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Theory of Binet formulas for Fibonacci and Lucas p-numbers

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Abstract

Modern natural science requires the development of new mathematical apparatus. The generalized Fibonacci numbers or Fibonacci p-numbers ($p = 0, 1, 2, 3, \ldots$), which appear in the "diagonal sums" of Pascal's triangle and are assigned in the recurrent form, are a new mathematical discovery. The purpose of the present article is to derive analytical formulas for the Fibonacci p-numbers. We show that these formulas are similar to the Binet formulas for the classical Fibonacci numbers. Moreover, in this article, there is derived one more class of the recurrent sequences, which is defined to be a generalization of the Lucas numbers (Lucas p-numbers).

1. Introduction

As is well known, the Golden Proportion $\tau = \frac{1+\sqrt{5}}{2}$ plays an increasingly important role in modern physical research [1–14]. A substantial number of researchers from various areas of science are inclined to believe that the Golden Proportion is one of the fundamental constants of the "physical world." As early as in Johannes Kepler's research the Golden Proportion was named as one of two treasures of geometry and compared it to Pythagorean Theorem. The outstanding American theoretical physicist, Richard Feynman (1918–1988), who is one of the founders of the quantum electrodynamics, expressed his admiration of the Golden Proportion in the following words: "What miracles exist in mathematics! According to my theory, the Golden Proportion of the ancient Greeks gives the minimal power condition of the butadiene[1] molecule."

The generalized Fibonacci or Fibonacci p-numbers [18] are one of the important mathematical discoveries of the modern Golden Section and Fibonacci numbers theory [15–34]. Let us define the *basic recurrence relation* for Fibonacci p-numbers. For any given p ($p = 0, 1, 2, 3, \ldots$) they are given by the following recurrent relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1). \tag{1}$$

Eq. (1) is then the basic recurrence relation.

It is necessary to note, that for various initial conditions

$$F_p(1) = a_1; F_p(2) = a_2; \ldots; F_p(p+1) = a_{p+1}, \tag{2}$$

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[1] A colorless, highly flammable hydrocarbon, C_4H_6, obtained from petroleum and used in the manufacture of synthetic rubber.

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where $a_1, a_2, \ldots, a_{p+1}$ are elements of the set of integers, real or complex numbers, we will obtain from (1) the infinite set of recurrent numerical sequences that relate to the class of the *recurrent Fibonacci p-series*.

In particular, if we take

$$F_p(1) = 1; F_p(2) = 1; \ldots; F_p(p) = 1; F_p(p+1) = 1, \tag{3}$$

then for these initial conditions, the basic recurrence relation (1) "generates" a class of the *Fibonacci p-numbers* that are "diagonal sums" of Pascal's triangle [18].

For different p, the basic recurrence relation (1) "generates" a number of the remarkable numerical sequences that are widely used in mathematics. For example, for the case $p = 0$, the recurrence relation (1) is reduced to the following:

$$F_0(n) = 2F_0(n-1), \tag{4}$$

which generates the sequence of the powers of two: $1, 2, 4, 8, 16, 32, \ldots$, for the given initial condition

$$F_0(1) = 1. \tag{5}$$

For the case $p = 1$, the basic recurrence relation (1) takes the following form:

$$F_1(n) = F_1(n-1) + F_1(n-2). \tag{6}$$

This recurrence relation for the initial conditions:

$$F_1(1) = 1, \quad F_1(2) = 1 \tag{7}$$

generates the classical Fibonacci numbers $F(n) = \{1, 1, 2, 3, 5, 8, 13, 21 \ldots\}$. Furthermore, given the initial conditions:

$$F_1(1) = 1, \quad F_1(2) = 3, \tag{8}$$

the relation (6) generates the classical Lucas numbers $L_1(n) = \{1, 3, 4, 7, 11, 18, 29, \ldots\}$.

It is known that the limit of the ratio of two adjacent Fibonacci numbers $F_1(n)$ (as well as the adjacent Lucas numbers $L_1(n)$ and the adjacent numbers of any numerical sequence that is given by the recurrence relation (6)) tends to the Golden Proportion, i.e.:

$$\lim_{n\to\infty} \frac{F_1(n)}{F_1(n-1)} = \tau = \frac{1+\sqrt{5}}{2}. \tag{9}$$

The Golden Proportion τ is a positive root of the following characteristic equation:

$$x^2 = x + 1, \tag{10}$$

that is also called the *Golden Section equation*.

Eq. (10) has two real roots:

$$x_1 = \tau_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = -\frac{1}{\tau_1} = \frac{1-\sqrt{5}}{2}. \tag{11}$$

Binet formulas are well known in the *Fibonacci numbers theory* [15–17]. These formulas allow all Fibonacci numbers $F_1(n)$ and Lucas numbers $L_1(n)$ to be represented by the roots x_1 and x_2 of Eq. (10):

$$F_1(n) = \frac{\tau_1^n - \left(-\frac{1}{\tau_1}\right)^n}{\sqrt{5}}; \tag{12}$$

$$L_1(n) = \tau_1^n + \left(-\frac{1}{\tau_1}\right)^n, \tag{13}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$.

Note that the recurrence relations (4), (6), Eq. (10), and the Binet formulas (12) and (13) are widely used for simulation of various physical and biological phenomena. In particular, they are used for the process of cell division [32,33] and in the description of Fibonacci's lattices on the surface of the "phyllotaxis" objects [22].

In recent years it has been shown that the Fibonacci p-numbers, given by (1), can be used for simulation of biological cell division [31,33] and the self-organizing systems [21]. Moreover, the connection of the Fibonacci p-numbers to Pascal's triangle has a special interest. It became a source of new mathematical and even philosophical discoveries based on the Fibonacci p-numbers, i.e. the "Law of structural harmony of systems" [21], the Generalized Principle of the Golden

Section [29] and others. In this vein, the basic goal of the present article is to derive the general analytical formulas that are similar to Binet formulas (12) and (13) for the Fibonacci p-numbers, given by (1). Undoubtedly, these formulas expand the mathematical apparatus of modern natural science, including physical research.

2. Some properties of the Fibonacci p-series

2.1. Extended Fibonacci and Lucas numbers

The *extended Fibonacci* $F_1(n)$ and *Lucas numbers* $L_1(n)$ that are given for positive and negative values of the discrete argument of n (see Table 1) are well-known.

As it follows from Table 1, the extended Fibonacci and Lucas numbers are infinite numerical sequences whose terms possess remarkable mathematical properties. For all the even ($n = 2k$) and odd ($n = 2k + 1$) values of the argument n, we have the following correlations for the Fibonacci numbers:

$$F_1(2k) = -F_1(-2k); \quad F_1(2k+1) = F_1(-2k-1), \tag{14}$$

and for the Lucas numbers:

$$L_1(2k) = L_1(-2k); \quad L_1(2k+1) = -L_1(-2k-1). \tag{15}$$

It is necessary to note, that the Binet formulas (12) and (13) give the extended Fibonacci and Lucas numbers.

2.2. Extended Fibonacci p-numbers

Let us consider the extended Fibonacci p-numbers that are given by the recurrence relation (1) at the initial conditions (3). As we saw earlier, this class of the Fibonacci p-numbers has direct relation to Pascal's triangle and sets the diagonal sums of Pascal's triangle.

For calculation of the Fibonacci p-numbers $F_p(0), F_p(-1), F_p(-2), \ldots, F_p(-p), \ldots, F_p(-2p+1), \ldots$, that correspond to negative or zero integer values of n, we will use the basic recurrence relation (1) and the initial conditions (3). Let us represent the Fibonacci p-number $F_p(p+1)$ according to (1):

$$F_p(p+1) = F_p(p) + F_p(0). \tag{16}$$

According to (3) $F_p(p) = F_p(p+1) = 1$, it then follows from (16) that $F_p(0) = 0$.

Continuing this process, i.e. representing the Fibonacci p-numbers $F_p(p), F_p(p-1), \ldots, F_p(2)$ in the form (1), we can get:

$$F_p(0) = F_p(-1) = F_p(-2) = \cdots = F_p(-p+1) = 0. \tag{17}$$

Let us represent the Fibonacci p-number $F_p(1)$ in the form:

$$F_p(1) = F_p(0) + F_p(-p). \tag{18}$$

Since $F_p(1) = 1$ and $F_p(0) = 0$, then it follows from (18) that

$$F_p(-p) = 1. \tag{19}$$

Representing the Fibonacci p-numbers $F_p(0), F_p(-1), \ldots, F_p(-p+1)$ in the form (1), we can get:

$$F_p(-p-1) = F_p(-p-2) = \cdots = F_p(-2p+1) = 0. \tag{20}$$

Continuing this process, we can get all the values of the Fibonacci p-numbers $F_p(n)$ for the negative values of n. Table 2 gives the values of the extended Fibonacci $p-$numbers for the cases $p = 1, 2, 3, 4, 5$.

Table 1
Extended Fibonacci and Lucas numbers

N	0	1	2	3	4	5	6	7	8
$F_1(n)$	0	1	1	2	3	5	8	13	21
$F_1(-n)$	0	1	−1	2	−3	5	−8	13	−21
$L_1(n)$	2	1	3	4	7	11	18	29	47
$L_1(-n)$	2	−1	3	−4	7	−11	18	−29	47

Table 2
Extended Fibonacci p-numbers

N	8	7	6	5	4	3	2	1	0	−1	−2	−3	−4	−5	−6	−7	−8	−9
$F_1(n)$	21	13	8	5	3	2	1	1	0	1	−1	2	−3	5	−8	13	−21	34
$F_2(n)$	9	6	4	3	2	1	1	1	0	0	1	0	−1	1	1	−2	0	2
$F_3(n)$	5	4	3	2	1	1	1	1	0	0	0	1	0	0	−1	1	0	1
$F_4(n)$	4	3	2	1	1	1	1	1	0	0	0	0	1	0	0	0	−1	1
$F_5(n)$	3	2	1	1	1	1	1	1	0	0	0	0	0	1	0	0	0	−1

2.3. Properties of the sums for the Fibonacci p-numbers

The following property for the Fibonacci p-numbers is proved in [18]:

$$F_p(1) + F_p(2) + F_p(3) + \cdots + F_p(n) = F_p(n+p+1) - 1. \tag{21}$$

Note that the formula (21) includes a number of the remarkable formulas of discrete mathematics. For example, for the case $p = 0$, this formula is reduced to the following well-known formula for the binary numbers:

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

As it was mentioned for the case $p = 1$, the Fibonacci p-numbers coincide with the classical Fibonacci numbers $F_1(n)$. Therefore, Eq. (21) reduces to the following formula:

$$F_1(1) + F_1(2) + F_1(3) + \cdots + F_1(n) = F_1(n+2) - 1,$$

that is well known from the Fibonacci numbers theory [15–17].

2.4. The Golden p-Sections

As it was mentioned, the ratio of the adjacent Fibonacci numbers $F_1(n)/F_1(n-1)$ tends to the Golden Proportion at the unlimited increase of n.

Let us consider the limit of the adjacent Fibonacci p-numbers $\frac{F_p(n)}{F_p(n-1)}$ for the case $n \to \infty$. To do this, let us introduce the following definition:

$$\lim_{n\to\infty} \frac{F_p(n)}{F_p(n-1)} = x. \tag{22}$$

Let us represent now the ratio of the adjacent Fibonacci p-numbers in the following form:

$$\frac{F_p(n)}{F_p(n-1)} = \frac{F_p(n-1) + F_p(n-p-1)}{F_p(n-p-1)} = 1 + \frac{1}{\frac{F_p(n-1)}{F_p(n-p-1)}} = 1 + \frac{1}{\frac{F_p(n-1)\cdot F_p(n-2)\cdot \cdots \cdot F_p(n-p)}{F_p(n-2)\cdot F_p(n-3)\cdot \cdots \cdot F_p(n-p-1)}}. \tag{23}$$

Taking into consideration the definition (22) for the case $n \to \infty$, we can replace the expression (23) with the following algebraic equation for the Fibonacci p-numbers:

$$x^{p+1} - x^p - 1 = 0, \tag{24}$$

note that Eq. (24) is the algebraic equation of the $(p + 1)$th degree and has $(p + 1)$ roots $x_1, x_2, x_3, \ldots, x_p, x_{p+1}$. Designate the positive root of the algebraic equation (24) as τ_p and, without loss of generality, let $x_1 = \tau_p$.

Let us examine the values of Eq. (24) for the different values of p. For the case $p = 0$, Eq. (24) is the trivial equation $x = 2$. For the case $p = 1$, Eq. (24) is simplifies to Eq. (10). Accordingly, the positive roots of Eq. (24) were named as the *generalized Golden Proportions* or the *Golden p-Proportions* [18].

The following identity connects the powers of the Golden p-Proportions τ_p and is a direct result of Eq. (24):

$$\tau_p^n = \tau_p^{n-1} + \tau_p^{n-p-1} = \tau_p \times \tau_p^{n-1}. \tag{25}$$

It is necessary to point out, that this identity is holds for all the roots $x_1, x_2, x_3, \ldots, x_p, x_{p+1}$ of Eq. (24). Hence, for the general case, we have:

$$x_k^n = x_k^{n-1} + x_k^{n-p-1} = x_k \times x_k^{n-1}, \tag{26}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$; x_k is the root of Eq. (24); $k = 1, 2, \ldots, p + 1$.

2.5. The properties of the roots of the characteristic equation

Theorem 1. *For the given integer* $p > 0$, *the following correlations for the roots of the characteristic equation* $x^{p+1} - x^p - 1 = 0$ *is valid*:

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1} = 1. \tag{27}$$

$$\begin{aligned}
&x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_p + x_1x_{p+1} + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1} = 0;\\
&x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1} = 0;\\
&\ldots;\\
&x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 0.
\end{aligned} \tag{28}$$

$$x_1x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = (-1)^p. \tag{29}$$

Proof. The characteristic equation (24) has the $p + 1$ roots $x_1, x_2, x_3, x_4, \ldots x_p, x_{p+1}$. Thus, by factoring algebraically, it is possible to write:

$$x^{p+1} - x^p - 1 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)\cdots(x - x_p)(x - x_{p+1}) = 0. \tag{30}$$

Let us remove the parentheses in (30). Then, for the even p, we have:

$$\begin{aligned}
x^{p+1} - x^p - 1 &= (x - x_1)(x - x_2)(x - x_3)(x - x_4)\cdots(x - x_p)(x - x_{p+1})\\
&= x^{p+1} - (x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1})x^p\\
&\quad + (x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_{p+1} + x_2x_3 + x_2x_4\\
&\quad + \cdots + x_2x_p + x_2x_{p+1} + \cdots x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1})x^{p-1}\\
&\quad - (x_1x_2x_3 + x_1x_3x_4 + \cdots + x_1x_px_{p+1} + x_2x_3x_4 + x_2x_3x_5\\
&\quad + \cdots + x_2x_px_{p+1} + \cdots + x_{p-1}x_px_{p+1})x^{p-2}\\
&\quad + (x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1})x^{p-3}\\
&\quad + \cdots + (x_1x_2x_3x_4\ldots x_{p-2}x_{p-1}x_p + x_1x_3x_4\ldots x_{p-1}x_px_{p+1}\\
&\quad + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1})x - x_1x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 0.
\end{aligned} \tag{31}$$

It follows from the comparison of (24) and (31):

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1} = 1;\\
&x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_{p+1} + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1} = 0;\\
&x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1} = 0;\\
&\ldots;\\
&x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 0;\\
&x_1x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 1.
\end{aligned}$$

For the odd p, we have:

$$\begin{aligned}
&(x - x_1)(x - x_2)(x - x_3)(x - x_4)\cdots(x - x_p)(x - x_{p+1})\\
&\quad = x^{p+1} - (x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1})x^p + (x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_{p+1}\\
&\qquad + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1})x^{p-1}\\
&\qquad - (x_1x_2x_3 + x_1x_3x_4 + \cdots + x_1x_px_{p+1} + x_2x_3x_4 + x_2x_3x_5 + \cdots + x_2x_px_{p+1} + \cdots + x_{p-1}x_px_{p+1})x^{p-2}\\
&\qquad + (x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1})x^{p-3} + \cdots - (x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p\\
&\qquad + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1})x + x_1x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 0.
\end{aligned} \tag{32}$$

It follows from the comparison of (24) and (32):

$$x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1} = 1;$$
$$x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_1x_{p+1} + x_2x_3 + x_2x_4 + \cdots + x_2x_p + x_2x_{p+1} + \cdots + x_{p-1}x_p + x_{p-1}x_{p+1} + x_px_{p+1} = 0;$$
$$x_1x_2x_3x_4 + x_1x_2x_3x_5 + \cdots + x_{p-2}x_{p-1}x_px_{p+1} = 0;$$
$$\ldots;$$
$$x_1x_2x_3x_4\cdots x_{p-2}x_{p-1}x_p + x_1x_3x_4\cdots x_{p-1}x_px_{p+1} + \cdots + x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = 0;$$
$$x_1x_2x_3x_4\cdots x_{p-1}x_px_{p+1} = -1. \qquad \square$$

Let us give some explanations regarding to the identities (27)–(29) that connect the roots of the characteristic equation (24). It is evident from identity (27) that the sum of the roots of Eq. (24) is identically equal to 1. Expression (28) gives the values for the every possible sums of the roots of the algebraic equation (24) taken by two, three, ..., p roots from the $(p + 1)$ roots of Eq. (24). According to (28), each of these sums is identically equal to 0! At last, the expression (28) sets the value of the product of all the roots of Eq. (24). According to (28) this product is equal to 1 (for the even p) or -1 (for the odd p).

These surprising properties of the roots of the algebraic equation (24) give us a right to rigorously consider the basic recurrence relation (1) and Fibonacci p-numbers that are diagonal sums of Pascal's triangle.

Let us consider the following expression for the roots of Eq. (24):

$$(x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1})^k, \tag{33}$$

where $k = 1, 2, \ldots, p$. Taking into consideration the identity (27) we can write:

$$(x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1})^k = 1. \tag{34}$$

On the other hand, the expression (33) can be factorized if we use the binomial (for the case $p = 1$), trinomial (for the case $p = 2$) and polynomial (for the arbitrary p) formulas in [34]. As it is known in [34], for the given k, the polynomial formula for (34) will include in itself the sum of all the kth powers of the characteristic equation (24) that are taken with the coefficient 1, that is,

$$x_1^k + x_2^k + x_3^k + x_4^k + \cdots + x_p^k + x_{p+1}^k, \tag{35}$$

and the sum of the products of every possible combination of two ($k = 2$), three ($k = 3$), or k roots of the characteristic equation (24), which are taken with the factors that are known as polynomial coefficients. Therefore, the next theorem follows from the reasoning above and expressions (33) and (34).

Theorem 2. *For the given integers $p = 1, 2, 3\ldots$ and $k = 1, 2, 3, \ldots, p$, the following identity is true for the roots of the characteristic equation $x^{p+1} - x^p - 1 = 0$:*

$$(x_1 + x_2 + x_3 + x_4 + \cdots + x_p + x_{p+1})^k = x_1^k + x_2^k + x_3^k + x_4^k + \cdots + x_p^k + x_{p+1}^k = 1. \tag{36}$$

3. Binet formulas for the Fibonacci and Lucas p-numbers

Let consider the Binet formulas (12) and (13). As we mentioned, these formulas give the Fibonacci and Lucas numbers via the roots of the algebraic equation (10).

For the given $p > 0$, using (12) and (13), we will derive the Binet formula that gives Fibonacci p-numbers in the form:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n, \tag{37}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the characteristic equation (24) that satisfy the identity (26) and $k_1, k_2, \ldots, k_{p+1}$ are some constant coefficients that depend on the initial terms of the Fibonacci p-series.

It follows from (17) that $F_p(0) = 0$ for the given $p > 0$. Therefore, we will consider the Fibonacci p-numbers given by the recurrence relation (1) for the following initial conditions:

$$F_p(0) = 0, F_p(1) = F_p(2) = \cdots = F_p(p) = 1. \tag{38}$$

Taking into consideration (37) and (38) it is possible to derive the following system of algebraic equations:

$$\begin{cases} F_p(0) = k_1 + k_2 + \cdots + k_{p+1} = 0; \\ F_p(1) = k_1x_1 + k_2x_2 + \cdots + k_{p+1}x_{p+1} = 1; \\ F_p(2) = k_1(x_1)^2 + k_2(x_2)^2 + \cdots + k_{p+1}(x_{p+1})^2 = 1; \\ \cdots; \\ F_p(p) = k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p = 1, \end{cases} \tag{39}$$

solving the system of the equations (39), we will get the numerical values of the coefficients $k_1, k_2, \ldots, k_{p+1}$.

4. The Derivation of the Binet formulas for the classical Fibonacci and Lucas numbers

We next apply the general formula (37) in order to derive the Binet formulas for the case $p = 1$. For this case, the characteristic equation (24) reduces to (10). As it was mentioned above, Eq. (10) has two roots $x_1 = \tau_1$ and $x_2 = -\frac{1}{\tau_1}$, where $\tau_1 = \frac{1+\sqrt{5}}{2}$.

Therefore, the formula (37), for the case $p = 1$, takes the following form:

$$F_1(n) = k_1(\tau_1)^n + k_2\left(-\frac{1}{\tau_1}\right)^n, \tag{40}$$

and, using (38) and (39), it is possible to derive the following system of the algebraic equations:

$$\begin{cases} F_1(0) = k_1 + k_2; \\ F_1(1) = k_1\tau_1 + k_2\left(-\frac{1}{\tau_1}\right), \end{cases} \tag{41}$$

where $F_1(0) = 0$ and $F_1(1) = 1$ according to (38).

Solving the system (41) we get: $k_1 = \frac{1}{\sqrt{5}}$ and $k_2 = -\frac{1}{\sqrt{5}}$. If we substitute k_1 and k_2 to (39), we get the well-known Binet formula for the classical Fibonacci numbers given by (12).

If we accept $k_1 = k_2 = 1$ in (40) we get the Binet formula for the classical Lucas numbers that is given by (13). This formula generates the Lucas series with the initial terms $L_1(0) = 2$ and $L_1(1) = 1$:

$$2, 1, 3, 4, 7, 11, 18, 29, \ldots \tag{42}$$

Let us make one important note regarding the identity $F_1(0) = 0$. This fact follows directly from the Binet formula (40). Indeed, according to (40), we have the following for the case $n = 0$:

$$F_1(0) = k_1(\tau)^0 + k_2(-1/\tau)^0 = k_1 + k_2 = \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}. \tag{43}$$

These simple calculations show that the identity $F_1(0) = 0$ follows from the fact that the sum of the coefficients k_1, k_2 in the expression (40) is identically equal to 0.

5. The Binet formulas for the Fibonacci and Lucas 2-numbers

5.1. Binet formula for the Fibonacci 2-numbers

Let us give $p = 2$ and use the above approach for the derivation of the Binet formula in order to calculate the Fibonacci 2-numbers. The recurrence relation (1), the initial conditions (38), and the algebraic equation (24) take the following forms respectively for the case $p = 2$:

$$F_2(n) = F_2(n-1) + F_2(n-3); \tag{44}$$

$$F_p(0) = 0, \quad F_p(1) = F_p(2) = 1; \tag{45}$$

$$x^3 = x^2 + 1. \tag{46}$$

Eq. (46) takes three roots, one is real root x_1 and two are complex conjugate roots x_2 and x_3:

$$x_1 = \frac{h}{6} + \frac{2}{3h} + \frac{1}{3} = 1.4655712319\ldots; \tag{47}$$

$$x_2 = -\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} - \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right) = -0.233\ldots - (0.793\ldots)\mathrm{i}; \tag{48}$$

$$x_3 = -\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} + \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right) = -0.233\ldots + (0.793\ldots)\mathrm{i}, \tag{49}$$

where

$$h = \sqrt[3]{116 + 12\sqrt{93}}. \tag{50}$$

Let point out that the real root x_1 of the algebraic equation (46) is an irrational number and equates to the golden 2-proportion τ_2 [18]. The value of h given by (50) is also irrational; therefore, the roots x_2 and x_3 are complex numbers, the real parts of which are irrational.

Using (37), it is possible to write the Binet formula for the Fibonacci 2-numbers in the following form:

$$F_2(n) = k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n. \tag{51}$$

Using (45) and (51), it is possible to derive the following system of algebraic equations:

$$\begin{cases} F_2(0) = k_1 + k_2 + k_3; \\ F_2(1) = k_1x_1 + k_2x_2 + k_3x_3; \\ F_2(2) = k_1x_1^2 + k_2x_2^2 + k_3x_3^2, \end{cases} \tag{52}$$

where $F_2(0) = 0$ and $F_2(1) = F_2(2) = 1$ according to (45).

Solving the system (52) we get:

$$k_1 = \frac{2h(h+2)}{(h^3+8)}, \tag{53}$$

$$k_2 = \frac{\left(-(h+2) + \mathrm{i}\sqrt{3}(h-2)\right)h}{(h^3+8)}, \tag{54}$$

$$k_3 = \frac{\left(-(h+2) - \mathrm{i}\sqrt{3}(h-2)\right)h}{(h^3+8)}. \tag{55}$$

Note that, due to the irrationality of h given by (50), the coefficient (53) is an irrational number. Therefore, the coefficients (54) and (55) are complex conjugate numbers, real parts of which are irrational.

Using (53)–(55), we can now write the following Binet formula for the Fibonacci 2-numbers:

$$F_2(n) = \frac{2h(h+2)}{(h^3+8)}\left(\frac{h}{6} + \frac{2}{3h} + \frac{1}{3}\right)^n + \frac{\left(-(h+2) + \mathrm{i}\sqrt{3}(h-2)\right)h}{(h^3+8)}\left(-\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} - \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right)\right)^n + \frac{\left(-(h+2) - \mathrm{i}\sqrt{3}(h-2)\right)h}{(h^3+8)}\left(-\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} + \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right)\right)^n. \tag{56}$$

It seems incredible at first sight that formula (56) is apparently a complicated combination of complex numbers with irrational real components, actually gives the integer Fibonacci 2-series $F_p(n)$ for any integer $n = 0, \pm1, \pm2, \pm3, \ldots$.

5.2. The Binet formula for the Lucas 2-numbers

As we mentioned above, the Binet formula (13), for the classical Lucas numbers, is a special case of formula (40) if the coefficients k_1 and k_2 in (40) are identically equal to 1. Similarly, if in formula (51), we take $k_1 = k_2 = k_3 = 1$, then we get the integer series that is called the *Lucas 2-series* $L_2(n)$.

Thus, the Lucas 2-series is given by the following formula:

$$L_2(n) = (x_1)^n + (x_2)^n + (x_3)^n, \tag{57}$$

or

$$L_2(n) = \left(\frac{h}{6} + \frac{2}{3h} + \frac{1}{3}\right)^n + \left(-\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} + \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right)\right)^n + \left(-\frac{h}{12} - \frac{1}{3h} + \frac{1}{3} - \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{3h}\right)\right)^n. \tag{58}$$

Let us calculate the initial terms of the series (57). For the case $n = 0$, it follows directly from (57):

$$L_2(0) = 3. \tag{59}$$

For the calculation of the Lucas 2-numbers $L_2(1)$, $L_2(2)$ we can use Theorem 2. Indeed, formula (57) is a special case ($p = 2$) of the more general formula (36). According to (36), we have:

$$L_2(1) = L_2(2) = 1. \tag{60}$$

Thus, the Lucas 2-series is the following numerical sequence that extends infinitely to the left and right of $L_2(0) = 3$, depending on n:

$$\ldots, 1, -5, 2, 3, -2, 0, 3, 1, 1, 4, 5, 6, 10, 15, 21, 31, 46, 67, 98, 144, \ldots \tag{61}$$

Note that every term $L_2(n)$ ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$) is determined according to the following recurrence relation:

$$L_2(n) = L_2(n-1) + L_2(n-3) \tag{62}$$

for the initial conditions (59) and (60).

6. The Binet formulas for the Fibonacci and Lucas 3-numbers

6.1. The Binet formula for the Fibonacci 3-numbers

For the case $p = 3$, the basic recurrence relation (1), the initial conditions (38), and the characteristic equation (24) take the following forms, respectively:

$$F_3(n) = F_3(n-1) + F_3(n-3); \tag{63}$$

$$F_3(0) = 0, \quad F_3(1) = F_3(2) = F_3(3) = 1; \tag{64}$$

$$x^4 = x^3 + 1. \tag{65}$$

Eq. (65) has four roots—two real roots, x_1 and x_2, and two complex conjugate roots, x_3 and x_4. The roots of Eq. (65) are irrational complex numbers that have complicated symbolic representation. Therefore, we will use their approximate numerical values:

$$x_1 = 1.38\ldots;$$
$$x_2 = -0.819\ldots;$$
$$x_3 = 0.219\ldots + \mathrm{i}0.914\ldots;$$
$$x_4 = 0.219\ldots - \mathrm{i}0.914\ldots.$$

It is necessary to point out that the root x_1 of the algebraic equation (65) is the golden 3-proportion τ_3 [18].

Formula (37) for the Fibonacci 3-numbers takes the following form:

$$F_3(n) = k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n + k_4(x_4)^n. \tag{66}$$

From (64) and (66), it is possible to derive the following system of equations:

$$\begin{cases} F_3(0) = k_1 + k_2 + k_3 + k_4; \\ F_3(1) = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4; \\ F_3(2) = k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + k_4x_4^2; \\ F_3(3) = k_1x_1^3 + k_2x_2^3 + k_3x_3^3 + k_4x_4^3, \end{cases} \tag{67}$$

where $F_3(0) = 0$ and $F_3(1) = F_3(2) = F_3(3) = 1$ according to (64).

Solving the system (67) we get the following coefficients:

$$k_1 = 0.3969\ldots; \quad k_2 = -0.1592\ldots; \quad k_3 = -0.1188\ldots - \mathrm{i}0.2045\ldots; \quad k_3 = -0.1188\ldots + \mathrm{i}0.2045\ldots \tag{68}$$

Therefore, using (68), we represent the Binet formula for the Fibonacci 3-numbers in the following numerical form:

$$F_3(n) = 0.3969 \cdot 1.38^n \cdot -0.1592(-0.819)^n + (-0.1188 - \mathrm{i}0.2045)(0.219 + \mathrm{i}0.914)^n + (-0.1188 + \mathrm{i}0.2045)(0.219 - \mathrm{i}0.914)^n. \tag{69}$$

6.2. Binet formula for the Lucas 3-numbers

If we accept $k_1 = k_2 = k_3 = k_4 = 1$ in (66) then we can get the following formula:

$$L_3(n) = (x_1)^n + (x_2)^n + (x_3)^n + (x_4)^n, \tag{70}$$

that gives the *Lucas 3-series* $L_3(n)$.

Using the given values for each x_i, we represent the formula (70) for the Lucas 3-numbers in the following numerical form:

$$L_3(n) = 1.38^n \cdot +(-0.819)^n + (0.219 + \mathrm{i}0.914)^n + (0.219 - \mathrm{i}0.914)^n. \tag{71}$$

Hence, by using (70) and Theorem 2, we calculate the initial terms of the Lucas 3-numbers. Here we have:

$$L_3(0) = 4, \tag{72}$$

and

$$L_3(1) = L_3(2) = L_3(3) = 1. \tag{73}$$

Consequently, the Lucas 3-series is:

$$\ldots, 4, -7, 3, 0, 4, -3, 0, 0, 4, 1, 1, 1, 5, 6, 7, 8, 13, 19, 26, 34, 47, 66, 92, \ldots \tag{74}$$

in which each term $L_3(n)$ (for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$) is determined according to the following recurrence relation:

$$L_3(n) = L_3(n-1) + L_3(n-4), \tag{75}$$

using the initial conditions (72) and (74).

7. The Binet formulas for the Fibonacci and Lucas 4-numbers

7.1. The Binet formulas for the Fibonacci 4-numbers

For the case $p = 4$ formulas (1), (38) and (24) take the following forms, respectively:

$$F_4(n) = F_4(n-1) + F_4(n-4); \tag{76}$$

$$F_4(0) = 0; \quad F_4(1) = F_4(2) = F_4(3) = F_4(4) = 1; \tag{77}$$

$$x^5 = x^4 + 1. \tag{78}$$

Eq. (78) has five roots, one real is root x_1 that coincides with the golden 4-proportion τ_4 [18] and two pairs of complex conjugate roots x_2, x_3 and x_4, x_5:

$$\begin{aligned}
x_1 &= \frac{h}{6} + \frac{2}{h} = 1.3247\ldots; \\
x_2 &= \frac{1}{2} - \frac{\mathrm{i}\sqrt{3}}{2} = 0.5 - \mathrm{i}\ 0.866\ldots; \\
x_3 &= \frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2} = 0.5 + \mathrm{i}\ 0.866\ldots; \\
x_4 &= -\frac{h}{12} - \frac{1}{h} - \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{h}\right) = -0.6623\ldots - \mathrm{i}\ 0.5623\ldots; \\
x_5 &= -\frac{h}{12} - \frac{1}{h} + \mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6} - \frac{2}{h}\right) = -0.6623\ldots + \mathrm{i}\ 0.5623\ldots,
\end{aligned} \tag{79}$$

where $h = \sqrt[3]{(108 + 12\sqrt{69})}$.

For this case, formula (37) for the Fibonacci 4-numbers takes the following form:

$$F_4(n) = k_1(x_1)^n + k_2(x_2)^n + k_3(x_3)^n + k_4(x_4)^n + k_5(x_5)^n. \tag{80}$$

From (77) and (78), it is possible to derive the following system of algebraic equations:

$$\begin{cases} F_4(0) = k_1 + k_2 + k_3 + k_4 + k_5; \\ F_4(1) = k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5; \\ F_4(2) = k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + k_4x_4^2 + k_5x_5^2; \\ F_4(3) = k_1x_1^3 + k_2x_2^3 + k_3x_3^3 + k_4x_4^3 + k_5x_5^3; \\ F_4(4) = k_1x_1^4 + k_2x_2^4 + k_3x_3^4 + k_4x_4^4 + k_5x_5^4, \end{cases} \tag{81}$$

where $F_4(0) = 0$ and $F_4(1) = F_4(2) = F_4(3) = F_4(4) = 1$ according to (77).

Solving the system (81) we get the approximate values of the coefficients:

$$\begin{aligned} &k_1 = 0.38095\ldots; \quad k_2 = -0.07133\ldots + \mathrm{i}0.2063\ldots; \quad k_3 = -0.07133\ldots - \mathrm{i}0.2063\ldots; \\ &k_4 = -0.1191\ldots + \mathrm{i}0.04577\ldots; \quad k_5 = -0.1191\ldots - \mathrm{i}0.04577\ldots \end{aligned} \tag{82}$$

Substituting the numerical values of the roots (79) and the coefficients (82) into formula (80), we can get the following Binet formula for the Fibonacci 4-numbers in the numerical form.

7.2. The Binet formula for the Lucas 4-numbers

If we accept $k_1 = k_2 = k_3 = k_4 = k_5 = 1$ in (80), then we get the following formula:

$$L_4(n) = (x_1)^n + (x_2)^n + (x_3)^n + (x_4)^n + (x_5)^n, \tag{83}$$

that gives the *Lucas 4-series.*

Substituting the expressions (79) into (83), we get formula (83) in the analytic form:

$$\begin{aligned} L_4(n) = {} & \left(\frac{h}{6}+\frac{2}{h}\right)^n + \left(\frac{1}{2}-\frac{\mathrm{i}\sqrt{3}}{2}\right)^n + \left(\frac{1}{2}+\frac{\mathrm{i}\sqrt{3}}{2}\right)^n + \left(-\frac{h}{12}-\frac{1}{h}-\mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{h}\right)\right)^n \\ & + \left(-\frac{h}{12}-\frac{1}{h}+\mathrm{i}\frac{\sqrt{3}}{2}\left(\frac{h}{6}-\frac{2}{h}\right)\right)^n. \end{aligned} \tag{84}$$

The initial terms of the Lucas 4-series can be calculated by using (83) and Theorem 2. Here we have:

$$L_4(0) = 5, \tag{85}$$

and

$$L_4(1) = L_4(2) = L_4(3) = L_4(4) = 1. \tag{86}$$

Hence, the following Lucas 4-series are:

$$\ldots, 5, -9, 4, 0, 0, 5, -4, 0, 0, 0, 5, 1, 1, 1, 1, 6, 7, 8, 9, 10, 16, 23, 31, 40, 50, 66, \ldots \tag{87}$$

in which each term $L_4(n)$ ($n = 0, \pm1, \pm2, \pm3, \ldots$) is determined by the recurrence formula:

$$L_4(n) = L_4(n-1) + L_4(n-5), \tag{88}$$

for the initial conditions (85) and (86).

8. The Binet formulas for the Fibonacci and Lucas p-numbers (a general case)

8.1. The Binet formula for the Fibonacci p-numbers

In the general case the Binet formula for the Fibonacci p-series has the form given by (37). The coefficients $k_1, k_2, k_3, \ldots, k_p, k_{p+1}$ in formula (37) can be calculated by means of the solution of the system (39).

Let us prove the following theorem.

Theorem 3. *For the given integer $p > 0$ any Fibonacci p-number $F_p(n)$ ($n = 0, \pm1, \pm2, \pm3, \ldots$) can be represented in the form*:

$$F_p(n) = k_1(x_1)^n + k_2(x_2)^n + \cdots + k_{p+1}(x_{p+1})^n,$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the characteristic equation (24), and $k_1, k_2, \ldots, k_{p+1}$ are constant coefficients, which are one solution of the system (39).

Proof. The initial terms of the recurrence Fibonacci p-series for the values $n = 0, 1, 2, \ldots, p$ are determined according to (39).

Let us represent the Fibonacci p-number $F_p(p + 1)$ using (37):

$$F_p(p+1) = k_1(x_1)^{p+1} + k_2(x_2)^{p+1} + \cdots + k_{p+1}(x_{p+1})^{p+1}. \tag{89}$$

Given the identity from Eq. (26) we now write (89) in the form:

$$F_p(p+1) = [k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p] + [k_1(x_1)^0 + k_2(x_2)^0 + \cdots + k_{p+1}(x_{p+1})^0]. \tag{90}$$

Therefore, according to (39), we have:

$$F_p(p+1) = F_p(p) + F_p(0), \tag{91}$$

i.e., the basic recurrence relation (1) is true for the Fibonacci p-number $F_p(p + 1)$.

Furthermore, by applying formula (37) for the Fibonacci p-numbers $F_p(p + 2), F_p(p + 3), \ldots, F_p(n), \ldots$, and using identity (26), it is easy to prove that the formula (37) is valid for all positive values of n.

Let us prove that the formula (37) is true for the negative values of $n = -1, -2, -3, \ldots$.

In order to do this, let us consider the formula (37) for the case $n = -1$:

$$F_p(-1) = k_1(x_1)^{-1} + k_2(x_2)^{-1} + \cdots + k_{p+1}(x_{p+1})^{-1}. \tag{92}$$

Let write identity (26) in the form:

$$x_k^{n-p-1} = x_k^n - x_k^{n-1}. \tag{93}$$

For the case $n = p$, identity (93) takes the form:

$$x_k^{-1} = x_k^p - x_k^{p-1}. \tag{94}$$

Using Eq. (94), we represent expression (92) by the following equation:

$$F_p(-1) = [k_1(x_1)^p + k_2(x_2)^p + \cdots + k_{p+1}(x_{p+1})^p] - [k_1(x_1)^{p-1} + k_2(x_2)^{p-1} + \cdots + k_{p+1}(x_{p+1})^{p-1}]. \tag{95}$$

Hence, by (39), we see that expression (95) is equivalent to:

$$F_p(-1) = F_p(p) - F_p(p-1) = 0,$$

that is, formula (95) gives the Fibonacci p-number $F_p(-1) = 0$.

Furthermore, considering formula (37) for the negative values $n = -2, -3, -4, \ldots$ and using (93), it is easy to prove that formula (37) is valid for all negative values n. □

Theorem 4. *For the given integer $p > 0$, the formula*

$$L_p(n) = (x_1)^n + (x_2)^n + \cdots + (x_{p+1})^n, \tag{96}$$

where $x_1, x_2, \ldots, x_{p+1}$ are the roots of the characteristic equation (24), gives the Lucas p-series $L_p(n)$ ($n = 0, \pm1, \pm2, \pm3, \ldots$), which can be given by the recurrence relation:

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \tag{97}$$

for the following initial conditions:

$$L_p(0) = p + 1, \tag{98}$$

$$L_p(1) = L_p(2) = \cdots = L_p(p) = 1. \tag{99}$$

Proof. Let us prove that Theorem 4 is true for the initial terms of the Lucas p-series. In fact, for the case $n = 0$, we write the formula (96) as:

$$L_p(0) = (x_1)^0 + (x_2)^0 + \cdots + (x_{p+1})^0 = 1 + 1 + \cdots + 1 = p + 1.$$

This proves that expression (98) is true.

We now consider the expression (96) for the cases $n = 1, 2, 3, \ldots, p$, that is,

$$L_p(1) = x_1 + x_2 + \cdots + x_{p+1};$$
$$L_p(2) = (x_1)^2 + (x_2)^2 + \cdots + (x_{p+1})^2;$$
$$L_p(3) = (x_1)^3 + (x_2)^3 + \cdots + (x_{p+1})^3;$$
$$\cdots;$$
$$L_p(p) = (x_1)^p + (x_2)^p + \cdots + (x_{p+1})^p.$$

According to Theorem 2, the expressions we previously considered are all identically equal to 1. This proves that the formula (96) is true for the cases $n = 1, 2, 3, \ldots, p$.

The validity of formula (96), for any Lucas p-number $L_p(n)$ ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$) given by the recurrence relation (97) with initial conditions (98) and (99), is proved similarly to Theorem 3 and is based on identities (26) and (93) that connect the roots of characteristic equation (24).

Thus, the basic result of Theorem 4 is the introduction of a new class of recurrent series, given by the recurrence relation (97) with initial conditions (98) and (99). The class is expressed by the formula (96). These recurrent series are a natural generalization of the classical Lucas numbers (42) that correspond to the case $p = 1$.

We calculate the initial terms of the Lucas p-series by using (96) and Theorem 2. Here we have:

$$L_p(0) = p + 1, \tag{100}$$

and

$$L_p(1) = L_p(2) = L_p(3) = \cdots = L_p(p) = 1. \tag{101}$$

Then we will represent the Lucas p-series by the numerical sequence, in which each term $L_p(n)$ ($n = 0, \pm 1, \pm 2, \pm 3, \ldots$) is determined according to the recurrence relation

$$L_p(n) = L_p(n-1) + L_p(n-p-1), \tag{102}$$

for the initial conditions (100) and (102). □

9. Conclusion and discussion

The preceding fundamental mathematical formulas, which allow us to express in a compact form some deep mathematical laws found in the Universe, play a great role in theoretical natural sciences. The majority of these formulas bear the names of their founders: "Euler's formulas", "Gauss' law", "Moivre's formulas", to name a few. Binet formulas (12) and (13), which express the connection of the Golden Section to the Fibonacci and Lucas numbers, belong to the category of such formulas. These formulas are named in the honor of the French mathematician *Jacques Philippe Marie Binet* (1786–1856) (of 19th century) who was elected a member of the Parisian Academy of sciences in 1843. He published many works on mechanics, mathematics, and astronomy. In mathematics, Binet introduced the notion of "the Beta-function", he considered the linear differential equations with the variable factors, and made essential contributions to the development of the matrix theory. In 1812, he discovered the rule of matrix multiplication. All this was sufficient to immortalize his name in the history of mathematics. However, we may consider the Binet formulas (12) and (13) his highest mathematical achievement.

The analysis of the formulas (12) and (13) allow us to feel true "aesthetic pleasure" and become confident in the power of human mind once again. In fact, we see from those formulas that the Fibonacci and Lucas numbers always have integer values. On the other hand, any degree of the Golden Proportion is an irrational number, that is, the right parts of formulas (12) and (13) are some combinations of irrational numbers.

For example, the Lucas number $L_2 = 3$, according to (13), can be represented in the form:

$$3 = \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1+\sqrt{5}}{2}\right)^{-2}, \tag{103}$$

and the Fibonacci number $F_5 = 5$ can be represented in the form:

$$5 = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^5 + \left(\frac{1+\sqrt{5}}{2}\right)^{-5}}{\sqrt{5}}. \tag{104}$$

Thus, the Binet formulas (12) and (13) express a connection of integers (the Fibonacci and Lucas numbers) to irrationals (the Golden Proportion powers). We further demonstrated in [25,27] that the Binet formulas are the basis of the hyperbolic Fibonacci and Lucas functions that has a strategic importance for modern theoretical natural sciences.

From such point of view, it is necessary to estimate the new mathematical formulas (37) and (96) that give the analytical form for the Fibonacci and Lucas p-numbers. It is also necessary to point out that the number of new Binet formulas for the Fibonacci and Lucas p-numbers obtained in the present article is theoretically infinite ($p = 1,2,3\ldots$), and the classical Binet formulas (12) and (13) are their special cases for $p = 1$.

The situation with the comprehension of the "physical sense" of the new Binet formulas, which, at first sight, seem incredible, is reminiscent of the situation in mathematics when complex numbers were introduced. It is well-known that a solution of the cubic equation

$$x^3 + px + q = 0, \tag{105}$$

is given by means of the following formula

$$x = u + v,$$

where

$$u = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}; \quad v = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}}; \quad uv = -\frac{p}{3}; \quad \Delta = \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

Using basic algebra, we see that there are three roots of Eq. (105), namely:

(1) For the case $\Delta > 0$ we get one real and two complex conjugate roots; for example, the equation $x^3 + 15x + 124 = 0$, for which $\Delta > 0$, has the roots: $x_1 = -4; x_{2,3} = 2 \pm 3i\sqrt{3}$.
(2) For the case $\Delta = 0$, $p \neq 0$, $q \neq 0$ the equation has three real roots, here two of them are coincident; for example the equation $x^3 - 12x + 16 = 0$ has the following roots: $x_1 = -4$; $x_{2,3} = 2$.
(3) For the case $\Delta < 0$ we have the most interesting case, so called "non-reducible" case, when we need to extract the root of the 3-d degree from complex numbers and the cubic roots u, v are complex numbers. Nevertheless, in this case Eq. (105) has the different real roots. For example, the equation $x^3 - 21x + 20 = 0$, for which

$$\Delta = -243; \quad u = \sqrt[3]{-10 + \sqrt{-243}}; \quad v = \sqrt[3]{-10 - \sqrt{-243}}, \tag{106}$$

has the real roots 1, 4, −5. The substitution of these roots into the corresponding algebraic equation will convince us of this.

In the 16th century, this fact seemed paradoxical for mathematicians. In fact, all coefficients of the equation $x^3 - 21x + 20 = 0$ and all its roots are real numbers but the intermediate calculations lead us to "imaginary", "false", "nonexistent" numbers such as numbers in (106). Mathematicians appeared to be in very difficult situation, as had happened to them repeatedly (since the discovery of irrational numbers). The complete neglect of the numbers such as (106) would have meant to refuse the general formulas for the solutions of the third degree algebraic equations, as well as other remarkable mathematical achievements. On the other hand, to recognize these persistently appearing "monstrous" numbers such as (106), now understood to be as valid as real numbers, was intolerable from the common sense point of view. Many mathematicians did not recognize the "monstrous" numbers such as (106) for a long time. For example, Descartes considered that there would never be a serious interpretation for the complex numbers and they were forever doomed to remain only as "imaginary numbers." We have used the term "imaginary numbers" since the 17th century. It is perpetual reminder of Descartes' skepticism. Others great mathematicians of that time, including Newton and Leibnitz, had the same opinion.

The complex numbers finally obtained recognition after the works of the French mathematician Abraham de Moivre (1667–1754), who is the author of the well-known Moivre's formulas. Based on Moivre's formulas, Euler established the validity of the following expressions for the trigonometric functions:

$$\cos x = \frac{e^{xi} + e^{-xi}}{2}, \quad \sin x = \frac{e^{xi} - e^{-xi}}{2i}. \tag{107}$$

Concerning the importance of these formulas, Euler wrote the following: "*From here it is clear, how imaginary quantities are led to the sine's and cosine's of the real arches*".

Probably, the similar statement is pertinent for the analytical formulas given in the present article for the Fibonacci and Lucas p-numbers.

We now return to the new Binet formulas, which we introduced in the present article. If we take into consideration the above thoughts regarding the complex numbers, we come to the conclusion that these formulas, similar to "Moivre's formulas" and "Euler's formulas" (107), touch upon some very deep numerical concepts. These concepts are the relationships between the integers (the Fibonacci and Lucas p-numbers), irrationals (the Golden p-Proportions), the complex numbers, and the binomial factors.

It is now difficult to predict in which part of science the above-introduced Binet formulas for the Fibonacci and Lucas p-numbers will have the most effective application. It is clear that the *theory of the Binet formulas*, which we stated in the present article, is a challenge to the branch of modern mathematics known as the *Fibonacci numbers theory* [15–17], which is actively developing. Given the concepts of *new Binet formulas*, which we introduced in the present article, the *hyperbolic Fibonacci and Lucas functions* [25,27], which are based on the Golden Section, and the *generalized Fibonacci matrixes* [26], which are based on the Fibonacci p-numbers, we can quite pertinently speak about creation of the new mathematical apparatus for scientific research. This apparatus has been recognized in modern science as *Harmony Mathematics* [23,24]. The authors are sure that the new mathematical apparatus will attract the attention of theoretical physicists if we take into consideration the active interest of physical science to the Fibonacci numbers and the Golden Section [1–14]. It is already possible to predict the application of the new Binet formulas in the new coding theory [30] that is based on the Fibonacci matrices [26], the elements of which are the Fibonacci p-numbers.

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