (\text{mod } 10),\ u_{10} = v_5 = \pm 5519292$;
(xix) $a = 13,\ |r| = 1,\ s = -2,\ b = *,\ u_{13} - 1$;
(xx) $a = 14, |r| = 1,\ s = -5, b \equiv 7 \ (\text{mod } 14), u_{14} = v_7 = \pm 559$;
(xxi) $a = 26,\ |r| = 1,\ s = -2,\ b \equiv 13 \ (\text{mod } 26),\ u_{26} = -v_{13} = \pm 181$.

Remark 2.7: By Theorem 2.6, if $u(r,s)$ is nondegenerate and $gcd(r,s) = 1$, then there exist only 12 possible indices n for which u_n can be Lucasian. It is noteworthy that for the Lucas sequences $u(\pm 1, -2)$, u_n is Lucasian for 10 of these indices, namely $n = 1,2,3,4,5,6,8,10,13,26$. The only other nondegenerate Lucas sequences $u(r,s)$ for which $gcd(r,s) = 1$ and u_n is Lucasian for 5 or more indices n are $u(\pm 1, -3)$ and $u(\pm 1, -5)$. For $u(\pm 1, -3)$, u_n is Lucasian for the 6 indices $n = 1,2,3,4,5,10$, while for $u(\pm 1, -5)$, u_n is Lucasian for the 5 indices $n = 1,2,4,7,14$.

Theorem 2.8: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s) = 1$. Let a and b be positive integers. Then $v_a | u_b$ if and only if one of the following conditions also holds. For convenience, the value of v_a is given in condition (ii)-(vii).

(i) $2a|b,\ r = *,\ s = *$;
(ii) $a = 1,\ |r| = 1,\ s = *,\ b = *,\ v_1 = r$;
(iii) $a = 2,\ r \equiv 1 \ (\text{mod } 2),\ s = (\pm 1 - r^2)/2,\ b = *,\ v_2 = \pm 1$;
(iv) $a = 2,\ r \equiv 0 \ (\text{mod } 2),\ s = (\pm 2 - r^2)/2,\ b \equiv 0 \ (\text{mod } 2),\ v_2 = \pm 2$;
(v) $a = 4,\ |r| = 1,\ s = -2,\ b = *,\ v_4 = 1$;
(vi) $a = 4,\ |r| = 2,\ s = -7,\ b \equiv 0 \ (\text{mod } 2),\ v_4 = 2$;
(vii) $a = 5,\ |r| = 2,\ s = -3,\ b \equiv 0 \ (\text{mod } 2),\ v_5 = \pm 2$.

Remark 2.9: Theorem 2.6 generalizes Theorems 2.1 and 2.2, which were proven in [4]. Theorem 2.8 generalizes Theorem 2.4 which was also proved in [4]. As contrasted to the proofs of Theorems 2.1, 2.2, and 2.4, the proofs of Theorems 2.6 and 2.7, which will be given in Section 4, do not treat the cases $D > 0$ and $D < 0$ separately.

The key result in proving Theorems 2.6 and 2.8 is Theorem 2.10 given below which is the main theorem of [1].

Theorem 2.10: Let $u(r,s)$ be a nondegenerate Lucas sequence for which $gcd(r,s) = 1$. Then u_n has a primitive prime divisor if $n > 30$. Moreover u_n has no primitive prime divisor only if $n = 1,2,3,4,5,6,7,8,10,12,13,18$, or 30.

Remark 2.11: Consider all nondegenerate Lucas sequences $u(r,s)$ for which $gcd(r,s) = 1$. Tables 1 and 3 of [1] list all terms $u_n, n \geq 1$, which have no primitive prime divisors. We note that in [1], the authors define a prime p to be a primitive prime divisor of u_n if $p | u_n$ but

$p \nmid Du_1u_2 \ldots u_{n-1}$. In contrast to this definition, we do not include p as a primitive prime divisor of u_n if $p|D$, but $p \nmid u_k$ for $1 \le k < n$.

For reference, Theorem 2.12 lists all degenerate Lucas sequences $u(r,s)$ and $v(r,s)$.

Theorem 2.12: The Lucas sequences $u(r,s)$ and $v(r,s)$ are degenerate if and only if one of the following conditions holds:

(i) $s = 0$.
(ii) $\alpha/\beta = 1$ and $D = r^2 + 4s = 0$.
(iii) $\alpha/\beta = -1$, $r = 0$ and $s = N$ for some non-zero integer N.
(iv) α/β is a primitive cube root of unity, $r = N$, and $s = -N^2$ for some non-zero integer N.
(v) α/β is a primitive fourth root of unity, $r = 2N$, and $s = -2N^2$ for some non-zero integer N.
(vi) α/β is a primitive sixth root of unity, $r = 3N$, and $s = -3N^2$ for some non-zero integer N.

Proof: This is proved in [9, p. 613]. □

3. NECESSARY LEMMAS AND DEFINITIONS

The following lemmas and definition will be needed for the proofs of Theorems 2.6 and 2.8. Lemmas 3.1, 3.2, 3.3, and 3.5 are well-known (and follow readily from (1), (3), and (4)).

Lemma 3.1: $u_{2n} = u_n v_n$.

Lemma 3.2:

$$u_n(-r,s) = (-1)^{n+1} u_n(r,s). \tag{5}$$

$$v_n(-r,s) = (-1)^n v_n(r,s). \tag{6}$$

It follows from Lemma 3.2 that, $u_n(-r,s)$ is Lucasian if and only if $u_n(r,s)$ is Lucasian.

Lemma 3.3: Consider the Lucas sequences $u(r,s)$ and $v(r,s)$. Then $u_n|u_{in}$ for all $i \ge 1$ and $v_n|v_{(2j+1)n}$ for all $j \ge 0$. □

Lemma 3.4: If u_a is not Lucasian, and $a|c$, then u_c is not Lucasian.

Proof: By Lemma 3.3, $u_a|u_c$. It is now evident that u_c is not Lucasian if u_a is not Lucasian. □

Lemma 3.5: Let $u(r,s)$ and $v(r,s)$ be Lucas sequences for which $2 \nmid gcd(r,s)$. □

(i) Suppose r is odd and s is even. Then $2 \nmid u_n$ and $2 \nmid v_n$ for $n \ge 1$.
(ii) Suppose r and s are both odd. Then $2|u_n$ if and only if $3|n$, and $2|v_n$ if and only if $3|n$.
(iii) Suppose r is even and s is odd. Then $2|u_n$ if and only if $2|n$, and $2|v_n$ for all $n \ge 0$.

Lemma 3.6: Let $v(r,s)$ be a Lucas sequence for which $2 \nmid gcd(r,s)$.

(i) If r and s are both odd and $2^k \parallel v_3$ for some positive integer k, then $2 \parallel v_n$ for $n \equiv 0 \pmod 6$ and $2^k \parallel v_n$ for $n \equiv 3 \pmod 6$. Recall that $2^k \parallel a$ if $2^k|a$, but $2^{k+1} \nmid a$.
(ii) If r is even and s is odd and $2^k \parallel v_1 = r$, then $2 \parallel v_{2n}$ and $2^k \parallel v_{2n+1}$ for all $n \ge 0$.

Proof: This is proved in [8]. □

Definition 3.7: For the Lucas sequence $u(r,s)$ the rank of appearance of the positive integer m in $u(r,s)$, denoted by $\omega(m)$, is the least positive integer n, if it exists, such that $m|u_n$. The rank of appearance of m in $v(r,s)$, denoted by $\overline{\omega}(m)$, is defined similarly.

Lemma 3.8: Let $u(r,s)$ and $v(r,s)$ be Lucas sequences for which $gcd(r,s) = 1$. Let p be an odd prime. If $\omega(p)$ is odd, then $\overline{\omega}(p)$ does not exist and $u_{\omega(p)}$ is not Lucasian.

Proof: This was proved by Carmichael [2, p. 47]. □

Lemma 3.9: Let $u(r,s)$ and $v(r,s)$ be Lucas sequences for which $gcd(r,s)=1$. Suppose that p is an odd prime and $\omega(p)=2n$. Then $\overline{\omega}(p)=n$.

Proof: This is proved in Proposition 2(iv) of [7]. □

Definition 3.10: The 2-valuation of the integer n, denoted by $[n]_2$ is the largest integer k such that $2^k|n$.

Lemma 3.11: Let $v(r,s)$ be a Lucas sequence for which $gcd(r,s)=1$. Suppose that u_a is Lucasian and that p and q are distinct odd prime divisors of u_a. Then $[\overline{\omega}(p)]_2=[(\overline{\omega}(q)]_2$.

Proof: This is proved in Proposition 2(ix) of [7]. □

Lemma 3.12: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s)=1$. Let a and b be positive integers and let $d=gcd(a,b)$.

(i) $gcd(u_a,u_b)=u_d$;

(ii) $gcd(v_a,v_b)=\begin{cases} v_d \text{ if } [a]_2=[b]_2, \\ 1 \text{ or } 2 \text{ otherwise;}\end{cases}$

(iii) $gcd(u_a,v_b)=\begin{cases} v_d \text{ if } [a]_2>[b]_2, \\ 1 \text{ or } 2 \text{ otherwise.}\end{cases}$

Proof: This is proved in [6] and [3, section 5]. □

Lemma 3.13: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s)=1$. Then $m|u_n$ if and only if $\omega(m)|n$. Moreover, if $m\geq 3$, then $m|v_n$ if and only if $\overline{\omega}(m)|n$ and $[\overline{\omega}(m)]_2=[n]_2$.

Proof: The results follow from Lemmas 3.3. and 3.12. □

Lemma 3.14: Let $u(r,s)$ be nondegenerate Lucas sequence for which $2\not| gcd(r,s)$. If r is even, s is odd, and $4|a$, then u_a is not Lucasian. If r and s are both odd and $6|a$, then u_a is not Lucasian. If r and s are both odd and $6|a$, the u_a is not Lucasian. If r and s are both odd, $4|u_3$, and $3|a$, then u_a is not Lucasian.

Proof: First suppose that r is even and s is odd. By Lemma 3.5 (iii), $2|v_n$ for all $n\geq 0$. Moreover, $u_2=v_1=r$. Suppose that $2^k \parallel r$. By Lemma 3.6 (ii), $2^{k+1}\not| v_n$ for any $n\geq 0$. However, $2|v_2$, and hence by Lemma 3.1, $2^{k+1}|u_4=u_2v_2$. Thus, u_4 is not Lucasian, and conseuqently by Lemma 3.4, u_a is not Lucasian if $4|a$.

Now suppose that both r and s are odd. By Lemma 3.5(ii), $2|u_3$ and $2|v_3$. Suppose that $2^k \parallel v_3$. By Lemma 3.6 (i), $2^{k+1}\not| v_n$ for any $n\geq 0$. However, $2^{k+1}|u_6=u_3v_3$. Therefore, u_6 is not Lucasian, and hence by Lemma 3.4, u_a is not Lucasian if $6|a$.

Finally, suppose that in addition to r and s both being odd, $4|u_3$. Since $u_3=r^2+s$, this can occur only if $s\equiv 3 \pmod 4$. However, then $v_3=r(r^2+3s)\equiv 2 \pmod 4$, and $2\parallel v_3$. By Lemma 3.6 (i), it follows that 4 is not a divisor of $v(r,s)$. Since $u_3|u_{3t}$ for all $t\geq 1$ by Lemma 3.3, we see that $4|u_{3t}$ for all $t\geq 1$. Hence, u_a is not Lucasian if $3|a$. □

Lemma 3.15: Let $u(r,s)$ and $v(r,s)$ be nondegenerate Lucas sequences for which $gcd(r,s)=1$. Then u_a is Lucasian for $a\geq 1$ if and only if at least one of the following four conditions holds:

(i) a is odd and $u_a=\pm 1$;

(ii) a is even and $u_{a/2}=\pm 1$;

(iii) a is even, $u_{a/2}$ is Lucasian, and $v_{a/2}=\pm 1$;

(iv) $a = 3$ and $u_a = \pm 2$.

Proof: We first show sufficiency. The sufficiency of (i) is obvious. The sufficiency of (ii) and (iii) follow from the fact that $u_a = u_{a/2}v_{a/2}$ by Lemma 3.1. The sufficiency of (iv) follows from the fact that by Lemma 3.5, $2|u_a$ for $a \geq 1$ only if $2|v_a$.

We now show necessity. Suppose first that a is odd, $u_a \neq \pm 1$, and u_a is Lucasian. If u_a has an odd prime divisor p, then by Lemma 3.13, $\omega(p)|a$ and hence $\omega(p)$ is odd. It now follows from Lemmas 3.4 and 3.8 that u_a is not Lucasian.

Now assume that $u_a \geq 2$ and u_a is a power of 2. By Lemma 3.5, we must have that r and s are both odd and $3|a$. Since $2|u_3 = r^2 + s$, we see that u_a has no primitive prime divisor if $a > 3$. It now follows from Theorem 2.10 that a must equal 3. By Lemma 3.14, u_a is not Lucasian if $4|u_3$. Hence, $a = 3$ and $u_a = \pm 2$. By our above discussion, we see that if a is odd and u_a is Lucasian, then either $u_a = \pm 1$ or $a = 3$ and $u_a = \pm 2$.

At this point, we assume that a is even, $u_{a/2} \neq \pm 1$, and $v_{a/2} \neq \pm 1$. Suppose that $2|u_{a/2}$. By Lemma 3.5, either r and s are both odd, or r is even and s is odd. If r and s are both odd, then $3|(a/2)$ and $2|v_{a/2}$ by Lemma 3.5. Suppose that $2^k \parallel v_3$. Then $2^{k+1}|u_6 = u_3v_3$, and hence $2^{k+1}|u_a$ by Lemma 3.3. Thus, u_a is not Lucasian by Lemma 3.6 (i). If r is even and s is odd, then $a/2$ is even by Lemma 3.5. Moreover, by Lemma 3.5 (iii), $2|v_n$ for all $n \geq 0$. Suppose that $2^k \parallel v_1$. Then $2^k|u_2 = r = v_1$, and hence $2^k|u_{a/2}$ by Lemma 3.3. Then $2^{k+1}|u_a = u_{a/2}v_{a/2}$, and u_a is not Lucasian by Lemma 3.6 (ii).

Next suppose that $2|v_{a/2}$. Since $2 \nmid u_{a/2}$ by our above arguments, we see by Lemma 3.5 that r is even, s is odd, and $a/2$ is odd. Then $u_{a/2}$ has an odd prime divisor. By our earlier discussion and Lemma 3.4, it follows that both $u_{a/2}$ and u_a are not Lucasian.

The only remaining case to consider is the one in which $u_{a/2}$ has an odd prime divisor p and $v_{a/2}$ has an odd prime divisor q. Since $gcd(u_{a/2}, v_{a/2}) = 1$ or 2 by Lemma 3.12 (iii), $p \neq q$. Suppose that $2^t \parallel a$. If $t = 1$, then $\omega(p)|(a/2)$ and $\omega(p)$ is odd by Lemma 3.13. Thus, both $u_{a/2}$ and u_a are not Lucasian by Lemmas 3.4 and 3.8. Consequently we must have that $t \geq 2$. By Lemmas 3.13 and 3.9, it follows that

$$[\overline{\omega}(p)]_2 \leq t - 2. \tag{7}$$

Since $q|v_{a/2}$, we see by Lemma 3.13 that

$$[\overline{\omega}(q)]_2 = t - 1. \tag{8}$$

Since $pq|u_a$ it now follows from Lemma 3.11 and (7) and (8) that u_a is not Lucasian. Hence, if a is even and u_a is Lucasian, then $u_{a/2} = \pm 1$ or $v_{a/2} = \pm 1$. We note further that if $v_{a/2} = \pm 1$ and u_a is Lucasian, then $u_{a/2}$ is also Lucasian since $u_a = u_{a/2}v_{a/2}$. Necessity is thus shown and our result follows. □

Remark 3.16: Suppose that $u(r,s)$ and $v(r,s)$ are nondegenerate, $gcd(r,s) = 1$, and u_a is Lucasian. We note that if $v_{a/2} = \pm 1$, then $u_a = u_{a/2}v_{a/2} = \pm u_{a/2}$. It now follows from Lemma 3.15 that if it is not the case that $a = 3$ and $u_a = \pm 2$, then either u_a or $u_{a/2}$ has no primitive prime divisor. In what follows, we will apply the results of [1], which determine all n such that u_n has no primitive prime divisor to find all a for which u_a is not Lucasian and it is not the case that $a = 3$ and $u_a = \pm 2$.

Lemma 3.17: Let $u(r,s)$ be a nondegenerate Lucas sequence for which $gcd(r,s) = 1$. If a is odd, then u_a is not Lucasian if $a \geq 12$, $a \neq 14$, and $a \neq 26$.

Proof: Suppose that a is odd, $a \geq 9$ and $a \neq 13$. By Lemma 3.15, if u_a were to be Lucasian, then u_a must equal ± 1, and consequently u_a would have no primitive prime divisor. It now follows from Theorem 2.10 that u_a is not Lucasian.

We now assume that $a = 18$, $a = 30$, or $a = 60$. By our earlier discussion, u_9 and u_{15} are both not Lucasian. Thus, u_a is not Lucasian by Lemma 3.4. Now assume that a is even and either $a = 22$ or $a \geq 28$. By our previous argument, we can assume that $a \neq 30$, $a \neq 36$, and $a \neq 60$. If u_a were to be Lucasian, then by Remark 3.16, either $u_{a/2}$ or u_a would have no primitive prime divisors. Therefore, by Theorem 2.10, u_a is not Lucasian in this case also.

We next suppose that $a = 20$ and u_a is Lucasian. By Theorem 2.10, u_{20} has a primitive prime divisor. Thus by Lemma 3.15 and Remark 3.16, $u_{10} = \pm 1$ and u_{10} has no primitive prime divisor. By Table 1 of [1], the only recurrences $u(r,s)$ for which u_{10} has no primitive prime divisor are $u(\pm 2, -3)$, $u(\pm 5, -7)$, and $u(\pm 5, -18)$. We see by inspection that in each case, $u_{10} \neq \pm 1$. Hence, u_{20} is not Lucasian.

Now assume that $a = 16$ and u_a is Lucasian. Since u_{16} has a primitive prime divisor by Theorem 2.10, we see by Lemma 3.15 and Remark 3.16 that $u_8 = \pm 1$. By Table 1 of [1], the only recurrences $u(r,s)$ for which u_8 has no primitive prime divisor are $u(\pm 2, -7)$ and $u(\pm 1, -2)$. In each case, we see by inspection that $u_8 \neq \pm 1$. Thus u_{16} is not Lucasian.

We finally suppose that $a = 12$ and u_a is Lucasian. By Lemma 3.15 either $u_6 = \pm 1$ or $v_6 = \pm 1$. First assume that $u_6 = \pm 1$. Since $u_6 = u_3 v_3$, $v_3 = r(r^2 = 3s) = \pm 1$. Thus, $r = \pm 1$. Then $r^2 + 3s = 3s + 1 = \pm 1$. Hence, $3s = -2$, which is impossible, or $3s = 0$ and $s = 0$, which is excluded by the assumption that $u(r,s)$ is nondegenerate. Therefore, by Lemma 3.15 and Remark 3.16, $v_6 = \pm 1$ and u_{12} has no primitive prime divisor. Since $v_6 = \pm 1$, it follows from Lemma 3.5 that r is odd and s is even. By Table 1 of [1], the only recurrences $u(r,s)$ for which r is odd and s is even and u_{12} has no primitive prime divisor are $u(\pm 1, -2)$ and $u(\pm 1, -4)$. In both cases, we see that for the corresponding Lucas sequence, $v(r,s)$, $v_6 \neq \pm 1$. Thus, u_{12} is not Lucasian. By Lemma 3.4, u_{24} is also not Lucasian. □

Lemma 3.18: Suppose that $u(r,s)$ and $v(r,s)$ are nondegenerate Lucas sequences for which $gcd(r,s) = 1$. Suppose that a is even, $u_{a/2} = \pm 1$, $|u_a| \geq 3$, and $\overline{\omega}(v_{a/2}) = a/2$. Then $u_a | v_b$ if and only if $b \equiv a/2 \pmod{a}$.

Proof: Since $u_a = u_{a/2} v_{a/2}$ by Lemma 3.1, $u_a = \pm v_{a/2}$. Note that $b \equiv a/2 \pmod{a}$ if and only if $b = (2k+1)(a/2)$ for some $k \geq 0$. The result now follows from Lemma 3.12 (ii). □

4. PROOFS OF THE MAIN THEOREMS

We are now ready for the proofs of Theorems 2.6 and 2.8.

Proof of Theorem 2.6: A substantial part of this proof deals with examining nondegenerate Lucas sequences $u(r,s)$ for which $gcd(r,s) = 1$ and either $u_{a/2}$ or u_a has no primitive prime divisor. Since there exist infinitely many such Lucas sequences when $a/2$ or $a = 1$, 2, 3, 4 or 6 by Table 3 of [1], we will treat these cases separately.

If $u_a = 1$, then clearly $u_a | v_n$ for all $n \geq 1$. The result for the case in which $a = 1$ now follows since $u_1 = 1$. Suppose that $a = 2$. Then $u_2 = r = v_1$. If $|r| = 1$ or 2, then $u_2 | v_n$ for all n by our above observation and by Lemma 3.5 (iii). If $|r| \geq 3$, then condition (iii) holds for Lemma 3.18.

Now assume that $a = 3$. By Lemma 3.15, u_3 is Lucasian only if $u_3 = \pm 1$ or $u_3 = \pm 2$. We note that $u_3 = r^2 + s = \pm 1$ if and only if $s = \pm 1 - r^2$, and condition (iv) is satisfied. If $u_3 = r^2 + s = \pm 2$, then $s = \pm 2 - r^2$. Hence, $s \equiv r \pmod 2$. Consequently, r and s are both odd since $gcd(r, s) = 1$. By Lemma 3.5 (ii), it follows that condition (v) holds.

Next suppose that $a = 4$. By Lemma 3.15 either $u_2 = r = \pm 1$ or $v_2 = r^2 + 2s = \pm 1$. If $u_2 = \pm 1$, then

$$u_4 = u_2 v_2 = \pm v_2 = \pm(r^2 + 2s) = \pm(2s + 1). \tag{9}$$

We claim that $|v_2| \geq 3$. Suppose that $v_2 = \pm 1$. By (9), either $2s + 1 = 1$ or $2s + 1 = -1$. If $2s + 1 = 1$, then $s = 0$, which is excluded since $u(r, s)$ and $v(r, s)$ are both nondegenerate. If $2s + 1 = -1$, then $u(r, s)$ and $v(r, s)$ would be degenerate by Theorem 2.12 (iv), contrary to assumption. Noting that $v_2 = 2s + 1$ is odd, we see that $|u_4| = |v_2| \geq 3$. Since $v_1 = r = \pm 1$, it follows from Lemma 3.18 that condition (vi) holds. If $v_2 = r^2 + 2s = \pm 1$, then

$$u_4 = u_2 v_2 = \pm u_2 = \pm r = \pm v_1. \tag{10}$$

Moreover, $v_2 = \pm 1$ if and only if $s = (\pm 1 - r^2)/2$ and $r \equiv 1 \pmod 2$. If $r = \pm 1$, then $s = 0$ which contradicts the fact that both $u(r, s)$ and $v(r, s)$ are nondegenerate. Hence, $|u_4| = |u_2| = |r| = |v_1| \geq 3$. We now see by Lemma 3.18 that condition (vii) holds.

We now assume that $a = 6$. By Lemma 3.15, either $u_3 = \pm 1$ or $v_3 = \pm 1$. However, by the treatment of the case $a = 12$ in the latter part of the proof of Lemma 3.17, we see that v_3 cannot equal ± 1. Thus $u_3 = \pm 1$. Then $u_6 = u_3 v_3 = \pm v_3$. We will show that $|v_3| \geq 3$ and $\overline{\omega}(v_3) = 3$. Since $u_3 = r^2 + s$, it will then follow from Lemma 3.18 that condition (xii) holds.

We note that $v_3 \neq 0$ since $v(r, s)$ is nondegenerate. Suppose that

$$0 < |v_3| = |r(r^2 + 3s)| \leq 2.$$

Then $r = \pm 1$ or $r = \pm 2$. Substituting these values into $s = 1 - r^2$, and noting that $s \neq 0$, we obtain a value for $|v_3| = |r(r^2 + 3s)| > 3$ in each case.

We now claim that $\overline{\omega}(v_3) = 3$. If $\overline{\omega}(v_3) < 3$, then by Lemmas 3.3 and 3.12, $\overline{\omega}(v_3) = 1$ and $|v_1| = |v_3|$. If $|v_1| = |v_3|$, then, since $r \neq 0$, $r^2 + 3s = \pm 1$. Since, also, $r^2 + s = \pm 1$, we have $s = 0$ or ± 1; but $s \neq 0$ by Theorem 2.12 (i), and $r^2 + s = \pm 1$ is not possible for an integer $r \neq 0$ if $s = \pm 1$. Hence, $|v_1| \neq |v_3|$, and it follows that condition (xii) holds.

We next suppose that $a = 8$. By Lemma 3.15, $u_4 = \pm 1$ or $v_4 = \pm 1$. First suppose that

$$u_4 = u_2 v_2 = r(r^2 + 2s) = \pm 1.$$

Then $r = \pm 1$ and $r^2 + 2s = 2s + 1 = \pm 1$. If $2s + 1 = 1$, then $s = 0$, which is excluded since $u(r, s)$ is nondegenerate. If $2s + 1 = -1$, then $r = \pm 1$ and $s = -1$, which is also excluded by Lemma 2.12 (iv) since $u(r, s)$ is nondegenerate. Now assume that $v_4 = \pm 1$. Then $u_8 = u_4 v_4 = \pm u_4$, and u_8 has no primitive prime divisor. By Table 1 of [1], the only recurrences $u(r, s)$ for which u_8 has no primitive prime divisor are $u(\pm 2, -7)$ and $u(\pm 1, -2)$. By inspection, we see that $v_4(\pm 2, -7) = 2$, while $v_4(\pm 1, -2) = 1$. We further observe that if $r = \pm 1$ and $s = -2$, then

$$u_8 = u_4 = \pm v_2 = \pm 3,$$

while

$$u_2 = v_1 = \pm 1.$$

It now follows from Lemma 3.18 that condition (xiv) holds.

We now assume that $a \geq 5$ and a is odd. By Lemmas 3.15 and 3.17, it follows $a = 5$, 7, or 13 and $u_a = \pm 1$. Then u_a has no primitive prime divisor. By examining Table 1 of [1] and evaluating the term u_a in all recurrences $u(r,s)$ for which u_a has no primitive prime divisor, we see that u_a is Lucasian if and only if one of the conditions (viii), (ix), (x), (xi), (xiii), or (xix) holds.

Finally, we suppose that $a \geq 10$ and a is even. By Lemmas 3.15 and 3.17, we see that $a = 10$, 14, or 26 and either $v_{a/2} = \pm 1$ or $u_{a/2} = \pm 1$. If $v_{a/2} = \pm 1$, then $u_a = u_{a/2}v_{a/2} = \pm u_{a/2}$, and u_a has no primitive prime divisor. By Theorem 2.10, u_a has a primitive prime divisor if $a = 14$ or 26. Thus, $v_{a/2}$ can equal ± 1 only if $a = 10$. From Table 1 or [1], we see that $u_{10}(r,s)$ has no primitive prime divisor only if $r = \pm 2,\ s = -3,\ r = \pm 5, s = -7$, or $r = \pm 5,\ s = -18$. In each case, we observe that $v_5(r,s) \neq \pm 1$. Hence, we must have that $u_{a/2} = \pm 1$. Since $a/2 \geq 5$ and $a/2$ is odd, we obtain all the values of r and s for which $u_{a/2} = \pm 1$ from conditions (viii) - (xi), (xiii), and (xix). By inspection of these recurrences $u(r,s)$ and the corresponding recurrences $v(r,s)$, we obtain the values of $u_a(r,s)$ and ascertain that $|u_a(r,s)| \geq 3$ and $\overline{\omega}(v_{a/2}(r,s)) = a/2$. It follows from Lemma 3.18 that conditions (xv) - (xviii), (xx), and (xxi) hold. The results now follow. □

Proof of Theorem 2.8: By Theorem 2.4, if $|v_a| \geq 3$, then $v_a | u_b$ if and only if $2a | b$. Thus, we can asume that $|v_a| = 1$ or 2. First suppose that $v_a = \pm 1$. Then $u_{2a} = \pm u_a$, and u_{2a} is Lucasian if u_a is Lucasian. Using this observation, the proof of Theorem 2.6 determines all instances in which $v_a = \pm 1$ in the course of finding all terms u_{2a} which are Lucasian. It follows from conditions (ii), (vii), and (xiv) of Theorem 2.6 that $v_a = \pm 1$ if and only if conditions (ii), (iii), or (v) of Theorem 2.8 hold.

We now assume that $v_a = \pm 2$. If $a = 1$ and $v_1 = r = \pm 2$, it follows from Lemma 3.5 (iii) that $v_a | u_b$ if and only if $2a | b$. Now suppose that $a = 2$ and $v_2 = r^2 + 2s = \pm 2$. It follows that $v_2 = \pm 2$ if and only if $s = (\pm 2 - r^2)/2$ and r is even. Lemma 3.5 (iii) now implies that $v_2 | u_b$ if and only if condition (iv) of Theorem 2.8 holds.

Next assume that $a = 3$ and $v_3 = r(r^2 + 3s) = \pm 2$. Then $|r| = 1$ or 2. If $r = \pm 1$, then

$$r^2 + 3s = 3s + 1 = \pm 2.$$

Then $3s = 1$, which is impossible, or $3s = -3$, which yields $r = \pm 1,\ s = -1$. However, this case is excluded by Theorem 2.12 (iv), since $v(r,s)$ is nondegenerate.

Finally, assume that $a \geq 4$. By Lemma 3.5, we see that $\omega(2) \leq 3$. Hence $u_{2a} = u_a v_a$ has no primitive prime divisor. From Table 1 of [1], we find that there are exactly 10 Lucas sequences $u(r,s)$ for which some term u_{2n} has no primitive prime divisor for $n \geq 4$. By examining each of these recurrences, we see that $v_a = \pm 2$ for $a \geq 4$ if and only if r and s have the values given in conditions (vi) and (vii) of Theorem 2.8. We note that by Lemma 3.5 (iii), if $|r| = 2$ then $2 | u_n$ if and only if $2 | n$. The result now follows. □

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REFERENCES

[1] Bilu, Yu, Hanrot, G. and Voutier, P.M. "Existence of Primitive Divisors of Lucas and Lehmer Numbers." *J. Reine Angew. Math.*, Vol. *539* (2001): pp. 75-122.
[2] Carmichael, R.D. "On the Numerical Factors of the Arithmetic Forms $\alpha^n \pm \beta^n$." *Ann. Math.*, (Second Series), Vol. *15* (1913): pp. 30-70.
[3] Hilton, P. and Pedersen, J. "Fibonacci and Lucas Numbers in Teaching and Research." *Journées Mathématiques & Informatique*, Vol. *3* (1991-1992): pp. 36-57.
[4] Hilton, P., Pedersen, J. and Somer, L. "On Lucasian Numbers." *The Fibonacci Quarterly*, Vol. *35.1* (1997): pp. 43-47.
[5] Lucas, E. "Théorie des Fonctions Numériques Simplement Périodiques." *American J. Math*, Vol. *1* (1878): pp. 184-220, 289-321.
[6] McDaniel, W.L. "The *G.C.D.* in Lucas Sequences and Lehmer Number Sequences." *The Fibonacci Quarterly*, Vol. *29.1* (1991): pp. 24-29.
[7] Somer, L. "Divisibility of Terms in Lucas Sequences of the Second Kind by their Subscripts." Applications of Fibonacci Numbers. Volume **6**. Edited by G.E. Bergum, A.N. Philippou, and A.F. Horadam. Dordrecht; Kluwer (1996), pp. 473-486.
[8] Somer, L. Solution to Problem B-825 Proposed by L. Somer. *The Fibonacci Quarterly*, Vol. *35.4* (1997): p. 376.
[9] Ward, M. "Prime Divisors of Second Order Recurring Sequences." *Duke Math J.*, Vol. *21* (1954): pp. 607-614.

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