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# IDENTITIES INVOLVING PARTIAL DERIVATIVES OF BIVARIATE FIBONACCI AND LUCAS POLYNOMIALS

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## 1. INTRODUCTION

The work of Filipponi and Horadam in [2] and [3] revealed that the first- and second-order derivative sequences of Lucas type polynomials defined by $u_{n+1}(x) = u_n(x) + u_{n-1}(x)$ yield some nice recurrence properties. More precisely, in [2] and [3], some identities involving first- and second-order derivative sequences of the Fibonacci polynomials $U_n(x)$ and the Lucas polynomials $V_n(x)$ are established. These results may also be extended to the $k^{\text{th}}$ derivative case, as conjectured in [3] and recently confirmed in [7]. See also [4]. Furthermore, Filipponi and Horadam [5] considered the partial derivative sequences of bivariate second-order recurrence polynomials.

In this paper we shall extend some of the results established in [5] and derive some identities involving the partial derivative sequences of the bivariate Fibonacci polynomials $U_n(x, y)$ and the bivariate Lucas polynomials $V_n(x, y)$ defined respectively by (cf. [5])

$$U_n(x, y) = xU_{n-1}(x, y) + yU_{n-2}(x, y),\ n \geq 2,\ U_0(x, y) = 0, U_1(x, y) = 1, \tag{1}$$

$$V_n(x, y) = xV_{n-1}(x, y) + yV_{n-2}(x, y),\ n \geq 2,\ V_0(x, y) = 2, V_1(x, y) = x. \tag{2}$$

Moreover, we shall establish some convolution-type identities as counterparts to those given in [7]. As may be seen, these results, together with those in [6], explain in some sense the "heredity" of linearity under differentiation.

Throughout the paper we use $U_n$ and $V_n$, respectively, to denote $U_n(x, y)$ and $V_n(x, y)$. The partial derivatives of $U_n$ and $V_n$ are defined by

$$U_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} U_n, \quad V_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j} V_n,\ k \geq 0, j \geq 0. \tag{3}$$

Using an argument similar to that given in [1] or by induction, one may easily obtain the combinatorial expressions of $U_n$ and $V_n$ in terms of $x$ and $y$. They are:

$$U_n = \sum_{i=0}^{[(n-1)/2]} \binom{n-i-1}{i} x^{n-2i-1} y^i,\ n \geq 1, \tag{4}$$

$$V_n = \sum_{i=0}^{[n/2]} \frac{n}{n-i} \binom{n-i}{i} x^{n-2i} y^i,\ n \geq 1, \tag{5}$$

where $[a]$ denotes the greatest integer not exceeding $a$.

The extension of the bivariate Fibonacci and Lucas polynomials through the negative subscripts yields

$$U_{-n} = -(-y)^{-n} U_n \quad \text{and} \quad V_{-n} = (-y)^{-n} V_n,\ n > 0. \tag{6}$$

## 2. SOME IDENTITIES INVOLVING $U_n^{(k,j)}$ AND $V_n^{(k,j)}$

***Theorem 1:*** Let $n$ be any integer and let $k, j \geq 0$. Then the following identities hold:

***(i)*** $V_n^{(k,j)} = yU_{n-1}^{(k,j)} + jU_{n-1}^{(k,j-1)} + U_{n+1}^{(k,j)}$,

***(ii)*** $U_n^{(k,j)} = xU_{n-1}^{(k,j)} + yU_{n-2}^{(k,j)} + kU_{n-1}^{(k-1,j)} + jU_{n-2}^{(k,j-1)}$,

***(iii)*** $V_n^{(k,j)} = xV_{n-1}^{(k,j)} + yV_{n-2}^{(k,j)} + kV_{n-1}^{(k-1,j)} + jV_{n-2}^{(k,j-1)}$,

***(iv)*** $V_n^{(k+1,j)} = nU_n^{(k,j)}$, $V_n^{(k,j+1)} = nU_{n-1}^{(k,j)}$. Hence, $U_n^{(k,j)} = U_{n-j}^{(k+j,0)}$, $V_n^{(k,j)} = nV_{n-j}^{(k+j,0)}/(n-j)$.

***Proof:***

***(i)*** It is easy to show by induction that $V_n = yU_{n-1} + U_{n+1}$ for any integer $n$. Hence, we have

$$V_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(yU_{n-1} + U_{n+1}) = \frac{\partial^j}{\partial y^j}(yU_{n-1}^{(k,0)}) + U_{n+1}^{(k,j)} = yU_{n-1}^{(k,j)} + jU_{n-1}^{(k,j-1)} + U_{n+1}^{(k,j)}.$$

***(ii)*** From (1), we see that

$$U_n^{(k,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(xU_{n-1} + yU_{n-2}) = \frac{\partial^k}{\partial x^k}(xU_{n-1}^{(0,j)}) + \frac{\partial^j}{\partial y^j}(yU_{n-2}^{(k,0)})$$
$$= xU_{n-1}^{(k,j)} + kU_{n-1}^{(k-1,j)} + yU_{n-2}^{(k,j)} + jU_{n-2}^{(k,j-1)}.$$

***(iii)*** This result can be proved by a method similar to that shown in (ii).

***(iv)*** We first prove the case $(k, j) = (1, 0)$. This can be done by induction on $n$. The identity trivially holds when $n = 0, 1$. Suppose that $V_{n-1}^{(1,0)} = (n-1)U_{n-1}$ and $V_{n-2}^{(1,0)} = (n-2)U_{n-2}$ for $n \geq 2$. Then

$$V_n^{(1,0)} = \frac{\partial}{\partial x}(xV_{n-1} + yV_{n-2}) = xV_{n-1}^{(1,0)} + yV_{n-2}^{(1,0)} + V_{n-1}$$
$$= (n-1)xU_{n-1} + (n-2)yU_{n-2} + yU_{n-2} + U_n = nU_n.$$

From (6), it follows that

$$V_{-n}^{(1,0)} = \frac{\partial}{\partial x}((-y)^{-n}V_n) = (-y)^{-n}V_n^{(1,0)} = -nU_{-n}.$$

Similarly, we can prove that $V_n^{(0,1)} = nU_{n-1}$ for any integer $n$. Thus,

$$V_n^{(k+1,j)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}V_n^{(1,0)} = n\frac{\partial^{k+j}}{\partial x^k \partial y^j}U_n = nU_n^{(k,j)} \tag{7}$$

and

$$V_n^{(k,j+1)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}V_n^{(0,1)} = \frac{\partial^{k+j}}{\partial x^k \partial y^j}(nU_{n-1}) = nU_{n-1}^{(k,j)}. \tag{8}$$

It now follows from (7) and (8) that $U_n^{(k,j+1)} = V_n^{(k+1,j+1)}/n = U_{n-1}^{(k+1,j)}$. Hence, $U_n^{(k,j)} = U_{n-j}^{(k+j,0)}$ and $V_n^{(k,j)} = nU_n^{(k-1,j)} = nU_{n-j}^{(k+j-1,0)} = nV_{n-j}^{(k+j,0)}/(n-j)$ by (7). □

As expected, (i)-(iv) reduce to Identities 1-4 in [7] when $y = 1$ and $j = 0$.

## 3. CONVOLUTION-TYPE IDENTITIES INVOLVING $U_n^{(k,j)}$ AND $V_n^{(k,j)}$

***Theorem 2:*** For any $k, j \geq 0$, we have:

***(a)*** $\sum_{i=0}^{n} U_i^{(k,j)} U_{n-i} = \frac{1}{k+j+1} U_n^{(k+1,j)}$,

***(b)*** $\sum_{i=0}^{n} U_i^{(k,j)} V_{n-i} = \frac{n+k+1}{k+j+1} U_n^{(k,j)}$,

***(c)*** $\sum_{i=0}^{n} V_i^{(k,j)} U_{n-i} = \left[\delta(0, k+j) + \frac{n(k+j)+j}{(k+j+1)(k+j)}\right] U_n^{(k,j)}$,

***(d)*** $\sum_{i=0}^{n} V_i^{(k,j)} V_{n-i} = [1+\delta(0,k+j)] V_n^{(k,j)} + \frac{(n-1)(k+j)+j}{(k+j+1)(k+j)} U_{n-1}^{(k,j)} + \frac{(n+1)(k+j)+j}{(k+j+1)(k+j)} U_{n+1}^{(k,j)}$,

where $\delta(s,r) = 1(0)$ if $s = (\neq)\, r$ is the Kronecker symbol.

***Proof:***

***(a)*** Let $A_n^{(k)} = \Sigma_{i=0}^{n} U_i^{(k,0)} U_{n-i}$. Now it may be shown by an induction argument that $U_i$ is a monic bivariate polynomial whose highest leading term is $x^{i-1}$, so that $U_0^{(k,0)} = U_1^{(k,0)} = \cdots = U_k^{(k,0)} = 0$ and $U_{k+1}^{(k,0)} = k!$ Therefore, $A_k^{(k)} = A_{k+1}^{(k)} = 0$ and $A_{k+2}^{(k)} = U_{k+1}^{(k,0)} U_1 = k! = U_{k+2}^{(k+1,0)} / (k+1)$. Assume $A_{n-1}^{(k)} = U_{n-1}^{(k+1,0)} / (k+1)$ and $A_{n-2}^{(k)} = U_{n-2}^{(k+1,0)} / (k+1)$ for $n \geq 2$. Then, from the assumption and Theorem 1(ii), we have

$$\begin{aligned} A_n^{(k)} &= \sum_{i=0}^{n} U_i^{(k,0)} U_{n-i} = \sum_{i=0}^{n-1} U_i^{(k,0)} (xU_{n-1-i} + yU_{n-2-i}) \\ &= xA_{n-1}^{(k)} + yA_{n-2}^{(k)} + U_{n-1}^{(k,0)}(yU_{-1}) \\ &= \frac{1}{k+1}(xU_{n-1}^{(k+1,0)} + yU_{n-2}^{(k+1,0)} + (k+1)U_{n-1}^{(k,0)}) = \frac{1}{k+1} U_n^{(k+1,0)}. \end{aligned} \tag{9}$$

From (9) and Theorem 1(iv), we have

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k,j)} U_{n-i} &= \sum_{i=0}^{n} U_{i-j}^{(k+j,0)} U_{n-i} = \sum_{r=0}^{n-j} U_r^{(k+j,0)} U_{n-j-r} \\ &= A_{n-j}^{(k+j)} = \frac{1}{k+j+1} U_{n-j}^{(k+j+1,0)} = \frac{1}{k+j+1} U_n^{(k+1,j)}. \end{aligned} \tag{10}$$

***(b)*** Using (10) and the fact that $V_n = yU_{n-1} + U_{n+1}$ for any integer $n$ [see the proof of Theorem 1(i)], we have

$$\begin{aligned} \sum_{i=0}^{n} U_i^{(k,j)} V_{n-i} &= \sum_{i=0}^{n} U_i^{(k,j)} (yU_{n-i-1} + U_{n+1-i}) \\ &= y\sum_{i=0}^{n-1} U_i^{(k,j)} U_{n-1-i} + U_n^{(k,j)}(yU_{-1}) + \sum_{i=0}^{n+1} U_i^{(k,j)} U_{n+1-i} \\ &= \frac{1}{k+j+1}(yU_{n-1}^{(k+1,j)} + U_{n+1}^{(k+1,j)}) + U_n^{(k,j)} \end{aligned}$$

$$=\frac{1}{k+j+1}(V_n^{(k+1,j)}-jU_{n-1}^{(k+1,j-1)})+U_n^{(k,j)}$$

$$=\frac{1}{k+j+1}(nU_n^{(k,j)}-jU_n^{(k,j)})+U_n^{(k,j)}=\frac{n+k+1}{k+j+1}U_n^{(k,j)}. \tag{11}$$

Using Theorem 1 and an argument similar to that of (a), it is easy to prove (c) and (d). Hence, the proofs are omitted here. □

Finally, we give two generalizations of identity (a) in Theorem 2. It is worth mentioning that (b)-(d) possess similar generalized forms.

***Theorem 3:*** Let $k, j, r, s\ge 0$. Then

$$\sum_{i=0}^{n}U_i^{(k,j)}U_{n-i}^{(r,s)}=\left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1}U_n^{(k+r+1,j+s)}.$$

***Proof:*** Let $A_n^{(k,j,r)}=\sum_{i=0}^{n}U_i^{(k,j)}U_{n-i}^{(r,0)}$. First, we show by induction on $r$ that

$$A_n^{(k,j,r)}=\left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1}U_n^{(k+r+1,j)}. \tag{12}$$

The case $r=0$ is just Theorem 2(a). Suppose the above identity is true for some $r\ge 0$. Since

$$\frac{\partial}{\partial x}A_n^{(k,j,r)}=A_n^{(k+1,j,r)}+A_n^{(k,j,r+1)}=\left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1}U_n^{(k+r+2,j)},$$

we get

$$A_n^{(k,j,r+1)}=\left\{\left[(k+j+r+1)\binom{k+j+r}{r}\right]^{-1}-\left[(k+1+j+r+1)\binom{k+1+j+r}{r}\right]^{-1}\right\}U_n^{(k+r+2,j)}$$

$$=\left[(k+j+r+2)\binom{k+j+r+1}{r+1}\right]^{-1}\left(\frac{k+j+r+2}{r+1}-\frac{k+j+1}{r+1}\right)U_n^{(k+r+2,j)}$$

$$=\left[(k+j+r+2)\binom{k+j+r+1}{r+1}\right]^{-1}U_n^{(k+r+2,j)}.$$

Therefore, we have

$$\sum_{i=0}^{n}U_i^{(k,j)}U_{n-i}^{(r,s)}$$

$$=\sum_{i=0}^{n}U_i^{(r,s)}U_{n-i}^{(k,j)}=\sum_{i=s}^{n}U_{i-s}^{(r+s,0)}U_{n-i}^{(k,j)} \quad \text{[from Theorem 1(iv)]}$$

$$=\sum_{i=0}^{n-s}U_i^{(r+s,0)}U_{n-s-i}^{(k,j)}=\sum_{i=0}^{n-s}U_i^{(k,j)}U_{n-s-i}^{(r+s,0)}$$

$$=\left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1}U_{n-s}^{(k+r+s+1,j)} \quad \text{[from (12)]}$$

$$=\left[(k+j+r+s+1)\binom{k+j+r+s}{r+s}\right]^{-1}U_n^{(k+r+1,j+s)} \quad \text{[from Theorem 2(iv)]}. \ \square$$

***Theorem 4:*** Let $k, j \geq 0$ and $t \geq 2$. Then

$$\sum_{i_1+i_2+\cdots+i_t=n} U_{i_1}^{(k,j)} U_{i_2}^{(k,j)} \cdots U_{i_t}^{(k,j)} = \prod_{i=2}^{t} \left[ (i\alpha - 1) \binom{i\alpha - 2}{\alpha} \right]^{-1} U_n^{(tk+t-1, tj)}$$

where $\alpha = k + j + 1$.

The proof of Theorem 4 can be carried out by induction on $t$ and is omitted here for the sake of brevity.

## 4. CONCLUDING REMARKS

The bivariate polynomials defined by (3) and (4) may be used to obtain identities for $k^{\text{th}}$ derivative sequences of Pell and Pell-Lucas polynomials by taking $x = 2$, $y = 1$, and $j = 0$ [6]. It is likely that this kind of bivariate treatment may also be extended to the bivariate integration sequences $\iint U_n \, dx \, dy$ to parallel some identities found in [4].

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