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Generalized Determinantal Identities Involving Lucas Polynomials

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Abstract

Determinants have played a significant part in various areas in mathematics. There are different perspectives on the study of determinants. Many problems on determinants of Fibonacci sequence and Lucas sequence appeared in various issues of the Fibonacci Quarterly. In this paper, we provide generalized determinantal identities involving Lucas polynomials and other polynomials in different orders. Entries of these determinants satisfying the recurrence relation of Lucas polynomials and other polynomials like Fibonacci polynomials, Chebyshev Polynomials, Pell polynomials, Pell-Lucas polynomials, Vieta-Lucas Polynomials.

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1. INTRODUCTION

There is a long tradition of using matrices and determinants to study Fibonacci numbers and Lucas numbers. Cahill and Narayan [11] show how Fibonacci and

Lucas numbers arise as determinants of some tridiagonal matrices. Bicknell – Johnson and Spears [10] use elementary matrix operations and determinants to generate classes of identities for generalized Fibonacci numbers. T. Koshy [14] explained two chapters on the use of matrices and determinants in Fibonacci numbers.

One may notice several practical and effective instruments for calculating determinants in the nice survey articles [5] and [6]. Much attention has been paid to the evaluation of determinants of matrices, especially when their entries are given recursively [6]. In this paper, we provide determinantal identities of Lucas polynomials. We are also establishing relations of Lucas polynomials with other polynomials in determinant form.

2. PRELIMINARIES

Before presenting our main theorems, we will need to introduce some known polynomials, results and notations.

Fibonacci sequence is defined as

$$F_n = F_{n-1} + F_{n-2},\ n \geq 2 \quad \text{with } F_0 = 0, F_1 = 1, \qquad [2.1]$$

Where F_n is a n^{th} number of sequence.

Lucas sequence is defined as

$$L_n = L_{n-1} + L_{n-2},\ n \geq 2 \quad \text{with } L_0 = 2, L_1 = 1, \qquad [2.2]$$

Where L_n is a n^{th} number of sequence.

Fibonacci polynomial [1] is defined as

$$f_{n+1}(x) = xf_n(x) + f_{n-1}(x),\ n \geq 2 \text{ with } f_0(x) = 0,\ f_1(x) = 1 \qquad [2.3]$$

Lucas polynomials [1] is defined as

$$l_{n+1}(x) = xl_n(x) + l_{n-1}(x), n \geq 2 \text{ with } l_0(x) = 2\ \ l_1(x) = x \qquad [2.4]$$

Chebyshev Polynomials [8] of first kind is defined as

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n \geq 2 \text{ with } T_0(x) = 1,\ T_1(x) = x \qquad [2.5]$$

Chebyshev Polynomials [8] of second kind is defined as

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),\ n \geq 2 \text{ with } U_0(x) = 1, U_1(x) = 2x \qquad [2.6]$$

Pell Polynomials [3] is defined as

$$P_{n+1}(x) = 2xP_n(x) + P_{n-1}(x),\ n \geq 2 \text{ with } P_0(x) = 0, P_1(x) = x \qquad [2.7]$$

Pell-Lucas Polynomials [3] is defined as

$$Q_{n+1}(x) = 2xQ_n(x) + Q_{n-1}(x),\ n \geq 2 \text{ with } Q_0(x) = 2, Q_1(x) = 2x \qquad [2.8]$$

Vieta-Lucas Polynomials [12] is defined as

$$\Omega_{n+1}(x) = x\Omega_n(x) - \Omega_{n-1}(x),\ n \geq 2 \text{ with } \Omega_0(x) = 2, \Omega_1(x) = x \qquad [2.9]$$

Some known results [1] are

$$2f_{n+m}(x) = f_n(x)l_m(x) + f_m(x)l_n(x), \qquad [2.10]$$

$$l_n(x) = f_{n+1}(x) + f_{n-1}(x), n \geq 2 \qquad [2.11]$$

$$\left(x^2+4\right) f_n^2(x) = l_n^2(x) - 4(-1)^n, \quad [2.12]$$

$$\left(x^2+4\right) f_n(x) = l_{n+1}(x) + l_{n-1}(x) \quad [2.13]$$

$$f_{n+1}(x)\, f_{n-1}(x) - f_n^2(x) = (-1)^n, \quad [2.14]$$

$$f_{n+m}(x) = f_n(x)\, f_{m-1}(x) + f_{n+1}(x)\, f_m(x) \quad [2.15]$$

3. MAIN DETERMINANTAL IDENTITIES

We define a family of Lucas polynomial as

$$S = \left\{ l_{n+p}(x), l_{n+q}(x), l_{n+q+r}(x), l_{n+s}(x), l_{n+s+r}(x) \right\},$$

Where n and p are non-negative integers, q and s are positive integers

With $0 \le$ p<q , q+1<s , r=1,

Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a + bx$ and

$$l_{n+q+r}(x) = xl_{n+q} + l_{n+p},\ l_{n+s}(x) = xl_{n+q+r} + l_{n+q},\ l_{n+s+r}(x) = xl_{n+s}(x) + l_{n+q+r}$$

We may also define family of other polynomials like Fibonacci polynomials, Chebyshev Polynomials, Pell polynomials, Pell-Lucas polynomials, Vieta-Lucas Polynomials in same manner.

Theorem 1: If n and p are non-negative integers, q is positive integer
With $0 \le$ p<q , r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q+r}(x) & l_{n+p}(x) & l_{n+q}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+p}(x) \end{vmatrix} = l_{n+p}^3(x) + l_{n+q}^3(x) + l_{n+q+r}^3(x) - 3l_{n+p}(x) l_{n+q}(x) l_{n+q+r}(x)$$

Proof: Let $\Delta = \begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q+r}(x) & l_{n+p}(x) & l_{n+q}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+p}(x) \end{vmatrix}$ [3.1]

Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a + bx$, Now

$$\Delta = \begin{vmatrix} a & b & a+bx \\ a+bx & a & b \\ b & a+bx & a \end{vmatrix} \tag{3.2}$$

Applying $R_1 + R_2 \to R_1$

$$\Delta = \begin{vmatrix} 2a+bx & a+b & a+b+bx \\ a+bx & a & b \\ b & a+bx & a \end{vmatrix} \tag{3.3}$$

Applying $C_1 - C_2 \to C_1$

$$\Delta = \begin{vmatrix} a+bx-b & a+b & a+b+bx \\ bx & a & b \\ b-a-bx & a+bx & a \end{vmatrix} \tag{3.4}$$

Applying $R_1 + R_3 \to R_1$

$$\Delta = \begin{vmatrix} 0 & 2a+b+bx & 2a+b+bx \\ bx & a & b \\ b-a-bx & a+bx & a \end{vmatrix} \tag{3.5}$$

Applying $C_2 - C_3 \to C_2$

$$\Delta = \begin{vmatrix} 0 & 0 & 2a+b+bx \\ bx & a-b & b \\ b-a-bx & bx & a \end{vmatrix} \tag{3.6}$$

Expand along first row, we get

$$\Delta = (a+bx)^3 + a^3 + b^3 - 3ab(a+bx)$$

Put $l_{n+p}(x) = a$, $l_{n+q}(x) = b$ and $l_{n+q+r}(x) = a+bx$, we get

$$\begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q+r}(x) & l_{n+p}(x) & l_{n+q}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+p}(x) \end{vmatrix} =$$

$$l_{n+p}^3(x) + l_{n+q}^3(x) + l_{n+q+r}^3(x) - 3l_{n+p}(x)l_{n+q}(x)l_{n+q+r}(x) \tag{3.7}$$

Corollary 1.1: If we put x = 1 in above result, for $0 \le p<q$, r=1, we get

$$\begin{vmatrix} L_{n+p} & L_{n+q} & L_{n+q+r} \\ L_{n+q+r} & L_{n+p} & L_{n+q} \\ L_{n+q} & L_{n+q+r} & L_{n+p} \end{vmatrix} = L_{n+p}^3 + L_{n+q}^3 + L_{n+q+r}^3 - 3L_{n+p}L_{n+q}L_{n+q+r}, \tag{3.8}$$

$= 2\left[L_n{}^3 + L_{n+1}{}^3 \right]$, It can be proved easily.

Theorem 2: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & 1 \\ l_{n+q}(x) & f_{n+q}(x) & 1 \\ l_{n+q+r}(x) & f_{n+q+r}(x) & 1 \end{vmatrix} = x\left[f_{n+p}(x)l_{n+q}(x) - f_{n+q}(x)l_{n+p}(x) \right] \qquad [3.9]$$

$$= \begin{cases} -2x, & if \quad n = 0,2,4... \\ 2x, & if \quad n = 1,3,5,... \end{cases}$$

Proof: Let $\Delta = \begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & 1 \\ l_{n+q}(x) & f_{n+q}(x) & 1 \\ l_{n+q+r}(x) & f_{n+q+r}(x) & 1 \end{vmatrix}$

Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a + bx$ and

$f_{n+p}(x) = c, f_{n+q}(x) = d$, Then by [2.2] $f_{n+q+r}(x) = c + dx$, Now

$$\Delta = \begin{vmatrix} a & c & 1 \\ b & d & 1 \\ a+bx & c+dx & 1 \end{vmatrix} \qquad [3.10]$$

Applying $R_1 - R_2 \to R_1$

$$\Delta = \begin{vmatrix} a-b & c-d & 0 \\ b & d & 1 \\ a+bx & c+dx & 0 \end{vmatrix} \qquad [3.11]$$

Applying $R_2 - R_3 \to R_2$

$$\Delta = \begin{vmatrix} a-b & c-d & 0 \\ b-(a+bx) & d-(c+dx) & 0 \\ a+bx & c+dx & 1 \end{vmatrix} \qquad [3.12]$$

Expand along third column, we get

$$\Delta = x(bc - ad) \qquad [3.13]$$

Put $l_{n+p}(x) = a, l_{n+q}(x) = b$, $l_{n+q+r}(x) = a + bx$ and

$f_{n+p}(x) = c$, $f_{n+q}(x) = d$, $f_{n+q+r}(x) = c + dx$, We get

$$\Delta = x\left[f_{n+p}(x)l_{n+q}(x) - f_{n+q}(x)l_{n+p}(x) \right] = = \begin{cases} -2x, & if \quad n = 0,2,4... \\ 2x, & if \quad n = 1,3,5,... \end{cases}$$

Corollary 2.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} L_{n+p} & F_{n+p} & 1 \\ L_{n+q} & F_{n+q} & 1 \\ L_{n+q+r} & F_{n+q+r} & 1 \end{vmatrix} = \begin{cases} -2, n = 0, 2, 4, \ldots \\ 2, n = 1, 3, 5, \ldots \end{cases} \quad [3.14]$$

It can be proved easily.

Theorem 3: If n and p are non-negative integers, q and s are positive integers With $0 \le p<q$, $q+1<s$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+s}(x) \\ l_{n+q+r}(x) & l_{n+s}(x) & l_{n+s+r}(x) \end{vmatrix} = 0$$

Proof: Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a + bx$ and $l_{n+q+r}(x) = xl_{n+q} + l_{n+p}$, $l_{n+s}(x) = xl_{n+q+r} + l_{n+q}$, $l_{n+s+r}(x) = xl_{n+s}(x) + l_{n+q+r}$

$$\text{Let } \Delta = \begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+s}(x) \\ l_{n+q+r}(x) & l_{n+s}(x) & l_{n+s+r}(x) \end{vmatrix} \quad [3.15]$$

Applying $C_1 + xC_2 \to C_1$

$$\Delta = \begin{vmatrix} l_{n+q+r}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+s}(x) & l_{n+q+r}(x) & l_{n+s}(x) \\ l_{n+s+r}(x) & l_{n+s}(x) & l_{n+s+r}(x) \end{vmatrix} \quad [3.16]$$

$$\Delta = \begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+q}(x) & l_{n+q+r}(x) & l_{n+s}(x) \\ l_{n+q+r}(x) & l_{n+s}(x) & l_{n+s+r}(x) \end{vmatrix} = 0 \quad \text{(two columns are identical)} \quad [3.17]$$

Corollary 3.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} L_{n+p} & L_{n+q} & L_{n+q+r} \\ L_{n+q} & L_{n+q+r} & L_{n+s} \\ L_{n+q+r} & L_{n+s} & L_{n+s+r} \end{vmatrix} = 0 \quad [3.18]$$

It can be proved easily.

Theorem 4: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} \left(l_{n+p}(x)+l_{n+q}(x)\right)^2 & l_{n+p}(x)l_{n+q+r}(x) & l_{n+q}(x)l_{n+q+r}(x) \\ l_{n+p}(x)l_{n+q+r}(x) & \left(l_{n+q}(x)+l_{n+q+r}(x)\right)^2 & l_{n+p}(x)l_{n+q}(x) \\ l_{n+q}(x)l_{n+q+r}(x) & l_{n+p}(x)l_{n+q}(x) & \left(l_{n+q+r}(x)+l_{n+p}(x)\right)^2 \end{vmatrix} =$$

$$2l_{n+p}(x)l_{n+q}(x)l_{n+q+r}(x)\left(l_{n+p}(x)+l_{n+q}(x)+l_{n+q+r}(x)\right)^3 \qquad [3.19]$$

Proof: Assume $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, then by [2.1] $l_{n+q+r}(x)=a+bx$
It can be proved same as Theorem: 1

Theorem 5: If n and p are non-negative integers, q is positive integer
With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) \\ l_{n+p}(x)-l_{n+q}(x) & l_{n+q}(x)-l_{n+q+r}(x) & l_{n+q+r}(x)-l_{n+p}(x) \\ l_{n+q}(x)+l_{n+q+r}(x) & l_{n+p}(x)+l_{n+q+2}(x) & l_{n+p}(x)+l_{n+q}(x) \end{vmatrix} =$$

$$l_{n+p}{}^3(x)+l_{n+q}{}^3(x)+l_{n+q+r}{}^3(x)-3l_{n+p}(x)l_{n+q}(x)l_{n+q+r}(x) \qquad [3.20]$$

Proof: Assume $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, then by [2.1] $l_{n+q+r}(x)=a+bx$
It can be proved same as Theorem: 1

Theorem 6: If n and p are non-negative integers, q and s are positive integers
With $0 \le p<q$, $q+1<s$, r=1, Prove that

$$\begin{vmatrix} l_{n+s}(x) & l_{n+q+r}(x) & l_{n+q}(x) & l_{n+p}(x) \\ l_{n+q+r}(x) & l_{n+s}(x) & l_{n+p}(x) & l_{n+q}(x) \\ l_{n+q}(x) & l_{n+p}(x) & l_{n+s}(x) & l_{n+q+r}(x) \\ l_{n+p}(x) & l_{n+q}(x) & l_{n+q+r}(x) & l_{n+s}(x) \end{vmatrix} =$$

$$x^2\left[l_{n+p}(x)+l_{n+q}(x)+l_{n+q+r}(x)+l_{n+s}(x)\right]\left[l_{n+q+r}{}^2(x)-l_{n+q}{}^2(x)\right] \qquad [3.21]$$

Proof: Assume $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, then by [2.1] $l_{n+q+r}(x)=a+bx$ and
$l_{n+q+r}(x)=xl_{n+q}+l_{n+p}$, $l_{n+s}(x)=xl_{n+q+r}+l_{n+q}$,
It can be proved same as Theorem: 1
Corollary 6.1: If we put x = 1 in above result, we get

$$\begin{vmatrix} L_{n+s}(x) & L_{n+q+r}(x) & L_{n+q}(x) & L_{n+p}(x) \\ L_{n+q+r}(x) & L_{n+s}(x) & L_{n+p}(x) & L_{n+q}(x) \\ L_{n+q}(x) & L_{n+p}(x) & L_{n+s}(x) & L_{n+q+r}(x) \\ L_{n+p}(x) & L_{n+q}(x) & L_{n+q+r}(x) & L_{n+s}(x) \end{vmatrix} \qquad [3.22]$$

$$= \left[L_{n+p}(x)+L_{n+q}(x)+L_{n+q+r}(x)+L_{n+s}(x)\right]L_{n+p}(x)\left[4L_{n+q}{}^2(x)-L_{n+p}{}^2(x)\right]$$

$= L_{2(n+p)+6}(x).L_{2(n+p)}(x)$,

It can be proved easily.

Theorem 7: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1 and $\alpha = l_{n+p}(x), \beta = l_{n+q}(x), \gamma = l_{n+q+r}(x)$, Prove that

$$\begin{vmatrix} \alpha\gamma+\beta^2 & \alpha^2 & \beta^2 & \alpha\gamma^2 & -\left(\alpha\gamma+\beta^2\right) \\ \alpha\gamma+\beta^2 & \alpha\gamma & -\alpha\gamma & \alpha\gamma^2 & \alpha\gamma+\beta^2 \\ 0 & 2\alpha\gamma & 2\alpha\gamma & -2\alpha\gamma^2 & 0 \\ \alpha\gamma+\beta^2 & -\alpha\gamma & \alpha\gamma & -\alpha\gamma^2 & \alpha\gamma+\beta^2 \\ -\left(\alpha\gamma+\beta^2\right) & \alpha^2 & \beta^2 & \alpha\gamma^2 & \alpha\gamma+\beta^2 \end{vmatrix} = -32\alpha^2\gamma^3\left(\alpha\gamma+\beta^2\right)$$

$$= -32 l_{n+p}{}^2(x) l_{n+q+r}{}^3(x)\left[l_{n+p}(x)l_{n+q+r}(x) + l_{n+q}{}^2(x)\right]^3 \qquad [3.23]$$

Proof: Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a+bx$ and $l_{n+q+r}(x) = xl_{n+q} + l_{n+p}$, $l_{n+s}(x) = xl_{n+q+r} + l_{n+q}$,

It can be proved same as Theorem: 1

Theorem 8: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & U_{n+p}(x) \\ l_{n+q}(x) & f_{n+q}(x) & 2U_{n+q}(x) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & U_{n+q+r}(x) \end{vmatrix} = 2U_n(x)\left[f_{n+p}(x)l_{n+q}(x) - l_{n+p}(x)f_{n+q}(x)\right]$$

[3.24]

Proof: Let $\Delta = \begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & U_{n+p}(x) \\ l_{n+q}(x) & f_{n+q}(x) & 2U_{n+q}(x) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & U_{n+q+r}(x) \end{vmatrix}$

Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1], $l_{n+q+r}(x) = a+bx$

$f_{n+p}(x) = c, f_{n+q}(x) = d$, Then by [2.2], $f_{n+q+r}(x) = c+dx$ and

$U_{n+p}(x) = j, U_{n+q}(x) = k$, Then by [2.6], $U_{n+q+r}(x) = 2xk - j$. Now

$$\Delta = \begin{vmatrix} a & c & j \\ b & d & 2k \\ a+bx & c+dx & 2xk-j \end{vmatrix}$$

Applying $R_1 + xR_2 \to R_1$

$$\Delta = \begin{vmatrix} a+bx & c+dx & j+2xk \\ b & d & 2k \\ a+bx & c+dx & 2xk-j \end{vmatrix} \qquad [3.25]$$

Applying $R_1 - R_3 \to R_1$

$$\Delta = \begin{vmatrix} 0 & 0 & 2j \\ b & d & 2k \\ a+bx & c+dx & 2xk-j \end{vmatrix} \qquad [3.26]$$

Expand along first row, we get

$$\Delta = 2j\left(bc-ad\right) \qquad [3.27]$$

Put $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, $f_{n+p}(x)=c$, $f_{n+q}(x)=d$ and $U_{n+p}(x)=j$,

$$\Delta = \begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & U_{n+p}\left(x\right) \\ l_{n+q}(x) & f_{n+q}(x) & 2U_{n+q}\left(x\right) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & U_{n+q+r}\left(x\right) \end{vmatrix} = 2U_n\left(x\right)\left[f_{n+p}(x)l_{n+q}(x) - l_{n+p}(x)f_{n+q}(x)\right]$$

Theorem 9: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & T_{n+p}\left(x\right) \\ l_{n+q}(x) & f_{n+q}(x) & 2T_{n+q}\left(x\right) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & T_{n+q+r}\left(x\right) \end{vmatrix} = 2T_n\left(x\right)\left[f_{n+p}(x)l_{n+q}(x) - l_{n+p}(x)f_{n+q}(x)\right] \qquad [3.28]$$

Proof: Assume $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, then by [2.1] $l_{n+q+r}(x)=a+bx$

$f_{n+p}(x)=c$, $f_{n+q}(x)=d$, Then by [2.2], $f_{n+q+r}(x)=c+dx$ and

$T_{n+p}(x)=j, T_{n+q}(x)=k$, Then by [2.7], $T_{n+q+r}(x)=2xk-j$

It can be proved same as Theorem: 8

Theorem 10: If n and p are non-negative integers, q is positive integer With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & P_{n+p}\left(x\right) \\ l_{n+q}(x) & f_{n+q}(x) & 2P_{n+q}\left(x\right) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & P_{n+q+r}\left(x\right) \end{vmatrix} = 0 \qquad [3.29]$$

Proof: $$\Delta = \begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & P_{n+p}\left(x\right) \\ l_{n+q}(x) & f_{n+q}(x) & 2P_{n+q}\left(x\right) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & P_{n+q+r}\left(x\right) \end{vmatrix}$$

Assume $l_{n+p}(x)=a$, $l_{n+q}(x)=b$, then by [2.1], $l_{n+q+r}(x)=a+bx$

$f_{n+p}(x)=c$, $f_{n+q}(x)=d$, Then by [2.2], $f_{n+q+r}(x)=c+dx$ and

$P_{n+p}(x)=j, P_{n+q}(x)=k$, Then by [2.7], $P_{n+q+r}(x)=2xk+j$, we get

$$\Delta = \begin{vmatrix} a & c & j \\ b & d & 2k \\ a+bx & c+dx & 2xk+j \end{vmatrix}$$

Applying $R_1 + xR_2 \to R_1$

$$\Delta = \begin{vmatrix} a+bx & c+dx & 2xk+j \\ b & d & 2k \\ a+bx & c+dx & 2xk+j \end{vmatrix} \qquad [3.30]$$

Here two rows are identical, $\Delta = 0$

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & P_{n+p}(x) \\ l_{n+q}(x) & f_{n+q}(x) & 2P_{n+q}(x) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & P_{n+q+r}(x) \end{vmatrix} = 0$$

Theorem 11: If n and p are non-negative integers, q is positive integer
With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & Q_{n+p}(x) \\ l_{n+q}(x) & f_{n+q}(x) & 2Q_{n+q}(x) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & Q_{n+q+r}(x) \end{vmatrix} = 0 \qquad [3.31]$$

Proof: Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1], $l_{n+q+r}(x) = a+bx$
$f_{n+p}(x) = c,\ f_{n+q}(x) = d$, Then by [2.2] , $f_{n+q+r}(x) = c+dx$ and
$Q_{n+p}(x) = j, Q_{n+q}(x) = k$, Then by [2.8] $Q_{n+q+r}(x) = 2xk+j$

It can be proved same as Theorem: 10

Theorem 12: If n and p are non-negative integers, q is positive integer
With $0 \le p<q$, r=1, Prove that

$$\begin{vmatrix} l_{n+p}(x) & f_{n+p}(x) & \Omega_{n+p}(x) \\ l_{n+q}(x) & f_{n+q}(x) & \Omega_{n+q}(x) \\ l_{n+q+r}(x) & f_{n+q+r}(x) & \Omega_{n+q+r}(x) \end{vmatrix} = 2\Omega_n(x)\left[f_{n+p}(x) l_{n+q}(x) - l_{n+p}(x) f_{n+q}(x) \right]$$

[3.32]

Proof: Assume $l_{n+p}(x) = a$, $l_{n+q}(x) = b$, then by [2.1] $l_{n+q+r}(x) = a+bx$

$f_{n+p}(x) = c,\ f_{n+q}(x) = d$, Then by [2.2] $f_{n+q+r}(x) = c+dx$ and
$\Omega_{n+p}(x) = j, \Omega_{n+q}(x) = k$, Then by [2.9] $T_{n+q+r}(x) = xk - j$

It can be proved same as Theorem: 8

4. CONCLUSION

This paper describes developed determinant identities of Lucas polynomials and derived relational identities of Lucas polynomials with others polynomials. Also extended the results in higher order determinants. These identities can be used to develop new identities of polynomials like Fibonacci polynomials, Chebyshev Polynomials, Pell polynomials, Pell-Lucas polynomials and Vieta-Lucas Polynomials.

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