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BINOMIALS TRANSFORMATION FORMULAE OF SCALED LUCAS NUMBERS

Abstract. The current paper represents a suplement for papers [7] and [8]. Many of the new summation formulae connecting Lucas numbers with binomials are presented here. All these relations are obtained by using definition and simple properties of the so called δ-Lucas numbers.

1. Introduction

The δ-Lucas numbers $\mathbb{A}_n(\delta)$ and $\mathbb{B}_n(\delta)$, $n \in \mathbb{N} \cup \{0\}$, $\delta \in \mathbb{C}$, have been defined in papers [7, 8] in the following way:

$$\mathbb{A}_n(\delta) := 2a_n(\delta) - b_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (-\delta)^k L_k, \tag{1}$$

$$\mathbb{B}_n(\delta) := a_n(\delta) + 2b_n(\delta) = -\sum_{k=0}^{n} \binom{n}{k} (-\delta)^k L_{k-1}, \tag{2}$$

where L_n are the Lucas numbers ($L_0 := 2$, $L_1 := 1$, $L_{n+2} = L_{n+1} + L_n$, $n = 0, 1, \ldots$), whereas $a_n(\delta)$ and $b_n(\delta)$ are the so called δ-Fibonacci numbers defined in paper [8] (see also paper [7]) by means of the relation:

$$(1 - \alpha\delta)^n = a_n(\delta) - \alpha b_n(\delta), \quad \delta \in \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}, \tag{3}$$

or, equivalently, by the relation:

$$(1 - \beta\delta)^n = a_n(\delta) - \beta b_n(\delta), \quad \delta \in \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\}, \tag{4}$$

where:

$$\alpha := \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \beta := \frac{1 - \sqrt{5}}{2}.$$

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We note that:

$$L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z}.$$

From (3) and (4) it follows that δ-Fibonacci numbers $a_n(\delta)$ and $b_n(\delta)$ satisfy the following recurrence relations (for $\mathbb{X} = a$ or $\mathbb{X} = b$, respectively):

$$\mathbb{X}_{n+2}(\delta) = (2-\delta)\mathbb{X}_{n+1}(\delta) + (\delta^2 + \delta - 1)\mathbb{X}_n(\delta). \tag{5}$$

And as follows from (1) and (2) δ-Lucas numbers also satisfy these recurrence relations.

Besides of the identities (1) and (2) the name "δ-Lucas numbers", taken for the polynomials $\mathbb{A}_n(\delta)$ and $\mathbb{B}_n(\delta)$, is additionally justified by the relations for the values of δ-Lucas numbers (deduced from (5)):

$$\mathbb{A}_n(1) = L_n, \quad \mathbb{B}_n(1) = L_{n+1}, \tag{6}$$

$$\mathbb{A}_n(-1) = L_{2n}, \quad \mathbb{B}_n(-1) = -L_{2n-1}, \tag{7}$$

$$\mathbb{A}_n(-2) = L_{3n}, \quad \mathbb{B}_n(-2) = -L_{3n-1}, \tag{8}$$

$$2^n\mathbb{A}_n\Big(-\frac{3}{2}\Big) = L_{4n}, \quad 2^n\mathbb{B}_n\Big(-\frac{3}{2}\Big) = -L_{4n-1}, \tag{9}$$

for every $n \in \mathbb{N} \cup \{0\}$. From this and from (1) and (2) for $\delta = 1, -1, -2, -\frac{3}{2}$, respectively, we receive eight formulae (30), (31) and the first equalities of (33), (34), (39), (40), (54) and (55), describing the operation result of the binomial transformation of scaled Lucas numbers for 4 different values of argument δ. This way of generating the formulae will be also used in section 2 (the general difficulty in applying this technique lies in the ability of generating independently the values $\mathbb{A}_n(\delta)$ and $\mathbb{B}_n(\delta)$ for the specific values of argument δ).

Another important identity, which will be used in the next section for generating the binomials transformation formulae of scaled Lucas numbers, is the following reducing identity (see (5.1) and (5.2) in [8]):

$$\zeta^n\mathbb{X}_n\Big(\frac{\delta}{\zeta}\Big) = \sum_{k=0}^{n}\binom{n}{k}(\zeta-1)^{n-k}\mathbb{X}_k(\delta), \tag{10}$$

for every $\zeta, \delta \in \mathbb{C}$, $\zeta \neq 0$, where $\mathbb{X}_k(\delta)$ denotes the linear combination of δ-Fibonacci numbers, i.e.:

$$\mathbb{X}_k(\delta) \equiv \varphi a_k(\delta) + \psi b_k(\delta),$$

for some $\varphi, \psi \in \mathbb{C}$. In view of (1) and (2) the relation (10) is also satisfied by δ-Lucas numbers.

Let us notice that the relation (10) can be given in two new ways as the mutually inverse relations:

$$r^n \mathbb{X}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (r-1)^{n-k} \mathbb{X}_k(r\delta), \tag{11}$$

$$\mathbb{X}_n(r\delta) = \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k \mathbb{X}_k(\delta), \tag{12}$$

for every $r, \delta \in \mathbb{C}$, $r \neq 0$. As a consequence, we obtain three special relations:

$$2^n \mathbb{X}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} \mathbb{X}_k(2\delta), \tag{13}$$

$$(-1)^n \mathbb{X}_n(-\delta) = \sum_{k=0}^{n} \binom{n}{k} (-2)^{n-k} \mathbb{X}_k(\delta), \tag{14}$$

$$(-1)^n \mathbb{X}_n(2\delta) = \sum_{k=0}^{n} \binom{n}{k} (-2)^{k} \mathbb{X}_k(\delta). \tag{15}$$

Moreover, from (10) for $\delta = r$ and $\zeta = \frac{r}{d}$ we obtain the relation:

$$r^n \mathbb{X}_n(d) = \sum_{k=0}^{n} \binom{n}{k} (r-d)^{n-k} d^k \mathbb{X}_k(r). \tag{16}$$

Hence, for $r = 1$, by (6), we obtain:

$$\mathbb{A}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^{n-k} \delta^k L_k, \tag{17}$$

$$\mathbb{B}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^{n-k} \delta^k L_{k+1}, \tag{18}$$

for $r = -1$, by (7), we have:

$$\mathbb{A}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (\delta+1)^{n-k} (-\delta)^k L_{2k}, \tag{19}$$

$$\mathbb{B}_n(\delta) = -\sum_{k=0}^{n} \binom{n}{k} (\delta+1)^{n-k} (-\delta)^k L_{2k-1}, \tag{20}$$

for $r=-2$, by (8), we get:

$$2^n \mathbb{A}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} (\delta+2)^{n-k} (-\delta)^k L_{3k}, \tag{21}$$

$$2^n \mathbb{B}_n(\delta) = -\sum_{k=0}^{n} \binom{n}{k} (\delta+2)^{n-k} (-\delta)^k L_{3k-1}, \tag{22}$$

for $r=-\frac{3}{2}$, by (9), we receive:

$$3^n \mathbb{A}_n(\delta) = \sum_{k=0}^{n} \binom{n}{k} \left(\delta+\frac{3}{2}\right)^{n-k} (-\delta)^k L_{4k}, \tag{23}$$

$$3^n \mathbb{B}_n(\delta) = -\sum_{k=0}^{n} \binom{n}{k} \left(\delta+\frac{3}{2}\right)^{n-k} (-\delta)^k L_{4k-1}. \tag{24}$$

At the end of this section we present one more auxiliary result, which will be used for generating many new identities.

Lemma 1. *Let $\alpha_{k,n}, \beta_{k,n} \in \mathbb{C}$, $k=0,1,\ldots,bn+c$, $b,c,n \in \mathbb{N}\cup\{0\}$, $a, r_0 \in \mathbb{Z}$. Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \{F, L\}$. If the following equalities:*

$$\mathbb{X}_{an+r} = \sum_{k=0}^{bn+c} \alpha_{k,n} \mathbb{Y}_{k+r} = \sum_{k=0}^{bn+c} \beta_{k,n} (-1)^{k-r} \mathbb{Z}_{k-r} \tag{25}$$

hold for $r=-1,0$ ($r=r_0$, r_0+1, respectively) and for every $n \in \mathbb{N}\cup\{0\}$, then these equalities hold for all $r\in\mathbb{Z}$ and $n\in\mathbb{N}\cup\{0\}$. Moreover, if we set:

$$U_n := \sum_{k=0}^{bn+c} \alpha_{k,n} \mathbb{Y}_{k-1} = \sum_{k=0}^{bn+c} \beta_{k,n} (-1)^{k+1} \mathbb{Z}_{k+1} \tag{26}$$

and

$$V_n := \sum_{k=0}^{bn+c} \alpha_{k,n} \mathbb{Y}_{k} = \sum_{k=0}^{bn+c} \beta_{k,n} (-1)^{k} \mathbb{Z}_{k}, \tag{27}$$

for every $n\in\mathbb{N}\cup\{0\}$, then we get:

$$F_{r-1}U_n + F_r V_n = \sum_{k=0}^{bn+c} \alpha_{k,n} \mathbb{Y}_{k+r} = \sum_{k=0}^{bn+c} \beta_{k,n} (-1)^{k-r} \mathbb{Z}_{k-r}, \tag{28}$$

for every $n\in\mathbb{N}\cup\{0\}$ and $r\in\mathbb{Z}$.

Proof. Only proof of the formula (28) will be given here. We will prove inductively that for every $m\in\mathbb{N}$ the equality (28) is an identity with respect to $n\in\mathbb{N}\cup\{0\}$ for every $r\in\mathbb{Z}$, $-m+1 \leqslant r \leqslant m$.

For $m = 1$ the above hypothesis results from (26) and (27). So, let us assume that the statement is true for some $m \in \mathbb{N}$. Then we find:

$$\begin{aligned}
F_{-m-1}U_n + F_{-m}V_n &= (F_{-m+1}U_n + F_{-m+2}V_n) - (F_{-m}U_n + F_{-m+1}V_n) \\
&= \sum_{k=0}^{bn+c} \alpha_{k,n}(\mathbb{Y}_{k-m+2} - \mathbb{Y}_{k-m+1}) \\
&= \sum_{k=0}^{bn+c} \beta_{k,n}((-1)^{k+m}\mathbb{Z}_{k+m-2} + (-1)^{k+m}\mathbb{Z}_{k+m-1}) \\
&= \sum_{k=0}^{bn+c} \alpha_{k,n}\mathbb{Y}_{k-m} = \sum_{k=0}^{bn+c} \beta_{k,n}(-1)^{k+m}\mathbb{Z}_{k+m}, \\
F_mU_n + F_{m+1}V_n &= (F_{m-1}U_n + F_mV_n) + (F_{m-2}U_n + F_{m-1}V_n) \\
&= \sum_{k=0}^{bn+c} \alpha_{k,n}(\mathbb{Y}_{k+m} + \mathbb{Y}_{k+m-1}) \\
&= \sum_{k=0}^{bn+c} \beta_{k,n}((-1)^{k-m}\mathbb{Z}_{k-m} - (-1)^{k-m}\mathbb{Z}_{k-m+1}) \\
&= \sum_{k=0}^{bn+c} \alpha_{k,n}\mathbb{Y}_{k+m+1} = \sum_{k=0}^{bn+c} \beta_{k,n}(-1)^{k-m-1}\mathbb{Z}_{k-m-1}.
\end{aligned}$$

Thus, in view of the inductive assumption the equality (28) holds for any $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$, $-m \leqslant r \leqslant m + 1$. It means, by virtue of the Mathematical Induction Rule, that the equality (28) is true for any $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$. ∎

We note that from Lemma 1 and identities (1), (2), (17), (18), (19)–(24) the following "mega-identity" can be generated:

$$\begin{aligned}
(29)\qquad F_{r-1}\mathbb{A}_n(\delta) + F_r\mathbb{B}_n(\delta) &= \sum_{k=0}^{n} \binom{n}{k} \delta^k (-1)^{k-r} L_{k-r} \\
= \sum_{k=0}^{n} \binom{n}{k} (1-\delta)^{n-k}\delta^k L_{k+r} &= \sum_{k=0}^{n} \binom{n}{k} (\delta+1)^{n-k}\delta^k(-1)^{k-r}L_{2k-r} \\
= 2^{-n}\sum_{k=0}^{n} \binom{n}{k} &(\delta+2)^{n-k}\delta^k(-1)^{k-r}L_{3k-r} \\
= 3^{-n}\sum_{k=0}^{n} \binom{n}{k} &\left(\delta+\frac{3}{2}\right)^{n-k}\delta^k(-1)^{k-r}L_{4k-r},
\end{aligned}$$

which holds for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

2. Summation formulae

In this section, by using the identities (1), (2), (17), (18), next Lemma 1 and formulae like (6)–(8) giving the values of δ-Lucas numbers for the specific values of argument δ, we can deduce the large set of identities of the summation nature. It will be the basic set of the binomials transformation formulae of scaled Lucas numbers.

From (1), (2) and (6) we get:

$$L_{n+1} = -\sum_{k=0}^{n} \binom{n}{k} (-1)^k L_{k-1}, \tag{30}$$

$$2L_{2n+1} = \sum_{k=0}^{2n} \binom{2n+1}{k} (-1)^k L_k, \tag{31}$$

$$L_{2n+1} = \frac{1}{n+1} + \sum_{k=1}^{2n} \binom{2n+1}{k-1} \frac{(-1)^k}{k} L_k, \tag{32}$$

whereas from (1), (2), (17), (18) and (7) we obtain:

$$L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_k = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k L_k, \tag{33}$$

$$L_{2n-1} = \sum_{k=0}^{n} \binom{n}{k} L_{k-1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^{k+1} L_{k+1}, \tag{34}$$

which, by Lemma 1, implies the general formula:

$$L_{2n+r} = \sum_{k=0}^{n} \binom{n}{k} L_{k+r} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^{k-r} L_{k-r}, \tag{35}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

Furthermore, from formulae (6), (7) and (10) for $\delta = \zeta = 1$ we find:

$$L_n = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^k L_{2k}, \tag{36}$$

$$L_{n+1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^{k+1} L_{2k-1}, \tag{37}$$

from which, in view of Lemma 1, we have:

$$L_{n+r} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} (-1)^{k-r} L_{2k-r}, \tag{38}$$

for every $r \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$. From (1), (2), (17), (18) and (8) for $\delta = -2$

we deduce:

$$L_{3n} = \sum_{k=0}^{n} \binom{n}{k} 2^k L_k = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-2)^k L_k, \tag{39}$$

$$L_{3n-1} = \sum_{k=0}^{n} \binom{n}{k} 2^k L_{k-1} = -\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-2)^k L_{k+1}. \tag{40}$$

Hence, using Lemma 1, we obtain:

$$L_{3n+r} = \sum_{k=0}^{n} \binom{n}{k} 2^k L_{k+r} = \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} 2^k (-1)^{k-r} L_{k-r}, \tag{41}$$

for every $r \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$. Whereas, from (6), (8) and (10) for $\delta = \zeta = -2$, we get:

$$(-2)^n L_n = \sum_{k=0}^{n} \binom{n}{k} (-3)^{n-k} L_{3k}, \tag{42}$$

$$(-2)^n L_{n+1} = -\sum_{k=0}^{n} \binom{n}{k} (-3)^{n-k} L_{3k-1}, \tag{43}$$

which, by Lemma 1, implies:

$$2^n (-1)^{n-r} L_{n-r} = \sum_{k=0}^{n} \binom{n}{k} (-3)^{n-k} L_{3k+r}, \tag{44}$$

for every $r \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$. We note that (44) is the inverse relation for the second identity (41). Furthermore, from (7), (8) and (10) for $\delta = -2$, $\zeta = 2$ we get:

$$2^n L_{2n} = \sum_{k=0}^{n} \binom{n}{k} L_{3k}, \tag{45}$$

$$2^n L_{2n-1} = \sum_{k=0}^{n} \binom{n}{k} L_{3k-1}, \tag{46}$$

which, by Lemma 1, implies:

$$2^n L_{2n+r} = \sum_{k=0}^{n} \binom{n}{k} (-1)^r L_{3k-r}, \tag{47}$$

for every $r \in \mathbb{Z}$ and $n \in \mathbb{N} \cup \{0\}$.

Next, from (1), (2) and Table 1 in [8], we obtain:

$$\mathbb{A}_n(2) = (1 + (-1)^n)5^{\lfloor n/2 \rfloor} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_k, \tag{48}$$

$$\mathbb{B}_n(2) = (3 - 2(-1)^n)5^{\lfloor n/2 \rfloor} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_{k-1}, \tag{49}$$

which, by Lemma 1, implies the general formula:

$$(-1)^{r+1}\big(L_{r-1} + F_r - F_{r+2}(-1)^n\big)5^{\lfloor n/2 \rfloor} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_{k-r}, \tag{50}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$. Furthermore, from (1), (2), (17), (18) one can deduce the identities:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} L_k &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 3^{n-k} L_k = 2^n \mathbb{A}_n\Big(-\frac{1}{2}\Big) \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_n & \text{if } n \text{ is even}, \\ 5^{1+\lfloor n/2 \rfloor} F_n & \text{if } n \text{ is odd}, \end{cases} \end{aligned} \tag{51}$$

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} L_{k-1} &= \sum_{k=0}^{n} \binom{n}{k} (-1)^{k-1} 3^{n-k} L_{k+1} = -2^n \mathbb{B}_n\Big(-\frac{1}{2}\Big) \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_{n-1} & \text{if } n \text{ is even}, \\ 5^{1+\lfloor n/2 \rfloor} F_{n-1} & \text{if } n \text{ is odd}. \end{cases} \end{aligned} \tag{52}$$

Hence, using Lemma 1, the following relations can be generated:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} L_{k+r} &= \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} (-1)^{k-r} L_{k-r} \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_{n+r} & \text{if } n \text{ is even}, \\ 5^{1+\lfloor n/2 \rfloor} F_{n+r} & \text{if } n \text{ is odd}, \end{cases} \end{aligned} \tag{53}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$. Moreover, from (1), (17), (2), (18) and (9), we get:

$$L_{4n} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 3^k L_k = \sum_{k=0}^{n} \binom{n}{k} 5^{n-k} (-3)^k L_k, \tag{54}$$

$$L_{4n-1} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 3^k L_{k-1} = -\sum_{k=0}^{n} \binom{n}{k} 5^{n-k} (-3)^k L_{k+1}. \tag{55}$$

Hence, using Lemma 1, the general identities follow:

$$L_{4n+r} = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} 3^k L_{k+r} = \sum_{k=0}^{n} \binom{n}{k} 5^{n-k} 3^k (-1)^{k-r} L_{k-r}, \tag{56}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

Now, from (8), (9) and (11), for $\delta = -\frac{3}{2}$, $\zeta = \frac{3}{4}$ we get:

$$(-3)^n L_{3n} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_{4k}, \tag{57}$$

$$(-3)^n L_{3n-1} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_{4k-1}, \tag{58}$$

which by Lemma 1 implies the following identity:

$$(-3)^n L_{3n+r} = \sum_{k=0}^{n} \binom{n}{k} (-2)^k L_{4k+r}, \tag{59}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

Furthermore, from (1), (17), (2) and (18), the next two series of identities can be deduced:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 3^k L_k = \sum_{k=0}^{n} \binom{n}{k} (-3)^k 4^{n-k} L_k &= 4^n \mathbb{A}_n\left(\frac{3}{4}\right) \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_{2n} & \text{if } n \text{ is even,} \\ 5^{1+\lfloor n/2 \rfloor} F_{2n} & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \tag{60}$$

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 3^k L_{k+1} = -\sum_{k=0}^{n} \binom{n}{k} (-3)^k 4^{n-k} L_{k-1} &= 4^n \mathbb{B}_n\left(\frac{3}{4}\right) \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_{2n+1} & \text{if } n \text{ is even,} \\ 5^{1+\lfloor n/2 \rfloor} F_{2n+1} & \text{if } n \text{ is odd.} \end{cases} \end{aligned} \tag{61}$$

Hence, using Lemma 1 again, the following general identities can be obtained:

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} 3^k L_{k+r} &= \sum_{k=0}^{n} \binom{n}{k} 3^k 4^{n-k} (-1)^{k-r} L_{k-r} \\ &= \begin{cases} 5^{\lfloor n/2 \rfloor} L_{2n+r} & \text{if } n \text{ is even,} \\ 5^{1+\lfloor n/2 \rfloor} F_{2n+r} & \text{if } n \text{ is odd,} \end{cases} \end{aligned} \tag{62}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

At the end of this section we will present few more identities connected with the complex arguments. By induction one can easily verify that:

$$\mathbb{A}_{2n}(-i) = -(2+i)^n i^{n+2} L_n = (2i-1)^n L_n, \tag{63}$$

$$\mathbb{A}_{2n+1}(-i) = -(2+i)^n i^{n+2}(L_n + iL_{n+1}) = (2i-1)^n (L_n + iL_{n+1}), \tag{64}$$

$$\mathbb{B}_{2n}(-i) = (2+i)^n i^{n+2} L_{n-1} = -(2i-1)^n L_{n-1}, \tag{65}$$

$$\mathbb{B}_{2n+1}(-i) = (2+i)^n i^{n+2}(L_{n-1} + iL_n) = -(2i-1)^n (L_{n-1} + iL_n). \tag{66}$$

Proof. Immediately from (5) we have

$$\mathbb{A}_{2n+2}(-i) = (2+i)\big(\mathbb{A}_{n+1}(-i) - \mathbb{A}_n(-i)\big), \tag{67}$$

for every $n = 0, 1, \ldots$. Whereas, from (1) we obtain

$$\mathbb{A}_0(-i) = 2, \qquad \mathbb{A}_1(-i) = 2 + i,$$

which is compatible with (63) and (64) for $n = 0$. Now, let assume that the equations (63) and (64) hold for every $n = 0, 1, \ldots, N$, where N is a fixed positive integer. Then, by (67), we have:

$$\begin{aligned}\mathbb{A}_{2N+2}(-i) &= (2+i)\big((2i-1)^N (L_N + iL_{N+1}) - (2i-1)^N L_N\big)\\ &= (2i-1)^N (2+i) i L_{N+1} = (2i-1)^{N+1} L_{N+1},\end{aligned}$$

and

$$\begin{aligned}\mathbb{A}_{2N+3}(-i) &= (2+i)\big((2i-1)^{N+1} L_{N+1} - (2i-1)^N (L_N + iL_{N+1})\big)\\ &= (2i-1)^N (2+i) i\big((2+i)L_{N+1} + iL_N - L_{N+1}\big)\\ &= (2i-1)^{N+1}\big(L_{N+1} + iL_{N+2}\big),\end{aligned}$$

which means, by virtue of the Mathematical Induction Rule, that (63) and (64) hold for every $n \in \mathbb{N} \cup \{0\}$. ∎

Using (1), (17), (2), (18) and (63)–(66), one can generate the formulae:

$$(2i-1)^n L_n = \sum_{k=0}^{2n} \binom{2n}{k} i^k L_k = \sum_{k=0}^{2n} \binom{2n}{k} (1+i)^{2n-k} (-i)^k L_k, \tag{68}$$

$$\begin{aligned}(2i-1)^n L_{n-1} &= \sum_{k=0}^{2n} \binom{2n}{k} i^k L_{k-1}\\ &= -\sum_{k=0}^{2n} \binom{2n}{k} (1+i)^{2n-k} (-i)^k L_{k+1},\end{aligned} \tag{69}$$

which, by Lemma 1, implies:

$$(2i-1)^n L_{n+r} = \sum_{k=0}^{2n} \binom{2n}{k} i^k L_{k+r} = \sum_{k=0}^{2n} \binom{2n}{k} (1+i)^{2n-k} i^k (-1)^{k-r} L_{k-r}, \tag{70}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$, and next:

$$(2i-1)^n (L_n + iL_{n+1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} i^k L_k = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (1+i)^{2n+1-k} (-i)^k L_k, \tag{71}$$

$$(2i-1)^n (L_{n-1} + iL_n) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} i^k L_{k-1} = -\sum_{k=0}^{2n+1} \binom{2n+1}{k} (1+i)^{2n+1-k} (-i)^k L_{k+1}, \tag{72}$$

which, using Lemma 1, implies:

$$(2i-1)^n (L_{n+r} + iL_{n+r+1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} i^k L_{k+r} = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (1+i)^{2n+1-k} i^k (-1)^{k-r} L_{k-r}, \tag{73}$$

for every $n \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{Z}$.

3. Application of the reduced formulae

For generating the binomials transformation formulae of scaled Lucas numbers one can also apply the following reduced formulae for δ-Lucas numbers (see formulae (39), (40) in [8]):

$$\mathbb{A}_{kn}(\delta) = a_k^n(\delta) \mathbb{A}_n \left(\frac{b_k(\delta)}{a_k(\delta)} \right), \tag{74}$$

$$\mathbb{B}_{kn}(\delta) = a_k^n(\delta) \mathbb{B}_n \left(\frac{b_k(\delta)}{a_k(\delta)} \right). \tag{75}$$

For example:
– for $\delta = 1$, from (1), (6) and the facts that $a_n(1) = F_{n+1}$, $b_n(1) = F_n$, we obtain:
(76)
$$L_{rn} = F_{r+1}^n \sum_{k=0}^{n} \binom{n}{k} \Big(-\frac{F_r}{F_{r+1}} \Big)^k L_k = \sum_{k=0}^{n} \binom{n}{k} (-F_r)^k F_{r+1}^{n-k} L_k,$$
(77)
$$L_{rn+1} = -F_{r+1}^n \sum_{k=0}^{n} \binom{n}{k} \Big(-\frac{F_r}{F_{r+1}} \Big)^k L_{k-1} = -\sum_{k=0}^{n} \binom{n}{k} (-F_r)^k F_{r+1}^{n-k} L_{k-1},$$
which, by Lemma 1, implies:
$$(78) \qquad L_{rn+s} = \sum_{k=0}^{n} \binom{n}{k} F_r^k F_{r+1}^{n-k} (-1)^{k-s} L_{k-s},$$
for every $r, n \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{Z}$;
– for $\delta = 2$,
$$(79) \qquad (1 + (-1)^{rn}) 5^{\lfloor rn/2 \rfloor - \lfloor r/2 \rfloor n} = \sum_{k=0}^{n} \binom{n}{k} ((-1)^r - 1)^k L_k,$$
which is attractive whenever r is odd, and
$$(80) \qquad (2(-1)^{rn} - 3) 5^{\lfloor rn/2 \rfloor - \lfloor r/2 \rfloor n} = \sum_{k=0}^{n} \binom{n}{k} (2(-1)^r - 3)^k L_{k-1};$$
– for $\delta = -i$,
$$(81) \qquad L_{rn} = \sum_{k=0}^{n} \binom{n}{k} F_r^k F_{r-1}^{n-k} L_k,$$
$$(82) \qquad L_{rn-1} = \sum_{k=0}^{n} \binom{n}{k} F_r^k F_{r-1}^{n-k} L_{k-1},$$
which, by Lemma 1, implies:
$$(83) \qquad L_{rn+s} = \sum_{k=0}^{n} \binom{n}{k} F_r^k F_{r-1}^{n-k} L_{k+s},$$
for every $r, n \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{Z}$. Moreover, we get:
$$(84) \qquad (2i - 1)^n (L_{2rn+r+n} + i L_{2rn+r+n+1}) = \sum_{k=0}^{2n+1} \binom{2n+1}{k} (F_r + i F_{r+1})^k (F_{r-1} + i F_r)^{2n+1-k} L_k.$$

We note that all the above formulae (76)–(84) give the complement of the respective formulae given in Section 6 in [8].

4. Final remark

Some of the identities presented in this paper are known. Let us present a list of these identities:

— the first of identities (33) – see [5],
— the first of identities (35) – see [5], for Gibonacci numbers discovered by Ruggles (1963) – according to [5],
— the identities (48) and (50) – discovered by Koshy 1998 – according to [5],
— the first of identities (39)–(41) follows from identity (239) in [1] for Gibonacci numbers.

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