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On generalized Fibonacci and Lucas polynomials

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ABSTRACT

Let $h(x)$ be a polynomial with real coefficients. We introduce $h(x)$-Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials and Byrd's Fibonacci polynomials and also the k-Fibonacci numbers, and we provide properties for these $h(x)$-Fibonacci polynomials. We also introduce $h(x)$-Lucas polynomials that generalize the Lucas polynomials and present properties of these polynomials. In the last section we introduce the matrix $Q_h(x)$ that generalizes the Q-matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ whose powers generate the Fibonacci numbers.

1. Introduction

In modern science there is a huge interest in the theory and application of the Golden Section and Fibonacci numbers [1–34]. The Fibonacci numbers F_n are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$, where $F_n = F_{n-1} + F_{n-2}$, $n \geqslant 2$, with the initial values $F_0 = 0$ and $F_1 = 1$.

Falcón and Plaza [11] introduced a general Fibonacci sequence that generalizes, among others, both the classical Fibonacci sequence and the Pell sequence. These general k-Fibonacci numbers $F_{k,n}$ are defined by $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$, $n \geqslant 2$, with the initial values $F_0 = 0$ and $F_1 = 1$. The Pell numbers are the 2-Fibonacci numbers. The k-Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the well-known 4-triangle longest-edge partition. On the other hand, in [12] the k-Fibonacci numbers were given in an explicit way and many properties were proven. In particular, the k-Fibonacci numbers were related with the so-called Pascal 2-triangle.

Polynomials also can be defined by Fibonacci-like recurrence relations. Such polynomials, called Fibonacci polynomials, were studied in 1883 by the Belgian mathematician Eugene Charles Catalan and the German mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by the recurrence relation

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geqslant 3, \tag{1.1}$$

where $F_1(x) = 1, F_2(x) = x$. The Fibonacci polynomials studied by Jacobsthal are defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x), \quad n \geqslant 3, \tag{1.2}$$

where $J_1(x) = J_2(x) = 1$. The Fibonacci polynomials studied by P.F. Byrd are defined by

$$\varphi_n(x) = 2x\varphi_{n-1}(x) + \varphi_{n-2}(x), \quad n \geqslant 2, \tag{1.3}$$

where $\varphi_0(x) = 0$, $\varphi_1(x) = 1$. The Lucas polynomials $L_n(x)$, originally studied in 1970 by Bicknell, are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x), n \geqslant 2, \tag{1.4}$$

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where $L_0(x) = 2, L_1(x) = x$.

In this paper, let $h(x)$ be a polynomial with real coefficients. We introduce $h(x)$-Fibonacci polynomials that generalize both Catalan's Fibonacci polynomials $F_n(x)$ and Byrd's Fibonacci polynomials $\varphi_n(x)$ and also the k-Fibonacci numbers $F_{k,n}$. In Section 2 we provide properties for the $h(x)$-Fibonacci polynomials. In Section 3, we introduce $h(x)$-Lucas polynomials that generalize the Lucas polynomials and present properties of these polynomials. In the last section, we introduce the matrix $Q_h(x)$ that generalizes the Q-matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ whose powers generate the Fibonacci numbers. In this paper, we exhibit some properties of the classical type for the $h(x)$-Fibonacci and $h(x)$-Lucas polynomials and the matrix $Q_h(x)$.

2. The $h(x)$-Fibonacci polynomials and their properties

Definition 2.1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Fibonacci polynomials $\{F_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x), \quad n \geqslant 1, \tag{2.1}$$

with initial conditions $F_{h,0}(x) = 0$, $F_{h,1}(x) = 1$.

For $h(x) = x$ we obtain Catalan's Fibonacci polynomials, and for $h(x) = 2x$ we obtain Byrd's Fibonacci polynomials. If $h(x) = k$, we obtain the k-Fibonacci numbers. For $k = 1$ and $k = 2$ we obtain the usual Fibonacci numbers and the Pell numbers.

The generating function $g_F(t)$ of the sequence $\{F_{h,n}(x)\}$ is defined by

$$g_F(t) = \sum_{n=0}^{\infty} F_{h,n}(x)t^n. \tag{2.2}$$

We consider $g_F(t)$ a formal power series. Therefore, we need not take care of the convergence of the series. For general material on generating functions we refer to the books [21,34].

Theorem 2.1.

$$g_F(t) = \frac{t}{1 - h(x)t - t^2}. \tag{2.3}$$

Proof. We have

$$\begin{aligned} & g_F(t) - h(x)tg_F(t) - t^2 g_F(t) \\ & = F_{h,0}(x) + tF_{h,1}(x) + \sum_{n=2}^{\infty} t^n \left[F_{h,n}(x) - h(x)F_{h,n-1}(x) - F_{h,n-2}(x)\right] = t. \end{aligned}$$

We thus obtain (2.3). □

Theorem 2.2. *Suppose that $h(x)$ is an odd polynomial (that is, $h(-x) = -h(x)$). Then $F_{h,n}(-x) = (-1)^{n+1}F_{h,n}(x)$ for $n \geqslant 0$.*

Proof. From (2.3) we obtain

$$\sum_{n=0}^{\infty} F_{h,n}(-x)(-t)^n = \frac{-t}{1 - h(x)t - t^2}$$

or

$$\sum_{n=0}^{\infty} (-1)^{n+1} F_{h,n}(-x)t^n = \frac{t}{1 - h(x)t - t^2}. \tag{2.4}$$

Applying (2.3) to the right-hand side of (2.4) we obtain

$$\sum_{n=0}^{\infty} (-1)^{n+1} F_{h,n}(-x)t^n = \sum_{n=0}^{\infty} F_{h,n}(x)t^n.$$

This proves Theorem 2.2. □

Binet's formulas are well known in the theory of Fibonacci numbers. These formulas can also be carried out for the $h(x)$-Fibonacci polynomials. Let $\alpha(x)$ and $\beta(x)$ denote the roots of the characteristic equation

$$v^2 - h(x)v - 1 = 0 \tag{2.5}$$

of the recurrence relation (2.1). Then

$$\alpha(x)=\frac{h(x)+\sqrt{h^2(x)+4}}{2},\quad \beta(x)=\frac{h(x)-\sqrt{h^2(x)+4}}{2}. \tag{2.6}$$

Note that $\alpha(x)+\beta(x)=h(x)$, $\alpha(x)\beta(x)=-1$ and $\alpha(x)-\beta(x)=\sqrt{h^2(x)+4}$.

We obtain the following Binet's formula for $F_{h,n}(x)$. □

Theorem 2.3. *For* $n\geqslant 0$,

$$F_{h,n}(x)=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}. \tag{2.7}$$

Proof. From the theory of difference equations we know that the general term of the $h(x)$-Fibonacci polynomials may be expressed in the form $F_{h,n}(x)=A\alpha^n(x)+B\beta^n(x)$ for some coefficients A and B. Using the values $n=0$ and $n=1$ we obtain $A=\frac{1}{\alpha(x)-\beta(x)}$ and $B=\frac{-1}{\alpha(x)-\beta(x)}$, which proves (2.7). □

Theorem 2.4. *For* $n\geqslant 1$,

$$F_{h,n}(x)=\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-i-1}{i}h^{n-2i-1}(x). \tag{2.8}$$

Proof. From Theorem 2.1 we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}F_{h,n}(x)t^n&=\frac{t}{1-(h(x)t+t^2)}=t\sum_{n=0}^{\infty}(h(x)t+t^2)^n\\&=t\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(h(x)t)^{n-i}(t^2)^i\\&=\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}h^{n-i}(x)t^{n+i+1}.\end{aligned}$$

Writing $n+i+1=m$ we obtain

$$\sum_{n=0}^{\infty}F_{h,n}(x)t^n=\sum_{m=0}^{\infty}\left[\sum_{i=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-i-1}{i}h^{m-2i-1}(x)\right]t^m.$$

This proves (2.8). □

Theorem 2.5. *For* $n\geqslant 1$,

$$F_{h,n}(x)=2^{1-n}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}h(x)^{n-2i-1}(h^2(x)+4)^i. \tag{2.9}$$

Proof. By (2.6) we have

$$\begin{aligned}\alpha^n(x)-\beta^n(x)&=2^{-n}\left[\left(h(x)+\sqrt{h^2(x)+4}\right)^n-\left(h(x)-\sqrt{h^2(x)+4}\right)^n\right]\\&=2^{-n}\left[\sum_{i=0}^{n}\binom{n}{i}h^{n-i}(x)\left(\sqrt{h^2(x)+4}\right)^i\right.\\&\qquad\left.-\sum_{i=0}^{n}\binom{n}{i}h^{n-i}(x)\left(-\sqrt{h^2(x)+4}\right)^i\right]\\&=2^{-n+1}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}h^{n-2i-1}(x)\left(\sqrt{h^2(x)+4}\right)^{2i+1}.\end{aligned}$$

Thus, by (2.7),

$$\begin{aligned}F_{h,n}(x)&=\frac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)}=\frac{\alpha^n(x)-\beta^n(x)}{\sqrt{h^2(x)+4}}\\&=2^{-n+1}\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2i+1}h^{n-2i-1}(x)\left(h^2(x)+4\right)^i.\end{aligned}$$ □

Theorem 2.6. *For $n \geqslant 1$,*

$$F_{h,n}(x) = i^{n-1} U_{n-1}\left(\frac{h(x)}{2i}\right), \tag{2.10}$$

where $i^2 = -1$ and $U_n(t) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n-j}{j} (2t)^{n-2j}$ is the Chebyshev polynomial of the second kind.

Proof. It is known that

$$\sum_{n=0}^{\infty} U_n(t) z^n = \frac{1}{1 - 2tz + z^2}.$$

Let $z = iy$ and $t = \frac{h(x)}{2i}$. Then we obtain

$$\sum_{n=0}^{\infty} i^n U_n\left(\frac{h(x)}{2i}\right) y^n = \frac{1}{1 - h(x)y - y^2}$$

or

$$\sum_{n=0}^{\infty} i^n U_n\left(\frac{h(x)}{2i}\right) y^{n+1} = \frac{y}{1 - h(x)y - y^2}.$$

Thus, by Theorem 2.1, we obtain (2.10). □

Theorem 2.7. *Let $D_n(x)$ denote the $n \times n$ tridiagonal matrix defined as*

$$D_n(x) = \begin{bmatrix} 1 & i & & & & \\ 0 & h(x) & i & & & \\ & i & h(x) & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot \\ & & & & \cdot & \cdot & i \\ & & & & & i & h(x) \end{bmatrix}, \quad n \geqslant 1, \tag{2.11}$$

where $i^2 = -1$. In addition, let $D_0(x) = [0]$. Then

$$\det D_n(x) = F_{h,n}(x), \quad n \geqslant 0. \tag{2.12}$$

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$, we have $\det D_1(x) = 1 = F_{h,1}(x)$ and $\det D_2(x) = h(x) = F_{h,2}(x)$.

Assume that $\det D_{n-1}(x) = F_{h,n-1}(x)$ and $\det D_{n-2}(x) = F_{h,n-2}(x)$, $n \geqslant 3$. Then

$$\det D_n(x) = h(x) \det D_{n-1}(x) - i^2 \det D_{n-2}(x) = h(x) F_{h,n-1}(x) + F_{h,n-2}(x).$$

From Definition 2.1 we obtain

$$\det D_n(x) = F_{h,n}(x). \qquad \square$$

3. The $h(x)$-Lucas polynomials

In this section, we introduce $h(x)$-Lucas polynomials and present their basic properties. We do not write down all proofs, since they are similar to those in Section 2.

Definition 3.1. Let $h(x)$ be a polynomial with real coefficients. The $h(x)$-Lucas polynomials $\{L_{h,n}(x)\}_{n=0}^{\infty}$ are defined by the recurrence relation

$$L_{h,n+1}(x) = h(x) L_{h,n}(x) + L_{h,n-1}(x), \quad n \geqslant 1, \tag{3.1}$$

with initial conditions $L_{h,0}(x) = 2$, $L_{h,1}(x) = h(x)$.

For $h(x) = x$ we obtain the Lucas polynomials, and for $h(x) = 1$, we obtain the usual Lucas numbers.

Theorem 3.1. *For $n \geqslant 1$,*

$$L_{h,n}(x) = F_{h,n+1}(x) + F_{h,n-1}(x). \tag{3.2}$$

Proof. We proceed by induction on n. For $n = 1$ and $n = 2$, we have $L_{h,1}(x) = h(x) = F_{h,2}(x) + F_{h,0}(x)$ and $L_{h,2}(x) = h^2(x) + 2 = F_{h,3}(x) + F_{h,1}(x)$. Now, assume that $L_{h,n-1}(x) = F_{h,n}(x) + F_{h,n-2}(x)$, $n \geqslant 3$. From Definitions 3.1 and 2.1,

$$\begin{aligned} L_{h,n}(x) &= h(x)L_{h,n-1}(x) + L_{h,n-2}(x) \\ &= h(x)\big(F_{h,n}(x) + F_{h,n-2}(x)\big) + F_{h,n-1}(x) + F_{h,n-3}(x) \\ &= \big(h(x)F_{h,n}(x) + F_{h,n-1}(x)\big) + \big(h(x)F_{h,n-2}(x) + F_{h,n-3}(x)\big) \\ &= F_{h,n+1}(x) + F_{h,n-1}(x). \qquad \square \end{aligned}$$

Corollary 3.1. *For $n \geqslant 1$,*

$$L_{h,n}(x) = h(x)F_{h,n}(x) + 2F_{h,n-1}(x). \tag{3.3}$$

Corollary 3.2. *For $n \geqslant 0$,*

$$L_{h,n}(x) = 2F_{h,n+1}(x) - h(x)F_{h,n}(x). \tag{3.4}$$

Theorem 3.2. *The generating function*

$$g_L(t) = \sum_{n=0}^{\infty} L_{h,n}(x)t^n \tag{3.5}$$

of the sequence $\{L_{h,n}(x)\}$ is given as

$$g_L(t) = \frac{2 - h(x)t}{1 - h(x)t - t^2}. \tag{3.6}$$

Theorem 3.3. *Suppose that $h(x)$ is an odd polynomial. Then $L_{h,n}(-x) = (-1)^n L_{h,n}(x)$.*

Theorem 3.4. *For $n \geqslant 0$, $L_{h,n}(x) = \alpha^n(x) + \beta^n(x)$, where $\alpha(x)$ and $\beta(x)$ are the roots of the characteristic Eq. (2.5).*

Corollary 3.3. *For $n \geqslant 0$,*

$$\alpha^n(x) = \frac{L_{h,n}(x) + \sqrt{h^2(x) + 4}F_{h,n}(x)}{2}, \tag{3.7}$$

$$\beta^n(x) = \frac{L_{h,n}(x) - \sqrt{h^2(x) + 4}F_{h,n}(x)}{2}. \tag{3.8}$$

Corollary 3.4. *For $n \geqslant 0$,*

$$L^2_{h,n}(x) - \left(h^2(x) + 4\right)F^2_{h,n}(x) = 4(-1)^n. \tag{3.9}$$

Corollary 3.5. *For $n \geqslant 0$,*

$$F_{h,2n}(x) = F_{h,n}(x)L_{h,n}(x). \tag{3.10}$$

Theorem 3.5. *For $n \geqslant 1$,*

$$L_{h,n}(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{n-i}\binom{n-i}{i} h^{n-2i}(x). \tag{3.11}$$

Theorem 3.6. *For $n \geqslant 1$,*

$$L_{h,n}(x) = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} h^{n-2i}(x)\left(h^2(x) + 4\right)^i. \tag{3.12}$$

Theorem 3.7. *For $n \geqslant 0$,*

$$L_{h,n}(x) = 2i^n T_n\left(\frac{h(x)}{2i}\right), \tag{3.13}$$

where $i^2 = -1$ and $T_n(t) = \frac{n}{2}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{n-j}\binom{n-j}{j}(2t)^{n-2j}$ is the Chebyshev polynomial of the first kind.

Theorem 3.8. *Let $R_n(x)$ denote the $n \times n$ tridiagonal matrix defined as*

$$R_n(x) = \begin{bmatrix} 2 & i & & & & & \\ 0 & \frac{h(x)}{2} & i & & & & \\ & i & h(x) & . & & & \\ & & . & . & . & & \\ & & & . & . & . & \\ & & & & . & . & i \\ & & & & & i & h(x) \end{bmatrix}, \quad n \geqslant 1, \tag{3.14}$$

where $i^2 = -1$. Then

$$\det R_n(x) = L_{h,n-1}(x), \quad n \geqslant 1. \tag{3.15}$$

4. The matrix $Q_h(x)$ and its properties

In this section, we introduce the matrix $Q_h(x)$ that plays the role of the Q-matrix. The Q-matrix is associated with the Fibonacci numbers and is defined as

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Definition 4.1. Let $Q_h(x)$ denote the 2×2 matrix defined as

$$Q_h(x) = \begin{bmatrix} h(x) & 1 \\ 1 & 0 \end{bmatrix}. \tag{4.16}$$

Theorem 4.1. *Let $n \geqslant 1$. Then*

$$Q_h^n(x) = \begin{bmatrix} F_{h,n+1}(x) & F_{h,n}(x) \\ F_{h,n}(x) & F_{h,n-1}(x) \end{bmatrix}. \tag{4.17}$$

Proof. We proceed by induction on n. The result holds for $n = 1$. Assume that it holds for $n = m$ $(\geqslant 1)$. Then

$$\begin{aligned} Q_h^{m+1}(x) = Q_h^m(x)Q_h(x) &= \begin{bmatrix} F_{h,m+1}(x) & F_{h,m}(x) \\ F_{h,m}(x) & F_{h,m-1}(x) \end{bmatrix}\begin{bmatrix} h(x) & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} h(x)F_{h,m+1}(x) + F_{h,m}(x) & F_{h,m+1}(x) \\ h(x)F_{h,m}(x) + F_{h,m-1}(x) & F_{h,m}(x) \end{bmatrix} \\ &= \begin{bmatrix} F_{h,m+2}(x) & F_{h,m+1}(x) \\ F_{h,m+1}(x) & F_{h,m}(x) \end{bmatrix}. \qquad \square \end{aligned}$$

Corollary 4.1. *Let $n \geqslant 1$. Then*

$$F_{h,n+1}(x)F_{h,n-1}(x) - F_{h,n}^2(x) = (-1)^n. \tag{4.18}$$

Proof. Since $\det Q_h(x) = -1$, then $\det Q_h^n(x) = (-1)^n$. By Theorem 4.1, $\det Q_h^n(x) = F_{h,n+1}(x)F_{h,n-1}(x) - F_{h,n}^2(x)$. We thus obtain (4.18). $\square$

Corollary 4.2. *Let $m, n \geqslant 0$. Then*

$$F_{h,m+n+1}(x) = F_{h,m+1}(x)F_{h,n+1}(x) + F_{h,m}(x)F_{h,n}(x). \tag{4.19}$$

In particular,

$$F_{h,2n+1}(x) = F_{h,n+1}^2(x) + F_{h,n}^2(x). \tag{4.20}$$

Proof. This result follows from the identity $Q_h^{m+n}(x) = Q_h^m(x)Q_h^n(x)$. □

Theorem 4.2. *The roots of the characteristic equation of* $Q_h^n(x)$ *are* $\alpha^n(x)$ *and* $\beta^n(x)$.

Proof. The characteristic polynomial of $Q_h^n(x)$ is

$$\det\left(Q_h^n(x) - \lambda I\right) = \det\begin{bmatrix} F_{h,n+1}(x) - \lambda & F_{h,n}(x) \\ F_{h,n}(x) & F_{h,n-1}(x) - \lambda \end{bmatrix}$$
$$= \lambda^2 - \lambda\left(F_{h,n+1}(x) + F_{h,n-1}(x)\right) + \left(F_{h,n+1}(x)F_{h,n-1}(x) - F_{h,n}^2(x)\right).$$

On the basis of Theorem 3.1 and Corollary 4.1,

$$\det\left(Q_h^n(x) - \lambda I\right) = \lambda^2 - \lambda L_{h,n}(x) + (-1)^n.$$

The roots of the characteristic equation are

$$\lambda = \frac{L_{h,n}(x) \pm \sqrt{L_{h,n}^2(x) - 4(-1)^n}}{2}.$$

From Corollary 3.4 we obtain

$$\lambda = \frac{L_{h,n}(x) \pm \sqrt{h^2(x) + 4}\, F_{h,n}(x)}{2}$$

and from Corollary 3.3 we obtain

$$\lambda = \alpha^n(x) \text{ or } \lambda = \beta^n(x). \quad \square$$

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