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Several identities involving the Fibonacci polynomials and Lucas polynomials

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Abstract
In this paper, the authors consider infinite sums derived from the reciprocals of the Fibonacci polynomials and Lucas polynomials. Then applying the floor function to the reciprocals of these sums, the authors obtain several new identities involving the Fibonacci polynomials and Lucas polynomials.
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1 Introduction

For any variable quantity x, the Fibonacci polynomials $F_n(x)$ and Lucas polynomials $L_n(x)$ are defined by $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$, $n \geq 0$, with the initial values $F_0(x) = 0$ and $F_1(x) = 1$; $L_{n+2}(x) = xL_{n+1}(x) + L_n(x)$, $n \geq 0$, with the initial values $L_0(x) = 2$ and $L_1(x) = x$. For $x = 1$, we obtain the usual Fibonacci numbers and Lucas numbers. Let $\alpha = \frac{1}{2}(x + \sqrt{x^2+4})$ and $\beta = \frac{1}{2}(x - \sqrt{x^2+4})$, then from the properties of second-order linear recurrence sequences, we have

$$F_n(x) = \frac{\alpha^n - \beta^n}{\sqrt{x^2+4}} \quad \text{and} \quad L_n(x) = \alpha^n + \beta^n.$$

Various authors studied the properties of the Fibonacci polynomials and Lucas polynomials and obtained many interesting results; see [1–3], and [4].

The so-called Fibonacci zeta function and Lucas zeta function defined by

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s} \quad \text{and} \quad \zeta_L(s) = \sum_{n=1}^{\infty} \frac{1}{L_n^s},$$

where the F_n and L_n denote the Fibonacci numbers and Lucas numbers, have been considered in several different ways. Navas [5] discussed the analytic continuation of these series. In [6] it is shown that for any positive distinct integer s_1, s_2, s_3, the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1, s_2, s_3 is even.

Ohtsuka and Nakamura [7] studied the partial infinite sums of reciprocal Fibonacci numbers and proved the following conclusions:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_{n-1}F_n - 1 & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-1}F_n & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Wu and Zhang [8] generalized these identities to the Fibonacci polynomials and Lucas polynomials. Similar properties were investigated in several different ways; see [9, 10], and [11]. Recently, some authors considered the nearest integer of the sum of reciprocal Fibonacci numbers and other famous sequences and obtained several new interesting identities; see [12] and [13]. Kilic and Arikan [14] defined a kth-order linear recursive sequence u_n for any positive integer p, q and $n > k$ as follows:

$$u_n = pu_{n-1} + qu_{n-2} + u_{n-3} + \cdots + u_{n-k},$$

and they proved that there exists a positive integer n_0 such that

$$\left\| \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\| = u_n - u_{n-1} \quad (n \geq n_0),$$

where $\| \cdot \|$ denotes the nearest integer. (Clearly, $\|x\| = \lfloor x + \frac{1}{2} \rfloor$.)

In this paper, we consider the subseries of infinite sums derived from the reciprocals of the Fibonacci polynomials and Lucas polynomials and prove the following.

Theorem 1 *For any positive integer x, n and even $a \geq 2$, we have*

(1) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{ak}(x)} \right)^{-1} \right\rfloor = F_{an}(x) - F_{an-a}(x) - 1 \quad (n \geq 1).$$

(2) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_{ak}(x)} \right)^{-1} \right\rfloor = L_{an}(x) - L_{an-a}(x) \quad (n \geq 1).$$

(3) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{ak}^2(x)} \right)^{-1} \right\rfloor = F_{an}^2(x) - F_{an-a}^2(x) - 1 \quad (n \geq 1).$$

(4) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_{ak}^2(x)} \right)^{-1} \right\rfloor = L_{an}^2(x) - L_{an-a}^2(x) + 1 \quad (n \geq 2).$$

Theorem 2 *For any positive integer x and odd $b \geq 1$, we have*

(1) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{bk}(x)} \right)^{-1} \right\rfloor = \begin{cases} F_{bn}(x) - F_{bn-b}(x) & \text{if } n \text{ is even and } n \geq 2; \\ F_{bn}(x) - F_{bn-b}(x) - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

(2) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_{bk}(x)} \right)^{-1} \right\rfloor = \begin{cases} L_{bn}(x) - L_{bn-b}(x) - 1 & \text{if } n \text{ is even and } n \geq 2; \\ L_{bn}(x) - L_{bn-b}(x) & \text{if } n \text{ is odd and } n \geq 3. \end{cases}$$

(3) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_{bk}^2(x)} \right)^{-1} \right\rfloor = \begin{cases} F_{bn}^2(x) - F_{bn-b}^2(x) & \text{if } n \text{ is even and } n \geq 2; \\ F_{bn}^2(x) - F_{bn-b}^2(x) - 1 & \text{if } n \text{ is odd and } n \geq 1. \end{cases}$$

Particularly, for $x \geq 2$, *we have*

(4) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{L_{bk}^2(x)} \right)^{-1} \right\rfloor = \begin{cases} L_{bn}^2(x) - L_{bn-b}^2(x) - 3 & \text{if } n \text{ is even and } n \geq 2; \\ L_{bn}^2(x) - L_{bn-b}^2(x) + 2 & \text{if } n \text{ is odd and } n \geq 3. \end{cases}$$

If $x = 1$, $b = 1$ (*respectively*, $x = 2$, $b = 1$) *or* $x = 1$, $b = 3$, *then from our theorems we can deduce the conclusions of* [7–10], *and* [11].

2 Proof of theorems

To complete the proof of our theorems, we need the following lemma.

Lemma *For any positive integer* x, m, *and* n,

$$F_m(x)F_n(x) = \frac{1}{x^2+4}\bigl(L_{m+n}(x) - (-1)^n L_{m-n}(x)\bigr), \tag{1}$$

$$L_m(x)L_n(x) = L_{m+n}(x) + (-1)^n L_{m-n}(x), \tag{2}$$

$$F_m(x)L_n(x) = F_{m+n}(x) + (-1)^n F_{m-n}(x) = F_{m+n}(x) - (-1)^m F_{n-m}(x), \tag{3}$$

$$F_{n-1}(x) + F_{n+1}(x) = L_n(x). \tag{4}$$

Proof We only prove identities (1) and (4), and other identities are proved similarly. For any positive integer x, m, and n, from the identity

$$F_n(x) = \frac{\alpha^n - \beta^n}{\sqrt{x^2+4}} \quad \text{and} \quad L_n(x) = \alpha^n + \beta^n,$$

we have

$$\begin{aligned} F_m(x)F_n(x) &= \frac{(\alpha^m - \beta^m)(\alpha^n - \beta^n)}{x^2+4} = \frac{(\alpha^{m+n} + \beta^{m+n} - \alpha^n\beta^m - \alpha^m\beta^n)}{x^2+4} \\ &= \frac{1}{x^2+4}\bigl(L_{m+n}(x) - (-1)^n L_{m-n}(x)\bigr), \end{aligned}$$

and

$$\begin{aligned} F_{n-1}(x) + F_{n+1}(x) &= \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{x^2+4}} + \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{x^2+4}} = \frac{\alpha^n(\alpha + \alpha^{-1}) - \beta^n(\beta + \beta^{-1})}{\sqrt{x^2+4}} \\ &= \frac{(\sqrt{x^2+4})\alpha^n + (\sqrt{x^2+4})\beta^n}{\sqrt{x^2+4}} = \alpha^n + \beta^n = L_n(x). \end{aligned}$$

This proves identities (1) and (4). □

Now we shall complete the proof of our theorems. We shall prove only Theorems 1(1), 1(3), 2(1), and 2(4), and other identities are proved similarly and omitted. First, we prove Theorem 1(1).

Proof of Theorem 1(1) Theorem 1(1) is equivalent to

$$\frac{1}{F_{an}(x) - F_{an-a}(x)} < \sum_{k=n}^{\infty} \frac{1}{F_{ak}(x)} \leq \frac{1}{F_{an}(x) - F_{an-a}(x) - 1}. \tag{5}$$

For any positive integer x, k and even $a \geq 2$, using identity (1), we have

$$\begin{aligned}
&\frac{1}{F_{ak}(x)} - \frac{1}{F_{ak}(x) - F_{ak-a}(x)} - \frac{1}{F_{ak+a}(x) - F_{ak}(x)} \\
&\quad = \frac{1}{F_{ak+a}(x) - F_{ak}(x)} - \frac{F_{ak-a}(x)}{F_{ak}(x)(F_{ak}(x) - F_{ak-a}(x))} \\
&\quad = \frac{F_{ak}^2(x) - F_{ak-a}(x)F_{ak+a}(x)}{F_{ak}(x)(F_{ak}(x) - F_{ak-a}(x))(F_{ak+a}(x) - F_{ak}(x))} \\
&\quad = \frac{L_{2a}(x) - 2}{F_{ak}(x)(x^2 + 4)(F_{ak}(x) - F_{ak-a}(x))(F_{ak+a}(x) - F_{ak}(x))}.
\end{aligned} \tag{6}$$

Since $F_n(x)$ and $L_n(x)$ are monotone increasing for n and a fixed positive integer x, we have $L_{2a}(x) - 2 > 0$, $F_{ak}(x) - F_{ak-a}(x) > 0$, and $F_{ak+a}(x) - F_{ak}(x) > 0$ for any positive integer x, k and even $a \geq 2$. Hence the numerator of the right-hand side of the above identity is positive for any positive integer x, k and even $a \geq 2$, so we get

$$\frac{1}{F_{ak}(x)} > \frac{1}{F_{ak}(x) - F_{ak-a}(x)} - \frac{1}{F_{ak+a}(x) - F_{ak}(x)}. \tag{7}$$

Using (7) repeatedly, we have

$$\begin{aligned}
\sum_{k=n}^{\infty} \frac{1}{F_{ak}(x)} &> \sum_{k=n}^{\infty} \left(\frac{1}{F_{ak}(x) - F_{ak-a}(x)} - \frac{1}{F_{ak+a}(x) - F_{ak}(x)} \right) \\
&= \frac{1}{F_{an}(x) - F_{an-a}(x)} - \frac{1}{F_{an+a}(x) - F_{an}(x)} + \frac{1}{F_{an+a}(x) - F_{an}(x)} \\
&\quad - \frac{1}{F_{an+2a}(x) - F_{an+a}(x)} + \frac{1}{F_{an+2a}(x) - F_{an+a}(x)} - \cdots \\
&= \frac{1}{F_{an}(x) - F_{an-a}(x)}.
\end{aligned} \tag{8}$$

On the other hand, we prove that for any positive integer x, k and even $a \geq 2$,

$$\frac{1}{F_{ak}(x)} < \frac{1}{F_{ak}(x) - F_{ak-a}(x) - 1} - \frac{1}{F_{ak+a}(x) - F_{ak}(x) - 1}. \tag{9}$$

Inequality (9) is equivalent to

$$\frac{F_{ak-a}(x) + 1}{F_{ak}(x)(F_{ak}(x) - F_{ak-a}(x) - 1)} > \frac{1}{F_{ak+a}(x) - F_{ak}(x) - 1},$$

or

$$F_{ak-a}(x)F_{ak+a}(x) - F_{ak-a}(x) + F_{ak+a}(x) - 1 > F_{ak}^2(x).$$

Using identity (1), the above inequality is equivalent to

$$\begin{aligned} &F_{ak+a+2}(x) + F_{ak+a-2}(x) - F_{ak-a+2}(x) - F_{ak-a-2}(x) \\ &\quad + 2F_{ak+a}(x) - 2F_{ak-a}(x) - L_{2a}(x) - L_2(x) > 0, \end{aligned}$$

or

$$\begin{aligned} &\big(F_{ak+a+2}(x) - L_{2a}(x)\big) + \big(F_{ak+a}(x) - F_{ak-a+2}(x)\big) \\ &\quad + \big(F_{ak+a-2}(x) - F_{ak-a-2}(x) - L_2(x)\big) + \big(F_{ak+a}(x) - 2F_{ak-a}(x)\big) > 0. \end{aligned} \tag{10}$$

Since $F_n(x)$ and $L_n(x)$ are monotone increasing for n and a fixed positive integer x, we have $F_{ak+a}(x) - F_{ak-a+2}(x) > 0$, $F_{ak+a-2}(x) - F_{ak-a-2}(x) - L_2(x) > 0$, and $F_{ak+a}(x) - 2F_{ak-a}(x) > 0$ for any positive integer x, k and even $a \geq 2$. Using identity (4), we have

$$\begin{aligned} F_{ak+a+2}(x) - L_{2a}(x) &> F_{2a+2}(x) - L_{2a}(x) \\ &= xF_{2a+1}(x) + F_{2a}(x) - F_{2a+1}(x) - F_{2a-1}(x) \\ &= (x-1)F_{2a+1}(x) + F_{2a}(x) - F_{2a-1}(x) > 0. \end{aligned}$$

Hence the numerator of each part in parentheses of the left-hand side of inequality (10) is positive, so inequality (10) holds for any positive integer x, k and even $a \geq 2$. Hence inequality (9) is true. Using (9) repeatedly, we have

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{F_{ak}(x)} &< \sum_{k=n}^{\infty} \left(\frac{1}{F_{ak}(x) - F_{ak-a}(x) - 1} - \frac{1}{F_{ak+a}(x) - F_{ak}(x) - 1} \right) \\ &= \frac{1}{F_{an}(x) - F_{an-a}(x) - 1}. \end{aligned} \tag{11}$$

Now inequality (5) follows from (8) and (11). This proves Theorem 1(1). □

Proof of Theorem 1(3) Now we prove Theorem 1(3). Theorem 1(3) is equivalent to

$$\frac{1}{F_{an}^2(x) - F_{an-a}^2(x)} < \sum_{k=n}^{\infty} \frac{1}{F_{ak}^2(x)} \leq \frac{1}{F_{an}^2(x) - F_{an-a}^2(x) - 1}. \tag{12}$$

For any positive integer x, k and even $a \geq 2$, using identities (1) and (2), we have

$$\begin{aligned} &\frac{1}{F_{ak}^2(x)} - \frac{1}{F_{ak}^2(x) - F_{ak-a}^2(x)} - \frac{1}{F_{ak+a}^2(x) - F_{ak}^2(x)} \\ &\quad = \frac{F_{ak}^4(x) - F_{ak-a}^2(x)F_{ak+a}^2(x)}{F_{ak}^2(x)(F_{ak}^2(x) - F_{ak-a}^2(x))(F_{ak+a}^2(x) - F_{ak}^2(x))} \\ &\quad = \frac{(F_{ak}^2(x) - F_{ak-a}(x)F_{ak+a}(x))(F_{ak}^2(x) + F_{ak-a}(x)F_{ak+a}(x))}{F_{ak}^2(x)(F_{ak}^2(x) - F_{ak-a}^2(x))(F_{ak+a}^2(x) - F_{ak}^2(x))} \\ &\quad = \frac{(L_{2a}(x) - 2)(F_{ak}^2(x) + F_{ak-a}(x)F_{ak+a}(x))}{F_{ak}^2(x)(x^2+4)(F_{ak}^2(x) - F_{ak-a}^2(x))(F_{ak+a}^2(x) - F_{ak}^2(x))}. \end{aligned} \tag{13}$$

Since $F_n(x)$ and $L_n(x)$ are monotone increasing for n and a fixed positive integer x, we have $L_{2a}(x) - 2 > 0$, $F^2_{ak}(x) - F^2_{ak-a}(x) > 0$, and $F^2_{ak+a}(x) - F^2_{ak}(x) > 0$ for any positive integer x, k and even $a \geq 2$. Hence the numerator of the right-hand side of the above identity is positive for any positive integer x, k and even $a \geq 2$, so we get

$$\frac{1}{F^2_{ak}(x)} > \frac{1}{F^2_{ak}(x) - F^2_{ak-a}(x)} - \frac{1}{F^2_{ak+a}(x) - F^2_{ak}(x)}. \tag{14}$$

Using (14) repeatedly, we have

$$\sum_{k=n}^{\infty} \frac{1}{F^2_{ak}(x)} > \sum_{k=n}^{\infty} \left(\frac{1}{F^2_{ak}(x) - F^2_{ak-a}(x)} - \frac{1}{F^2_{ak+a}(x) - F^2_{ak}(x)} \right) = \frac{1}{F^2_{an}(x) - F^2_{an-a}(x)}. \tag{15}$$

On the other hand, we prove that for any positive integer x, k and even $a \geq 2$,

$$\frac{1}{F^2_{ak}(x)} < \frac{1}{F^2_{ak}(x) - F^2_{ak-a}(x) - 1} - \frac{1}{F^2_{ak+a}(x) - F^2_{ak}(x) - 1}. \tag{16}$$

Inequality (16) is equivalent to

$$\frac{F^2_{ak-a}(x) + 1}{F^2_{ak}(x)(F^2_{ak}(x) - F^2_{ak-a}(x) - 1)} > \frac{1}{F^2_{ak+a}(x) - F^2_{ak}(x) - 1},$$

or

$$F^2_{ak-a}(x)F^2_{ak+a}(x) - F^4_{ak}(x) + F^2_{ak+a}(x) - F^2_{ak-a}(x) - 1 > 0.$$

Using identities (1) and (2), the above inequality is equivalent to

$$\left(x^2 + 2\right)L_{2ak+2a}(x) + 4L_{2ak}(x) - \left(x^2 + 6\right)L_{2ak-2a}(x) - 4 + L_{4a}(x) - L_4(x) > 0. \tag{17}$$

Since $L_n(x)$ are monotone increasing for n and a fixed positive integer x, we have $L_{4a}(x) - L_4(x) > 0$ for any positive integer x and even $a \geq 2$. On the other hand, we have

$$\begin{aligned} &\left(x^2 + 2\right)L_{2ak+2a}(x) + 4L_{2ak}(x) - \left(x^2 + 6\right)L_{2ak-2a}(x) - 4 \\ &\quad > \left(x^2 + 6\right)L_{2ak}(x) - \left(x^2 + 6\right)L_{2ak-2a}(x) - 4 > \left(x^2 + 6\right) - 4 > 0. \end{aligned}$$

Hence the numerator of the left-hand side of inequality (17) is positive, so inequality (17) holds for any positive integer x, k and even $a \geq 2$. Hence inequality (16) is true. Using (16) repeatedly, we have

$$\begin{aligned} \sum_{k=n}^{\infty} \frac{1}{F^2_{ak}(x)} &< \sum_{k=n}^{\infty} \left(\frac{1}{F^2_{ak}(x) - F^2_{ak-a}(x) - 1} - \frac{1}{F^2_{ak+a}(x) - F^2_{ak}(x) - 1} \right) \\ &= \frac{1}{F^2_{an}(x) - F^2_{an-a}(x) - 1}. \end{aligned} \tag{18}$$

Now inequality (12) follows from (15) and (18). This proves Theorem 1(3). □

Proof of Theorem 2(1) First we consider the case that $n = 2m \geq 2$ is even. At this time, for any odd $b \geq 1$, Theorem 2(1) is equivalent to

$$\frac{1}{F_{2mb}(x) - F_{2mb-b}(x) + 1} < \sum_{k=2m}^{\infty} \frac{1}{F_{bk}(x)} \leq \frac{1}{F_{2mb}(x) - F_{2mb-b}(x)}. \tag{19}$$

Now we prove that for any positive integer x, k and odd $b \geq 1$,

$$\frac{1}{F_{2bk}(x)} + \frac{1}{F_{2bk+b}(x)} < \frac{1}{F_{2bk}(x) - F_{2bk-b}(x)} - \frac{1}{F_{2bk+2b}(x) - F_{2bk+b}(x)}. \tag{20}$$

Inequality (20) is equivalent to

$$\frac{F_{2bk+2b}(x)}{F_{2bk+b}(x)(F_{2bk+2b}(x) - F_{2bk+b}(x))} < \frac{F_{2bk-b}(x)}{F_{2bk}(x)(F_{2bk}(x) - F_{2bk-b}(x))}.$$

Using identities (1) and (3), the above inequality is equivalent to

$$\bigl(F_{2bk+4b}(x) - F_{2bk+b}(x)\bigr) + \bigl(2F_{2bk+2b}(x) - 2F_{2bk-b}(x)\bigr) + \bigl(F_{2bk}(x) - F_{2bk-3b}(x)\bigr) > 0. \tag{21}$$

Since $F_n(x)$ is monotone increasing for n and a fixed positive integer x, we have $F_{2bk+4b}(x) - F_{2bk+b}(x) > 0$, $2F_{2bk+2b}(x) - 2F_{2bk-b}(x) > 0$, and $F_{2bk}(x) - F_{2bk-3b}(x) > 0$ for any positive integer x, k and odd $b \geq 1$. Hence the numerator of each part in parentheses of the left-hand side of inequality (21) is positive, so inequality (21) holds for any positive integer x, k and odd $b \geq 1$. Hence inequality (20) is true. Using (20) repeatedly, we have

$$\begin{aligned}
\sum_{k=2m}^{\infty} \frac{1}{F_{bk}(x)} &= \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk}(x)} + \frac{1}{F_{2bk+b}(x)} \right) \\
&< \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk}(x) - F_{2bk-b}(x)} - \frac{1}{F_{2bk+2b}(x) - F_{2bk+b}(x)} \right) \\
&= \frac{1}{F_{2bm}(x) - F_{2bm-b}(x)}.
\end{aligned} \tag{22}$$

On the other hand, we prove that for any positive integer x, k and odd $b \geq 1$,

$$\frac{1}{F_{2bk}(x)} + \frac{1}{F_{2bk+b}(x)} > \frac{1}{F_{2bk}(x) - F_{2bk-b}(x) + 1} - \frac{1}{F_{2bk+2b}(x) - F_{2bk+b}(x) + 1}. \tag{23}$$

Inequality (23) is equivalent to

$$\frac{F_{2bk+2b}(x) + 1}{F_{2bk+b}(x)(F_{2bk+2b}(x) - F_{2bk+b}(x) + 1)} > \frac{F_{2bk-b}(x) - 1}{F_{2bk}(x)(F_{2bk}(x) - F_{2bk-b}(x) + 1)},$$

or

$$\begin{aligned}
&F_{2bk}(x)\bigl(F_{2bk+2b}(x) + 1\bigr)\bigl(F_{2bk}(x) - F_{2bk-b}(x) + 1\bigr) \\
&\quad > F_{2bk+b}(x)\bigl(F_{2bk-b}(x) - 1\bigr)\bigl(F_{2bk+2b}(x) - F_{2bk+b}(x) + 1\bigr).
\end{aligned}$$

Using identities (1) and (3), the above inequality is equivalent to

$$\begin{aligned} &L_{4bk+3b}(x) - L_{4bk-b}(x) - F_{2bk+4b}(x) - 2F_{2bk+2b}(x) + F_{2bk+b+2}(x) \\ &\quad + 3F_{2bk+b}(x) - 2L_{2b}(x) - 4 + F_{2bk+b-2}(x) + F_{2bk+2}(x) + F_{2bk}(x) + F_{2bk-2}(x) \\ &\quad + 2F_{2bk-b}(x) + F_{2bk-3b}(x) > 0. \end{aligned} \tag{24}$$

Since $F_n(x)$ and $L_n(x)$ are monotone increasing for n and a fixed positive integer x, using identity (4), we have

$$\begin{aligned} &L_{4bk+3b}(x) - L_{4bk-b}(x) - F_{2bk+4b}(x) - 2F_{2bk+2b}(x) \\ &\quad = F_{4bk+3b+1}(x) + F_{4bk+3b-1}(x) - F_{4bk-b+1}(x) - F_{4bk-b-1}(x) - F_{2bk+4b}(x) - 2F_{2bk+2b}(x) \\ &\quad > \left(2x^2 + x + 4\right)F_{4bk+3b-3}(x) - 5F_{4bk-b+1}(x) > 0, \end{aligned}$$

and

$$\begin{aligned} &F_{2bk+b+2}(x) + 3F_{2bk+b}(x) - 2L_{2b}(x) - 4 \\ &\quad = xF_{2bk+b+1}(x) - 4 + 4F_{2bk+b}(x) - 2F_{2b+1}(x) - 2F_{2b-1}(x) > 0. \end{aligned}$$

Hence the numerator of each part in parentheses of the left-hand side of inequality (24) is positive, so inequality (24) holds for any positive integer x, k and odd $b \geq 1$. Hence inequality (23) is true. Using (23) repeatedly, we have

$$\begin{aligned} \sum_{k=2m}^{\infty} \frac{1}{F_{bk}(x)} &= \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk}(x)} + \frac{1}{F_{2bk+b}(x)} \right) \\ &> \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk}(x) - F_{2bk-b}(x) + 1} - \frac{1}{F_{2bk+2b}(x) - F_{2bk+b}(x) + 1} \right) \\ &= \frac{1}{F_{2bm}(x) - F_{2bm-b}(x) + 1}. \end{aligned} \tag{25}$$

Now inequality (19) follows from (22) and (25).

Similarly, we can consider the case that $n = 2m + 1 \geq 1$ is odd. At this time, for any odd $b \geq 1$, Theorem 2(1) is equivalent to the inequality

$$\begin{aligned} \frac{1}{F_{2bm+b}(x) - F_{2bm}(x)} &< \sum_{k=2m+1}^{\infty} \frac{1}{F_{bk}(x)} \\ &\leq \frac{1}{F_{2bm+b}(x) - F_{2bm}(x) - 1}. \end{aligned} \tag{26}$$

First we can prove that for any positive integer x, k and odd $b \geq 1$,

$$\frac{1}{F_{2bk+b}(x)} + \frac{1}{F_{2bk+2b}(x)} < \frac{1}{F_{2bk+b}(x) - F_{2bk}(x) - 1} - \frac{1}{F_{2bk+3b}(x) - F_{2bk+2b}(x) - 1}. \tag{27}$$

Inequality (27) is equivalent to

$$\frac{F_{2bk+3b}(x)-1}{F_{2bk+2b}(x)(F_{2bk+3b}(x)-F_{2bk+2b}(x)-1)} < \frac{F_{2bk}(x)+1}{F_{2bk+b}(x)(F_{2bk+b}(x)-F_{2bk}(x)-1)}.$$

Using identities (1) and (3), the above inequality is equivalent to

$$\begin{aligned} &L_{4bk+5b}(x)-L_{4bk+b}(x)-F_{2bk+5b}(x)-2F_{2bk+3b}(x)+\bigl(L_2(x)+3\bigr)F_{2bk+2b}(x) \\ &\quad -\bigl(L_2(x)+1\bigr)F_{2bk+b}(x)+F_{2bk}(x)+2F_{2bk-2b}(x)+2L_{2b}(x)+4>0. \end{aligned} \tag{28}$$

Since $F_n(x)$ and $L_n(x)$ are monotone increasing for n and a fixed positive integer x, using identity (4), we have

$$\begin{aligned} &L_{4bk+5b}(x)-L_{4bk+b}(x)-F_{2bk+5b}(x)-2F_{2bk+3b}(x) \\ &\quad = F_{4bk+5b+1}(x)+F_{4bk+5b-1}(x)-F_{4bk+b+1}(x)-F_{4bk+b-1}(x)-F_{2bk+5b}(x)-2F_{2bk+3b}(x) \\ &\quad > \bigl(x^3+3x\bigr)F_{4bk+5b-2}(x)+\bigl(x^2+2\bigr)F_{4bk+5b-3}(x)-5F_{4bk+b+1}(x)>0, \end{aligned}$$

and

$$\bigl(L_2(x)+3\bigr)F_{2bk+2b}(x)-\bigl(L_2(x)+1\bigr)F_{2bk+b}(x)>0.$$

Hence the numerator of each part in parentheses of the left-hand side of inequality (28) is positive, so inequality (28) holds for any positive integer x, k and odd $b \geq 1$. Hence inequality (27) is true. Using (27) repeatedly, we have

$$\begin{aligned} \sum_{k=2m+1}^{\infty}\frac{1}{F_{bk}(x)} &= \sum_{k=m}^{\infty}\left(\frac{1}{F_{2bk+b}(x)}+\frac{1}{F_{2bk+2b}(x)}\right) \\ &< \sum_{k=m}^{\infty}\left(\frac{1}{F_{2bk+b}(x)-F_{2bk}(x)-1}-\frac{1}{F_{2bk+3b}(x)-F_{2bk+2b}(x)-1}\right) \\ &= \frac{1}{F_{2bm+b}(x)-F_{2bm}(x)-1}. \end{aligned} \tag{29}$$

On the other hand, we prove that for any positive integer x, k and odd $b \geq 1$,

$$\frac{1}{F_{2bk+b}(x)}+\frac{1}{F_{2bk+2b}(x)} > \frac{1}{F_{2bk+b}(x)-F_{2bk}(x)}-\frac{1}{F_{2bk+3b}(x)-F_{2bk+2b}(x)}. \tag{30}$$

Inequality (30) is equivalent to

$$\frac{F_{2bk+3b}(x)}{F_{2bk+2b}(x)(F_{2bk+3b}(x)-F_{2bk+2b}(x))} > \frac{F_{2bk}(x)}{F_{2bk+b}(x)(F_{2bk+b}(x)-F_{2bk}(x))}.$$

Using identities (1) and (3), the above inequality is equivalent to

$$F_{2bk+5b}(x)+2F_{2bk+3b}(x)-F_{2bk+2b}(x)-F_{2bk+b}(x)-2F_{2bk}(x)-2F_{2bk-2b}(x)>0. \tag{31}$$

Since $F_n(x)$ is monotone increasing for n and a fixed positive integer x, using identity (4), we have

$$\begin{aligned}
&F_{2bk+5b}(x) + 2F_{2bk+3b}(x) - F_{2bk+2b}(x) - F_{2bk+b}(x) - 2F_{2bk}(x) - 2F_{2bk-2b}(x) \\
&\quad > xF_{2bk+5b-1}(x) + F_{2bk+5b-2}(x) + 2xF_{2bk+3b-1}(x) + 2F_{2bk+3b-2}(x) - 6F_{2bk+2b}(x) \\
&\quad > (3x+3)F_{2bk+3b-2}(x) - 6F_{2bk+2b}(x) > 0.
\end{aligned}$$

Hence the numerator of each part in parentheses of the left-hand side of inequality (31) is positive, so inequality (31) holds for any positive integer x, k and odd $b \geq 1$. Hence inequality (30) is true. Using (30) repeatedly, we have

$$\begin{aligned}
\sum_{k=2m+1}^{\infty} \frac{1}{F_{bk}(x)} &= \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk+b}(x)} + \frac{1}{F_{2bk+2b}(x)} \right) \\
&> \sum_{k=m}^{\infty} \left(\frac{1}{F_{2bk+b}(x) - F_{2bk}(x)} - \frac{1}{F_{2bk+3b}(x) - F_{2bk+2b}(x)} \right) \\
&= \frac{1}{F_{2bm+b}(x) - F_{2bm}(x)}.
\end{aligned} \tag{32}$$

Combining (29) and (32), we deduce inequality (26). This proves Theorem 2(1). □

Proof of Theorem 2(4) First we consider the case that $n = 2m \geq 2$ is even. At this time, for any odd $b \geq 1$, Theorem 2(4) is equivalent to

$$\frac{1}{L_{2bm}^2(x) - L_{2bm-b}^2(x) - 2} < \sum_{k=2m}^{\infty} \frac{1}{L_{bk}^2(x)} \leq \frac{1}{L_{2bm}^2(x) - L_{2bm-b}^2(x) - 3}. \tag{33}$$

Now we prove that for any positive integer k, $x \geq 2$ and odd $b \geq 1$,

$$\frac{1}{L_{2bk}^2(x)} + \frac{1}{L_{2bk+b}^2(x)} > \frac{1}{L_{2bk}^2(x) - L_{2bk-b}^2(x) - 2} - \frac{1}{L_{2bk+2b}^2(x) - L_{2bk+b}^2(x) - 2}. \tag{34}$$

Inequality (34) is equivalent to

$$\frac{L_{2bk+2b}^2(x) - 2}{L_{2bk+b}^2(x)(L_{2bk+2b}^2(x) - L_{2bk+b}^2(x) - 2)} > \frac{L_{2bk-b}^2(x) + 2}{L_{2bk}^2(x)(L_{2bk}^2(x) - L_{2bk-b}^2(x) - 2)}.$$

Using identity (2), the above inequality is equivalent to

$$\begin{aligned}
&4L_{8bk+4b}(x) - 4L_{8bk}(x) + 2L_{8bk+2b}(x) - L_{4bk+8b}(x) + 6L_{4bk+4b}(x) \\
&\quad - 3L_{4bk+2b}(x) - L_{4bk}(x) + 6L_{4bk-2b}(x) - L_{4bk-6b}(x) > 0.
\end{aligned} \tag{35}$$

Since $L_n(x)$ is monotone increasing for n and a fixed positive integer x, for any positive integer k, $x \geq 2$ and odd $b \geq 1$, we have $4L_{8bk+4b}(x) - 4L_{8bk}(x) > 0$, $2L_{8bk+2b}(x) - L_{4bk+8b}(x) > 0$, $6L_{4bk+4b}(x) - 3L_{4bk+2b}(x) - L_{4bk}(x) > 0$, $6L_{4bk-2b}(x) - L_{4bk-6b}(x) > 0$.

Hence the numerator of the left-hand side of inequality (35) is positive, so inequality (35) holds for any positive integer k, $x \geq 2$ and odd $b \geq 1$. Hence inequality (34) is true. Using (34) repeatedly, we have

$$\begin{aligned}
&\sum_{k=2m}^{\infty} \frac{1}{L_{bk}^2(x)} \\
&\quad = \sum_{k=m}^{\infty} \left(\frac{1}{L_{2bk}^2(x)} + \frac{1}{L_{2bk+b}^2(x)} \right) \\
&\quad > \sum_{k=m}^{\infty} \left(\frac{1}{L_{2bk}^2(x) - L_{2bk-b}^2(x) - 2} - \frac{1}{L_{2bk+2b}^2(x) - L_{2bk+b}^2(x) - 2} \right) \\
&\quad = \frac{1}{L_{2bm}^2(x) - L_{2bm-b}^2(x) - 2}. \qquad (36)
\end{aligned}$$

On the other hand, we prove that for any positive integer k, $x \geq 2$ and odd $b \geq 1$,

$$\frac{1}{L_{2bk}^2(x)} + \frac{1}{L_{2bk+b}^2(x)} < \frac{1}{L_{2bk}^2(x) - L_{2bk-b}^2(x) - 3} - \frac{1}{L_{2bk+2b}^2(x) - L_{2bk+b}^2(x) - 3}. \qquad (37)$$

Inequality (37) is equivalent to

$$\frac{L_{2bk+2b}^2(x) - 3}{L_{2bk+b}^2(x)(L_{2bk+2b}^2(x) - L_{2bk+b}^2(x) - 3)} > \frac{L_{2bk-b}^2(x) + 3}{L_{2bk}^2(x)(L_{2bk}^2(x) - L_{2bk-b}^2(x) - 3)}.$$

Using identity (3), the above inequality is equivalent to

$$\begin{aligned}
&L_{8bk+6b}(x) - 4L_{8bk+4b}(x) - 2L_{8bk+2b}(x) + 4L_{8bk}(x) - L_{8bk-2b}(x) + L_{4bk+8b}(x) \\
&\quad - 6L_{4bk+4b}(x) + 6L_{4bk+2b}(x) + 4L_{4bk}(x) - 6L_{4bk-2b}(x) + L_{4bk-6b}(x) > 0. \qquad (38)
\end{aligned}$$

Since $L_n(x)$ is monotone increasing for n and a fixed positive integer x, for any positive integer k, $x \geq 2$ and odd $b \geq 1$, we have

$$\begin{aligned}
&L_{8bk+6b}(x) - 4L_{8bk+4b}(x) - 2L_{8bk+2b}(x) \\
&\quad = \left(x^2 + 1\right)L_{8bk+6b-1}(x) + xL_{8bk+6b-3}(x) - 4L_{8bk+4b}(x) - 2L_{8bk+2b}(x) \\
&\quad > \left(x^2 - 3\right)L_{8bk+4b}(x) + (x - 2)L_{8bk+2b}(x) > 0,
\end{aligned}$$

and

$$\begin{aligned}
&L_{4bk+8b}(x) - 6L_{4bk+4b}(x) \\
&\quad = \left(x^2 + 1\right)L_{4bk+8b-1}(x) + xL_{4bk+8b-3}(x) - 6L_{4bk+4b}(x) \\
&\quad > 7L_{4bk+8b-3}(x) - 6L_{4bk+4b}(x) > 0,
\end{aligned}$$

and $4L_{8bk}(x) - L_{8bk-2b}(x) > 0$, $4L_{4bk}(x) - 6L_{4bk-2b}(x) > 0$.

Hence the numerator of the left-hand side of inequality (38) is positive, so inequality (38) holds for any positive integer k, $x \geq 2$ and even $b \geq 1$. Hence inequality (37) is true.

Using (37) repeatedly, we have

$$\begin{aligned}\sum_{k=2m}^{\infty}\frac{1}{L_{bk}^2(x)} &= \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk}^2(x)}+\frac{1}{L_{2bk+b}^2(x)}\right)\\ &< \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk}^2(x)-L_{2bk-b}^2(x)-3}-\frac{1}{L_{2bk+2b}^2(x)-L_{2bk+b}^2(x)-3}\right)\\ &= \frac{1}{L_{2bm}^2(x)-L_{2bm-b}^2(x)-3}. \qquad (39)\end{aligned}$$

Now inequality (33) follows from (36) and (39).

Similarly, we can consider the case that $n = 2m + 1 \geq 3$ is odd. At this time, for any odd $b \geq 1$, Theorem 2(4) is equivalent to the inequality

$$\frac{1}{L_{2bm+b}^2(x)-L_{2bm}^2(x)+3} < \sum_{k=2m+1}^{\infty}\frac{1}{L_{bk}^2(x)} \leq \frac{1}{L_{2bm+b}^2(x)-L_{2bm}^2(x)+2}. \qquad (40)$$

Now we prove that for any positive integer k, $x \geq 2$ and odd $b \geq 1$,

$$\frac{1}{L_{2bk+b}^2(x)}+\frac{1}{L_{2bk+2b}^2(x)} > \frac{1}{L_{2bk+b}^2(x)-L_{2bk}^2(x)+3}-\frac{1}{L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+3}. \qquad (41)$$

Inequality (41) is equivalent to

$$\frac{L_{2bk+3b}^2(x)+3}{L_{2bk+2b}^2(x)(L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+3)} > \frac{L_{2bk}^2(x)-3}{L_{2bk+b}^2(x)(L_{2bk+b}^2(x)-L_{2bk}^2(x)+3)}.$$

Using identity (2), the above inequality is equivalent to

$$\begin{aligned}&L_{8bk+10b}(x)-4L_{8bk+8b}(x)+4L_{8bk+4b}(x)-L_{8bk+2b}(x)-L_{4bk+10b}(x)\\ &\quad+6L_{4bk+6b}(x)-4L_{4bk+4b}(x)-4L_{4bk+2b}(x)+6L_{4bk}(x)-L_{4bk-4b}(x)>0. \qquad (42)\end{aligned}$$

Since $L_n(x)$ is monotone increasing for n and a fixed positive integer x, for any positive integer k, $x \geq 2$, and odd $b \geq 1$, we have

$$\begin{aligned}L_{8bk+10b}(x)-4L_{8bk+8b}(x) &= (x^2+1)L_{8bk+10b-2}(x)+xL_{8bk+10b-3}(x)-4L_{8bk+8b}(x)\\ &> 5L_{8bk+8b}(x)-4L_{8bk+8b}(x)+xL_{8bk+10b-3}(x)>0,\end{aligned}$$

and

$$\begin{aligned}&6L_{4bk+6b}(x)-4L_{4bk+4b}(x)-4L_{4bk+2b}(x)\\ &\quad=6xL_{4bk+6b-1}(x)+6L_{4bk+6b-2}(x)-4L_{4bk+4b}(x)-4L_{4bk+2b}(x)>0,\end{aligned}$$

and $6L_{4bk}(x) - L_{4bk-4b}(x) > 0$, $4L_{8bk+4b}(x) - L_{8bk+2b}(x) - L_{4bk+10b}(x) > 0$.

Hence the numerator of the left-hand side of inequality (42) is positive, so inequality (42) holds for any positive integer k, $x \geq 2$ and odd $b \geq 1$. Hence inequality (41) is true.

Using (41) repeatedly, we have

$$\begin{aligned}\sum_{k=2m+1}^{\infty}\frac{1}{L_{bk}^2(x)} &= \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk+b}^2(x)}+\frac{1}{L_{2bk+2b}^2(x)}\right)\\ &> \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk+b}^2(x)-L_{2bk}^2(x)+3}-\frac{1}{L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+3}\right)\\ &= \frac{1}{L_{2bm+b}^2(x)-L_{2bm}^2(x)+3}.\end{aligned}\tag{43}$$

On the other hand, we prove that for any positive integer k, $x \geq 2$ and odd $b \geq 1$,

$$\frac{1}{L_{2bk+b}^2(x)}+\frac{1}{L_{2bk+2b}^2(x)} < \frac{1}{L_{2bk+b}^2(x)-L_{2bk}^2(x)+2}-\frac{1}{L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+2}.\tag{44}$$

Inequality (44) is equivalent to

$$\frac{L_{2bk+3b}^2(x)+2}{L_{2bk+2b}^2(x)(L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+2)} > \frac{L_{2bk}^2(x)-2}{L_{2bk+b}^2(x)(L_{2bk+b}^2(x)-L_{2bk}^2(x)+2)}.$$

Using identity (2), the above inequality is equivalent to

$$\begin{aligned}&4L_{8bk+8b}(x)-4L_{8bk+4b}(x)+L_{4bk+10b}(x)-6L_{4bk+6b}(x)\\ &\quad+L_{4bk+4b}(x)+L_{4bk+2b}(x)-6L_{4bk}(x)+6L_{4bk-4b}(x)>0.\end{aligned}\tag{45}$$

It is clear that inequality (45) holds for any positive integer k, $x \geq 2$ and odd $b \geq 1$. So, inequality (44) is true. Using (44) repeatedly, we have

$$\begin{aligned}\sum_{k=2m+1}^{\infty}\frac{1}{L_{bk}^2(x)} &= \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk+b}^2(x)}+\frac{1}{L_{2bk+2b}^2(x)}\right)\\ &< \sum_{k=m}^{\infty}\left(\frac{1}{L_{2bk+b}^2(x)-L_{2bk}^2(x)+2}-\frac{1}{L_{2bk+3b}^2(x)-L_{2bk+2b}^2(x)+2}\right)\\ &= \frac{1}{L_{2bm+b}^2(x)-L_{2bm}^2(x)+2}.\end{aligned}\tag{46}$$

Combining (43) and (46), we deduce inequality (40). This proves Theorem 2(4). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WZ proposed if we could obtain some generalizations of [8, 10], and [11]. ZW obtained the theorems and completed the proof. All authors read and approved the final manuscript.

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