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On Some Magnified Fibonacci Numbers Modulo a Lucas Number

Ram Krishna Pandey
Department of Mathematics
Indian Institute of Technology Patna
Patliputra Colony, Patna - 800013
India
ram@iitp.ac.in

Abstract

Let, as usual, F_n and L_n denote the nth Fibonacci number and the nth Lucas number, respectively. In this paper, we consider the Fibonacci numbers $F_2, F_3, \ldots, F_t$. Let $n \geq 1$ be an integer such that $4n+2 \leq t \leq 4n+5$ and $m = F_{2n+2} + F_{2n+4} = L_{2n+3}$. We prove that the integers $F_2F_{2n+2}, F_3F_{2n+2}, \ldots, F_tF_{2n+2}$ modulo m all belong to the interval $[F_{2n+1}, 3F_{2n+2}]$. Furthermore, the endpoints of the interval $[F_{2n+1}, 3F_{2n+2}]$ are obtained only by the integers F_4F_{2n+2} and $F_{4n+2}F_{2n+2}$, respectively.

1 Introduction

Let $m_1, m_2, \ldots$ be positive integers. Cusick and Pomerance [4] discuss the quantity κ where

$$\kappa := \sup_{x\in(0,1)} \min_i \|xm_i\|.$$

Here, for $x \in \mathbb{R}$, $\|x\|$ is distance to the nearest integer. Observe that if we have a finite number of integers $m_1, m_2, \ldots, m_n$ and

$$\kappa_1 := \max_{\substack{m=m_j+m_l \\ 1\leq k\leq m/2}} \frac{1}{m} \min_i |km_i|_m,$$

where $|x|_m$ denotes the absolute value of the absolutely least remainder of x mod m, then by a remark of Haralambis [5] we have that $\kappa = \kappa_1$. The quantity κ has become an important quantity now; it is involved in the well-known *Lonely Runner conjecture*. This conjecture, due to Bienia et al. [2], is stated as follows:

Suppose n runners having nonzero distinct constant speeds run laps on a unit-length circular track. Then there is a time at which all the n runners are simultaneously at least $1/(n+1)$ units from their common starting point.

The original form of this conjecture, as given by Wills [8] and Cusick [3], is as follows:

Suppose $m_1, m_2, \ldots, m_n$ be n positive integers. Then $\kappa \geq 1/(n+1)$.

This is an interpretation also due to Liu and Zhu [7]. In the present paper we show that if the integers $m_1, m_2, \ldots, m_n$ in the definition of κ are $F_2, F_3, \ldots, F_t$ and $n \geq 1$ be an integer such that $4n+2 \leq t \leq 4n+5$, then

$$\kappa \geq \frac{F_{2n+1}}{F_{2n+2}+F_{2n+4}}.$$

This also confirms the lonely runner conjecture in the case where the speeds of the runners are $F_2, F_3, \ldots, F_t$.

2 Main Results

Let $M = \{F_2, F_3, \ldots, F_t\}$. Let $n \geq 1$ be an integer such that $4n+2 \leq t \leq 4n+5$ and $m = F_{2n+2} + F_{2n+4} = L_{2n+3}$. The above bound about κ may be obtained using Theorem 5 and κ_1. The lonely runner conjecture is also satisfied in the special case $n = 0$. In this special case the set of speeds may be $\{F_2\}$, $\{F_2, F_3\}$, $\{F_2, F_3, F_4\}$, and $\{F_2, F_3, F_4, F_5\}$. Taking $x = \frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{4}$ in the definition of κ for the sets $\{F_2\}$, $\{F_2, F_3\}$, $\{F_2, F_3, F_4\}$, and $\{F_2, F_3, F_4, F_5\}$, respectively, we see that the lonely runner conjecture is satisfied.

In this section, we use the following identities and these identities may be found in Koshy [6].

1. Cassini's identity: $F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1}$.
2. $F_n^2 + F_{n+1}^2 = F_{2n+1}$.
3. d'Ocagne's identity: $F_mF_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n}$.
4. $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$.
5. $F_{2n+1} - 1 = \sum_{i=1}^{n} F_{2i}$.

Lemma 1. *(a)* $F_2F_{2n+2} \equiv F_{4n+5}F_{2n+2} \pmod{m}$.

(b) $F_2F_{2n+2} \equiv -F_{4n+4}F_{2n+2} \pmod{m}$.

(c) $F_3F_{2n+2} \equiv F_{4n+3}F_{2n+2} \pmod{m}$.

(d) $F_4F_{2n+2} \equiv -F_{4n+2}F_{2n+2} \pmod{m}$.

Proof. (a) We have

$$\begin{aligned}
mF_{2n+2} &= (F_{2n+2}+F_{2n+4})F_{2n+2}\\
&= F_{2n+2}^2+F_{2n+4}F_{2n+2}\\
&= F_{2n+2}^2+(F_{2n+3}+F_{2n+2})(F_{2n+3}-F_{2n+1})\\
&= F_{2n+2}^2+F_{2n+3}^2+F_{2n+3}F_{2n+2}-F_{2n+3}F_{2n+1}-F_{2n+2}F_{2n+1}\\
&= F_{2n+2}^2+F_{2n+3}^2+(F_{2n+2}+F_{2n+1})F_{2n+2}-F_{2n+3}F_{2n+1}-F_{2n+2}F_{2n+1}\\
&= F_{2n+2}^2+F_{2n+3}^2+F_{2n+2}^2-F_{2n+3}F_{2n+1}\\
&= F_{2n+2}^2+F_{2n+3}^2+(-1)^{2n+1} \quad \text{(using Cassini's identity)}\\
&= F_{4n+5}-1 \quad \left(\text{using the identity} F_n^2+F_{n+1}^2=F_{2n+1}\right).
\end{aligned}$$

Hence we get

$$F_{4n+5}F_{2n+2}=(1+mF_{2n+2})F_{2n+2}\equiv F_{2n+2}=F_2F_{2n+2} \pmod{m}.$$

(b) We have

$$\begin{aligned}
F_2F_{2n+2} &\equiv F_{4n+5}F_{2n+2} \pmod{m} \quad \text{(by (a))}\\
&= (F_{4n+6}-F_{4n+4})F_{2n+2}\\
&= F_{4n+6}F_{2n+2}-F_{4n+4}F_{2n+2}\\
&= (L_{2n+3}F_{2n+3})F_{2n+2}-F_{4n+4}F_{2n+2} \quad (\text{since } L_nF_n=F_{2n})\\
&\equiv -F_{4n+4}F_{2n+2} \pmod{m} \quad (\text{as } m=L_{2n+3}).
\end{aligned}$$

(c) $F_3F_{2n+2}=(F_2+F_1)F_{2n+2}\equiv(F_{4n+5}-F_{4n+4})F_{2n+2}\equiv F_{4n+3}F_{2n+2} \pmod{m}$.

(d) $F_4F_{2n+2}=(F_3+F_2)F_{2n+2}\equiv(F_{4n+3}-F_{4n+4})F_{2n+2}\equiv -F_{4n+2}F_{2n+2} \pmod{m}$.

□

Lemma 2. $F_{r+5}F_{2n+2}=(F_{r+4}+F_{r+3})F_{2n+2}\equiv \epsilon F_{4n+1-r}F_{2n+2} \pmod{m}$ *for each* $0\le r\le 2n-2$, *where*

$$\epsilon=\begin{cases}+1, & \text{if } r \text{ is even;}\\ -1, & \text{if } r \text{ is odd.}\end{cases}$$

Proof. Using the recurrence relation $F_{n+1}=F_n+F_{n-1}$ for $n\ge 1$ and Lemma 1, the proof follows by induction on r. □

Thus in order to examine the integers $F_2F_{2n+2}, F_3F_{2n+2}, \ldots, F_{4n+5}F_{2n+2}$ modulo m, it is sufficient to examine only the integers $F_2F_{2n+2}, F_3F_{2n+2}, \ldots, F_{2n+3}F_{2n+2}$ modulo m.

Lemma 3. *(a)* $F_2F_{2n+2}\equiv F_{2n+2} \pmod{m}$.

(b) $F_3F_{2n+2}\equiv -F_{2n+3} \pmod{m}$.

(c) $F_4F_{2n+2}\equiv -F_{2n+1} \pmod{m}$.

(d) $F_5F_{2n+2}\equiv 2F_{2n+2}-F_{2n+1} \pmod{m}$.

Proof. (a) $F_2F_{2n+2} \equiv F_{2n+2} \pmod{m}$ (as $F_2 = 1$).

(b) $F_3F_{2n+2} = 2F_{2n+2} \equiv -(F_{2n+1} + F_{2n+2}) = -F_{2n+3} \pmod{m}$.

(c) $F_4F_{2n+2} = (F_2 + F_3)F_{2n+2} \equiv F_{2n+2} - F_{2n+3} = -F_{2n+1} \pmod{m}$.

(d) $F_5F_{2n+2} = 5F_{2n+2} \equiv 2F_{2n+2} - F_{2n+1} \pmod{m}$.

□

We observe that in all above four congruences if R is the remainder then we have, $F_{2n+1} \leq |R| \leq \frac{m}{2}$, i.e., we have that the positive remainder is always in the interval $[F_{2n+1}, 3F_{2n+2}]$.

Inductively, we can prove that

$$F_{2n+3-k} = F_{2n-1-k}F_5 + F_{2n-2-k}F_4$$

for each $0 \leq k \leq 2n-3$. Hence we have

$$\begin{aligned}
F_{2n+3-k}F_{2n+2} &= (F_{2n-1-k}F_5 + F_{2n-2-k}F_4)F_{2n+2} \\
&\equiv F_{2n-1-k}(2F_{2n+2} - F_{2n+1}) - F_{2n-2-k}F_{2n+1} \pmod{m} \quad \text{(using (c), (d) of Lemma 3)} \\
&= F_{2n-1-k}F_{2n+2} + F_{2n-1-k}(F_{2n+1} + F_{2n}) - F_{2n-1-k}F_{2n+1} - F_{2n-2-k}F_{2n+1} \\
&= F_{2n-1-k}F_{2n+2} + F_{2n-1-k}F_{2n} - F_{2n-2-k}F_{2n+1} \\
&= F_{2n-1-k}F_{2n+2} + (-1)^{2n-2-k}F_{2+k} \quad \text{(using d'Ocagne's identity)}\,.
\end{aligned}$$

Thus we get

$$F_{2n+3-k}F_{2n+2} \equiv F_{2n-1-k}F_{2n+2} + \epsilon F_{2+k} \pmod{m}, \ldots \tag{1}$$

where

$$\epsilon = \begin{cases} +1, & \text{if } k \text{ is even;} \\ -1, & \text{if } k \text{ is odd.} \end{cases}$$

Lemma 4. *Let* $R_k = F_{2n-1-k}F_{2n+2} + \epsilon F_{2+k}$ *for each* $0 \leq k \leq 2n-3$*, where*

$$\epsilon = \begin{cases} +1, & \textit{if } k \textit{ is even;} \\ -1, & \textit{if } k \textit{ is odd.} \end{cases}$$

Then $F_{2n+1} < |R_k|_m < \frac{m}{2}$.

Proof. We partition the set $\{0, 1, 2, \ldots, 2n-3\}$ into the sets A_2, A_3, A_4, and A_5 where

$$A_i = \left\{2n - 1 - i - 4t : 0 \leq t \leq \left\lfloor \frac{2n-1-i}{4} \right\rfloor \right\},$$

for each $2 \leq i \leq 5$. Notice that

$$\epsilon = \begin{cases} +1, & \text{if } k \in A_3 \cup A_5; \\ -1, & \text{if } k \in A_2 \cup A_4. \end{cases}$$

We consider the following cases:

Case I: ($k \in A_2$). Clearly (1) implies that

$$F_{6+4t}F_{2n+2} \equiv F_{2+4t}F_{2n+2} - F_{2n-1-4t} \pmod{m}.$$

$t = 0 \Rightarrow F_6F_{2n+2} \equiv F_2F_{2n+2} - F_{2n-1} \equiv F_{2n+2} - F_{2n-1} \pmod{m}$. Observe that $F_{2n+1} < F_{2n+2} - F_{2n-1} < m/2$.

$t = 1 \Rightarrow F_{10}F_{2n+2} \equiv F_6F_{2n+2} - F_{2n-5} \equiv F_{2n+2} - F_{2n-1} - F_{2n-5} \pmod{m}$. Observe that $F_{2n+1} < F_{2n+2} - F_{2n-1} - F_{2n-5} < m/2$.

Inductively, we let $t = \lfloor \frac{2n-3}{4} \rfloor$. In this case $k = 1$ or $k = 3$ depending on n is even or odd. If $k = 1$, then we have

$F_{2n+2}F_{2n+2} \equiv F_{2n-2}F_{2n+2} - F_3 \equiv F_{2n+2} - (F_{2n-1} + F_{2n-5} + \cdots + F_3) \pmod{m}$. Observe that $F_{2n+1} < F_{2n+2} - (F_{2n-1} + F_{2n-5} + \cdots + F_3) < m/2$ as we have the identity $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$.

If $k = 3$, then we have

$F_{2n}F_{2n+2} \equiv F_{2n-4}F_{2n+2} - F_5 \equiv F_{2n+2} - (F_{2n-1} + F_{2n-5} + \cdots + F_5) \pmod{m}$. Observe that $F_{2n+1} < F_{2n+2} - (F_{2n-1} + F_{2n-5} + \cdots + F_5) < m/2$ as we have the identity $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$.

Case II: ($k \in A_3$). Clearly (1) implies that

$$F_{7+4t}F_{2n+2} \equiv F_{3+4t}F_{2n+2} + F_{2n-2-4t} \pmod{m}.$$

$t = 0 \Rightarrow F_7F_{2n+2} \equiv F_3F_{2n+2} + F_{2n-2} \equiv -F_{2n+3} + F_{2n-2} \pmod{m}$. Observe that $F_{2n+1} < |-F_{2n+3} + F_{2n-2}| < m/2$.

$t = 1 \Rightarrow F_{11}F_{2n+2} \equiv F_7F_{2n+2} + F_{2n-6} \equiv -F_{2n+3} + F_{2n-2} + F_{2n-6} \pmod{m}$. Observe that $F_{2n+1} < |-F_{2n+3} + F_{2n-2} + F_{2n-6}| < m/2$.

Inductively, we let $t = \lfloor \frac{2n-4}{4} \rfloor$. In this case $k = 0$ or $k = 2$ depending on n is even or odd. If $k = 0$, then we have

$F_{2n+3}F_{2n+2} \equiv F_{2n-1}F_{2n+2} + F_2 \equiv -F_{2n+3} + (F_{2n-2} + F_{2n-6} + \cdots + F_2) \pmod{m}$. Observe that $F_{2n+1} < |-F_{2n+3} + (F_{2n-2} + F_{2n-6} + \cdots + F_2)| < m/2$ as we have $F_{2n+3} - (F_{2n-2} + F_{2n-6} + \cdots + F_2) = F_{2n+1} + F_{2n+2} - (F_{2n-2} + F_{2n-6} + \cdots + F_2)$ and the identity $F_{2n+1} - 1 = \sum_{i=1}^{n} F_{2i}$

If $k = 2$, the result follows from the above case $k = 0$.

Case III: ($k \in A_4$). Clearly (1) implies that

$$F_{8+4t}F_{2n+2} \equiv F_{4+4t}F_{2n+2} - F_{2n-3-4t} \pmod{m}.$$

$t = 0 \Rightarrow F_8F_{2n+2} \equiv F_4F_{2n+2} - F_{2n-3} \equiv -F_{2n+1} - F_{2n-3} \pmod{m}$. Observe that $F_{2n+1} < F_{2n+1} + F_{2n-3} < m/2$.

$t = 1 \Rightarrow F_{12}F_{2n+2} \equiv F_8F_{2n+2} - F_{2n-7} \equiv -(F_{2n+1} + F_{2n-3} + F_{2n-7}) \pmod{m}$. Observe that $F_{2n+1} < F_{2n+1} + F_{2n-3} + F_{2n-7} < m/2$.

Inductively, we let $t = \lfloor \frac{2n-5}{4} \rfloor$. In this case $k = 1$ or $k = 3$ depending on n is odd or even. If $k = 1$, then we have

$F_{2n+2}F_{2n+2} \equiv F_{2n-2}F_{2n+2} - F_3 \equiv -(F_{2n+1} + F_{2n-3} + F_{2n-7} + \ldots + F_3) \pmod{m}$. Observe that $F_{2n+1} + F_{2n-3} + F_{2n-7} + \ldots + F_3 < m/2$ as we have the identity $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$.

If $k = 3$. It follows from the above case $k = 1$.

Case IV: ($k \in A_5$) Clearly (1) implies that

$$F_{9+4t}F_{2n+2} \equiv F_{5+4t}F_{2n+2} + F_{2n-4-4t} \pmod{m}.$$

$t = 0 \Rightarrow F_9F_{2n+2} \equiv F_5F_{2n+2} + F_{2n-4} \equiv 2F_{2n+2} - F_{2n+1} + F_{2n-4} \pmod m$. Observe that $F_{2n+1} < 2F_{2n+2} - F_{2n+1} + F_{2n-4} < m/2$.

$t = 1 \Rightarrow F_{13}F_{2n+2} \equiv F_9F_{2n+2} + F_{2n-8} \equiv 2F_{2n+2} - F_{2n+1} + F_{2n-4} + F_{2n-8} \pmod m$. Observe that $F_{2n+1} < 2F_{2n+2} - F_{2n+1} + F_{2n-4} + F_{2n-8} < m/2$.

Inductively, we let $t = \lfloor \frac{2n-6}{4} \rfloor$. In this case $k = 0$ or $k = 2$ depending on n is odd or even. If $k = 0$, then we have

$F_{2n+3}F_{2n+2} \equiv F_{2n-1}F_{2n+2} + F_2 \equiv 2F_{2n+2} - F_{2n+1} + (F_{2n-4} + F_{2n-8} + \ldots + F_2) \pmod m$. Observe that $F_{2n+1} < 2F_{2n+2} - F_{2n+1} + (F_{2n-4} + F_{2n-8} + \ldots + F_2) < m/2$ as we have $2F_{2n+2} - F_{2n+1} + (F_{2n-4} + F_{2n-8} + \ldots + F_2) = F_{2n+2} + F_{2n} + (F_{2n-4} + F_{2n-8} + \ldots + F_2) < F_{2n+2} + F_{2n+1} - 1 < m/2$ using the identity $F_{2n+1} - 1 = \sum_{i=1}^{n} F_{2i}$

If $k = 2$. It follows from the above case $k = 0$. □

Theorem 5. *Let $F_2, F_3, \ldots, F_t$ be the Fibonacci numbers and let $n \geq 1$ be an integer such that $4n + 2 \leq t \leq 4n + 5$ and $m = F_{2n+2} + F_{2n+4} = L_{2n+3}$. Then the integers $F_2F_{2n+2}, F_3F_{2n+2}, \ldots, F_tF_{2n+2}$ modulo m all belong to the interval $[F_{2n+1}, 3F_{2n+2}]$. Moreover, both the endpoints of the interval $[F_{2n+1}, 3F_{2n+2}]$ are obtained only by the integers F_4F_{2n+2} and $F_{4n+2}F_{2n+2}$, respectively.*

Proof. Lemma 3 and Lemma 4 together prove the theorem. □

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